Franz-Viktor Kuhlmann *210 McLean Hall, Tel. 6111*

Abstract Algebra — Math 363.3 (01) T1, 1999–2000

Final Exam (Tuesday, December 21, 1999, in class)

Time: 3 hours

This is a "closed book" examination. — No calculators.
Show all your work to receive full credit.

Every problem is worth 10 marks. So the maximum number of marks is 100. The bonus marks are counted as long as your total does not exceed 100.

1) Prove by induction that the following holds for all positive integers n:

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3} .$$

2) a) Which of the following relations are equivalence relations (explain!): $x \sim y$ if

(i) x is the brother of y, or $x = y$ (for x and y human beings),
(ii) x is cousin of y, or $x = y$ (for x and y human beings),
(iii) x divides y (for x and y integers),
(iv) $x = y$ (for x and y integers),
(v) x and y have the same number of elements (for x and y sets).

b) Give the definition of "partition".

c) Determine the smallest positive representative of the following congruence class in $\mathbb{Z}/7\mathbb{Z}$:

$$\overline{750 \cdot 13 - 777 \cdot 12345} .$$

d) Find the smallest positive representative for the congruence class of

i) 60 modulo 21 ii) 1110 modulo 11 iii) -27 modulo 11 .

e) [**4 bonus marks**] Prove that the following is an equivalence relation for the elements of a given ring with 1: $x \sim y$ if there is a unit u such that $x = uy$. (What do we need the 1 for?)

3) a) State Lagrange's Theorem.

b) What is the order of S_6 ?

c) Is there a subgroup of order 7 in S_6 ? Explain!

d) What is the order of $\mathbb{Z}/n\mathbb{Z}$? Write down the cosets in $\mathbb{Z}/3\mathbb{Z}$. They are equivalence classes of which equivalence relation?

e) What are the possible orders of subgroups of $\mathbb{Z}/6\mathbb{Z}$?

/...2

4) **a)** Compute the following permutations:

$$(i)\ \begin{pmatrix} 1\,2\,3\,4\,5 \\ 5\,4\,3\,1\,2 \end{pmatrix}\begin{pmatrix} 1\,2\,3\,4\,5 \\ 5\,2\,1\,4\,3 \end{pmatrix}$$

$$(ii)\ \ (1\,5\,2\,4\,3)(1\,3\,4) \quad (iii)\ \ (4\,5)(4\,5)(2\,4\,5)\,.$$

b) Write in cycle notation:

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 5\,3\,4\,2\,1\,6 \end{pmatrix}, \quad \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 6\,5\,4\,3\,2\,1 \end{pmatrix}.$$

c) Is the permutation $(1\,3\,5)(2\,5)(1\,3)$ even or odd? Explain.

d) Let μ be any odd permutation. Prove that also μ^{-1} is odd.

e) [**2 bonus marks**] Write as a product of transpositions, then compute: $(1\,3\,2)(1\,3)(1\,3\,2)(1\,2\,3)$.

5) **a)** Determine the greatest common divisor of the polynomials $x^3 + 8$ and $x^3 - 7x - 6$ in $\mathbb{Q}[x]$.

b) Do the same for the polynomials $x^3 + \overline{3}$ and $x^3 + \overline{3}x - \overline{1}$ in $\mathbb{Z}/5\mathbb{Z}[x]$.

c) [**3 bonus marks**] Determine polynomials $f(x), g(x) \in \mathbb{Z}/5\mathbb{Z}[x]$ such that $f(x)(x^3+\overline{3})+g(x)(x^3+\overline{3}x-\overline{1})$ is the greatest common divisor of $x^3+\overline{3}$ and $x^3 + \overline{3}x - \overline{1}$.

6) Draw a regular hexagon (6 equal sides, 6 equal angles) and enumerate the six vertices clockwise from 1 to 6. Let G denote the group of symmetries of the hexagon.

a) Write down all rotations of the hexagon in cycle notation.

b) Determine:

$$G_{\{1\}}\,, \quad G_{\{1,2,5\}}\,, \quad G_{(\{1,2,5\})}\,, \quad G_{(\{1,3,5\})}\,, \quad G_{(\{1,6\})}\,.$$

Write every symmetry appearing in these groups in cycle notation and say which symmetry it is.

c) [**3 bonus marks**] List all possible generators of the group of rotations of the regular hexagon.

7) **a)** Find the greatest common divisor of -139 and -39. Find integers m and n such that $m(-139) + n(-39) = (-139, -39)$.

b) Determine the multiplicative inverse of $\overline{39}$ in Z_{139}.

c) Determine the multiplicative inverses of the units $\overline{2}$ (hint: 140 is even) and $\overline{138}$ in Z_{139}.

/...3

8) a) Is $\overline{77777}$ a unit in $\mathbb{Z}/49\mathbb{Z}$? Explain!

b) Determine all units together with their inverses, and all zero-divisors in $\mathbb{Z}/9\mathbb{Z}$.

9) a) Give the definition of "subgroup".

b) State one of the equivalent forms of the subgroup criterion. Use it to show that $n\mathbb{Z}$ is a subgroup of $(\mathbb{Z}, +)$ for every $n \in \mathbb{N}$.

c) Prove that for every $n \in \mathbb{N}$, A_n is a subgroup of S_n .

10) Let $(G, \cdot)$ be any group (not necessarily abelian!), and $a, b, c \in G$.

a) Determine the inverse of $ab^{-1}ca^{-1}bca$.

b) Solve for x:

$$axbc^{-1} = b\,.$$

c) Assume that $(G, \cdot)$ is abelian. Which of the following elements are equal to the identity for all possible choices of $a, b, c \in G$?
i) $abc^{-1}a^{-1}cb^{-1}$, ii) $ac^{-1}ba^{-1}ca$, iii) $ab^2c^{-1}a^{-1}bc^3a$,
iv) $c^{-1}ab^2c^{-1}a^{-1}b^{-1}c^2b^{-1}$.

d) Make Cayley tables of $(\mathbb{Z}/3\mathbb{Z}, +)$ and of $(\mathbb{Z}/3\mathbb{Z}, \cdot)$. Are both of them groups? Explain!

11) [10 bonus marks] Let T be a set of points in the plane (that is, $T \subseteq \mathbb{R} \times \mathbb{R}$). Further, let $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be a translation.

a) Define the mapping μ^k for every $k \in \mathbb{Z}$. (Hint: use induction for positive k, and use $k = -(-k)$ for negative k.)

b) Prove that $\mu^k(T) = T$ for all $k \in \mathbb{Z}$.

c) Show: if T is infinite, then it does not admit any non-trivial translation as a symmetry.

d) Show that the subset $\mathbb{Z} \times \mathbb{Z}$ of $\mathbb{R} \times \mathbb{R}$ has the symmetries μ_1 and μ_2 defined by $\mu_1(x, y) = (x + 1, y)$ and $\mu_1(x, y) = (x, y + 1)$. What are the inverses of these translations? What is $\mu_1^k \circ \mu_2^\ell(x, y)$ for $k, \ell \in \mathbb{Z}$?

e) Show that $(\{\mu_1^k \circ \mu_2^\ell \mid k, \ell \in \mathbb{Z}\}$ is an abelian subgroup of the motions of the plane.

* * * THE END * * *