Research Proposal [1]

A) Local uniformization

Local uniformization is a local form of resolution of singularities, and as such, is of valuation theoretical nature. As resolution of singularities is not known in positive characteristic, and since the original approach of Zariski used local uniformization as a prerequisite for resolution of singularities, it is an important open problem to prove local uniformization in positive characteristic. In [1], we prove local uniformization for Abhyankar places. In [19], local uniformization is proved for every place after a finite (Galois or purely wild) extension of the function field. In both papers, the results are generalized to the arithmetic case, but there is still a lot of room for improvement. I am planning to intensify my joint research with H. Knaf on open questions in the arithmetic case, and on the problem of simultaneous local uniformization of finitely many places.

I wish to refine my methods used in [1] and [19] in order to prove (or disprove) that local uniformization holds in positive characteristic without extension of the function field. A general question in this connection is to determine when and how local uniformization can be pulled down through algebraic extensions. If this would be known for extensions within the henselization, then my approach would yield an easy conceptual proof of Zariski's original local uniformization result in characteristic 0 ([Z]). For positive characteristic, stronger pull down results would be needed. I am planning to work on these questions together with O. Piltant and H. Knaf.

But there could also be a second way of refining the methods. The proofs in [1] and [19] use two deep results in the theory of valued function fields. One of them is the "Generalized Stability Theorem" [20], a generalization of the Grauert–Remmert Stability Theorem ([BGR]). The other is the theorem on "Henselian Rationality" which I will mention again in E) below. The method of proof for both is closely related to methods developed by Abhyankar ([A1], [A2], [A3]). The idea is to give suitable normal forms for "minimal purely wild extensions". Via ramification theory, these can be reduced to Artin-Schreier extensions, which are much easier to handle. A promising direction of research is to determine in which way the necessary hypotheses for these theorems and techniques can be relaxed. I will closely examine this question when I revise the proofs of my thesis for my book [30] and for the paper [32].

In the case of Henselian Rationality, Kaplansky's theory of immediate extensions plays a crucial role. One important question is whether there is some reasonable generalization of this theory to the case of extensions of transcendence degree > 1.

In the case of Henselian Rationality, I also proved a pull down principle in my thesis; the proof may lead the way to similar pull down principles for local uniformization.

A "test case" for local uniformization is the following question (discussed in [15]): Suppose $F|K$ admits a rational place and K is large, in the sense of [P]. Is K existentially closed in F? This is true if K is perfect. It would also be true if every rational place of $F|K$ admits local uniformization. The question has already attracted the interest of other

[1] The number citations refer to my list of contributions (Form 100 *and* part I of this proposal), and the letter citations to the references (Form 101).

model theorists, as I posed it at the Euro-Conference in Model Theory and Applications (in honour of A. MacIntyre's 60th birthday) at Ravello in May of 2002.

B) The defect

Through my work, it has become very clear that the reason for our problems in positive characteristic (for local uniformization as well as for the model theory of valued fields, see below) is the valuation theoretical phenomenon of "defect" (also called "ramification deficiency"). Even very simple fields, like rational function fields in two variables, can admit finite extensions with non-trivial defect if they are endowed with not so simple valuations. In [2] I construct such examples. In a recent paper ([CP]), Cutkosky and Piltant exploit one of these examples to study so-called "Simultaneous Resolution" for finite extensions. They have quite a different way of presenting this example. When Piltant visited me for the month of September 2002, we started to investigate the various ways of presenting such examples, their properties and what they tell us about resolution of singularities. It turned out that a classification of Artin-Schreier extensions with defect that I developed in my Ph.D. thesis can shed some light on phenomena that Cutkosky and Piltant observed in their work. Therefore, I will study in joint work with Piltant Artin-Schreier extensions with defect over valued rational function fields [33].

C) Extensions of valuations to rational function fields

In spite of my comprehensive paper [8], there are still open problems about the extensions of valuations to rational function fields. I plan to work on such questions together with S. Khanduja and H. Knaf. There is also one mysterious open problem in [8] concerning the value group and residue field obtained through power series. I plan to consider this problem in joint work with M. Vaquié and O. Piltant.

The biggest open problem is to compare the different ways of describing extensions of valuations: MacLane's and Vaquié's way using key polynomials, my own (and other researcher's) way using pseudo Cauchy sequences, generating sequences, and an approach due to Ron Brown. I learnt more about the latter approach (which works over maximal fields) when R. Brown visited our group recently for 7 weeks. During a visit of M. Vaquie in April/May of this year, I started to work with him on the problem of translating from key polynomials to pseudo Cauchy sequences, and vice versa. I am also planning to start a collaboration with L. Ghezzi and other young researchers I met at the summer school in Ottawa to discuss these questions in relation to their work. In particular, we want to study the example given in [CP] from this point of view.

My papers [4] and [8] show that rational function fields in several variables can have very nasty valuations which are yet trivial on the ground field. In view of the open problem of local uniformization in positive characteristic, it is important to know more about such valuations which allow algebraic extensions with defect. There are many open questions about such valuations, some of which are mentioned in [6]. I plan to work on these interesting and important problems, probably in collaboration with the above mentioned researchers.

D) Resolution of singularities

Suppose we have proved local uniformization in general. How can local resolutions be pieced together to obtain a global resolution? I intend to study classical approaches to this problem in detail. I wish to work on this problem in joint work with S. D. Cutkosky, H. Knaf, V. Cossart, B. Teissier and O. Piltant.

E) Model theory of valued fields

In my dissertation, I introduced and studied tame fields (which comprise all those perfect valued fields which are interesting for model theoretical reasons). I showed that they satisfy the Ax–Kochen–Ershov Theorem. The proof uses the same main valuation theoretical principles as my proof for local uniformization. Roughly speaking, the new results completed our knowledge about the model theory of perfect valued fields of positive characteristic. However, it remains to find a natural invariant relative to which tame fields admit quantifier elimination.

Compared to all classes of valued fields for which the Ax–Kochen–Ershov Theorem had been known before, the proof for the class of tame fields is by far more difficult. This is so because tame fields in general do not admit any sort of hulls that are unique up to isomorphism and could be used for the proof. Instead, I had to prove Henselian Rationality, which is a technically demanding structure theorem on the henselizations of valued function fields over tame fields.

The success of the new idea of proof generates hope that by refining the methods I will be able to determine the elementary properties of non-perfect valued fields, such as the power series field $\mathbb{F}_p((t))$. My goal is to find out whether adding the new axiom scheme developed in [3] leads to a complete axiomatization. For this, it is necessary to study further the structure theory of valued function fields — as outlined under A), B) and C) — and the structure induced by additive polynomials on non-perfect valued fields.

In joint work with Lou van den Dries I have shown in [12] that the images of all additive polynomials on $\mathbb{F}_p((t))$ have the optimal approximation property; this is, so to say, the elementary content of spherical completeness. This result leads to an axiom scheme much nicer than the one given in [13]. We make use of the fact that $\mathbb{F}_p((t))$ is locally compact (which is not an elementary property). I expect that the result holds in general for all maximal valued fields. But as local compactness is not at hand in general, it seems that a proof for this would have to rely on difficult combinatorial methods (which may turn out to be some sort of valuation theoretical analogue of Gröbner bases). However, the maximal fields are of particular interest in the model theory of valued fields, so we should make an effort to understand their elementary properties.

The result of [12] can also be proven by a result of Ershov about what he calls "extremal fields". Extremality is an elementary property, but apart from Ershov's result hardly anything is known about it. We do not even know whether maximal fields are extremal, not even in characteristic 0! It would be surprising if this were not true, but there is so far no obvious proof. I plan to work on this important and very natural notion that has not been noticed until lately by valuation theorists.

F) Nonarchimedean ordered exponential fields, Hardy fields

I want to further investigate the structure of nonarchimedean ordered fields with exponential functions by means of their natural valuation. This is important because among such fields there are the nonstandard models of real exponentiation as well as Hardy fields, which encode the asymptotic behaviour of functions on the reals.

Valuation theory has turned out to be a powerful tool in the study of nonarchimedean ordered exponential fields. Salma Kuhlmann has written a book about this approach and its results in the Fields Institute Monograph Series [K].

In particular, after having investigated the value groups of nonarchimedean ordered exponential fields, we now wish to continue our work on the residue fields of such fields, with respect to arbitrary convex valuations. As shown in the mentioned monograph, knowledge about these residue fields leads to proofs of some interesting results of [DMM] without the tool of truncation closed embeddings in LE-series fields. There are some unsolved mysteries about theories involving generalized power series; solving them could lead to generalizations of the mentioned results.

I also would like to work on some unsolved problems about "asymptotic couples" (work of Aschenbrenner and van den Dries). These are structures describing the interaction between valuation and logarithmic derivation on Hardy fields. They are closely related to (or, in some sense, a more precise version of) "contraction groups" which I introduced and studied in earlier papers (see also the appendix of [K]).

G) Spherically complete differential and difference fields, *D*-fields, and other structures

In the spirit of [5], I want to work on further applications of the Main Theorem of that paper. In particular, I am interested in spherically complete valued differential, difference and D-fields, extending the work of Rosenlicht ([Rs1], [Rs2]) and Scanlon ([S1], [S2]). I want to determine the relation to work by Robba and others (e.g., Hensel's Lemma for differential operators, cf. [Ro]). I also intend to study closely recent work by Prieß-Crampe and Ribenboim (based on [Pc] and [PR]). It appears to be somewhat complementary to my own paper [5]. Therefore, putting both approaches together may lead to interesting new ideas. In particular, refinements of the differential Hensel's Lemmas for both the Rosenlicht type and the Scanlon type differential valued fields would be desirable. In the Rosenlicht case, this could be achieved when one restricts the scope to situations where one has a better control of the relation between valuation and derivation.

Furthermore, I want to explore possible applications of the Main Theorem of [5] in all situations where spherically complete spaces (induced by spherically complete fields such as the p-adics) are around. I believe that a number of more or less classical results in such situations can be proved more elegantly using this theorem.

H) Further research topics

Apart from the above mentioned problems, I am planning to work on various subjects in valuation theory, field theory, and the theory of ordered structures.

Since 1995 I am interested in the valuation theory of cuts in ordered fields and ordered abelian groups. In the past years I have learnt about some new and challenging problems, which I will investigate in [28].

During the preparation of [8], I came across the problem of proving algebraic independence of elements in immediate extensions of valued fields. I am working on far-reaching generalizations ([34]) of classical results of MacLane and Schilling ([MS]). This is connected with problems recently studied by other researchers in the setting of power series. I am planning to dive deeper into this area.

Together with H. Lombardi and H. Perdry I have worked on algorithmic aspects of valued fields. This work shall be continued. Under my guidance, Perdry developed a nice proof for the continuity of roots, which I will prepare for inclusion in my book. I plan to work further with him on similar problems.

I) Papers to be submitted to refereed journals within the next year

My first immediate project is to finish the following papers and to submit them to refereed journals: [28] and

[31] Knaf, H. – Kuhlmann, F.-V.: *Every place admits local uniformization after a finite purely wild extension of the function field*

[32] Kuhlmann, F.-V.: *Elimination of ramification II: henselian rationality of valued function fields*

[33] Kuhlmann, F.-V.: *The model theory of tame valued fields*

[34] Kuhlmann, F.-V.: *Algebraic independence of elements in completions and maximal immediate extensions of valued fields*

[35] Khanduja, S. K. – Knaf, H. – Kuhlmann, F.-V.: *The uniqueness property for value-transcendental function fields*

[36] Kuhlmann, F.-V. – Piltant, O.: *Artin–Schreier extensions with non-trivial defect*

J) Book in preparation

Several chapters of my book are already available on my web site. I intend to complete the book within 2007 or 2008. The book is based on my habilitation thesis and on my doctoral thesis. It takes up the tradition of the books of Ribenboim [Ri] and Endler [E], but at least half of it will consist of new results, most of them obtained through my own research. Main aspects of this book will be:

- modern exposition of the classical results of valuation theory
- presentation of many facts that are used in ongoing research, but are hard to find in the literature
- thorough study of valuations with positive residue characteristic
- description of the interrelation between algebra and model theory of valued fields
- presentation of the valuation theory of local uniformization and resolution of singularities and its relation to the model theory of valued fields
- formulation of important open problems and possible approaches to their solution.

K) Other projects

- Together with M. Bremner, S. Kuhlmann and M. Marshall, I am maintaining the Research Centre "Algebra, Logic and Computation" at the University of Saskatchewan, in close cooperation with our colleagues from the University of Regina. Main goal of this Research Centre is the promotion of research and training of highly qualified personnel. This will be achieved through invitation of researchers, research seminars and colloquia, conferences and summer schools, and other forms of dissemination (e.g., internet fora). Based on what our group (S. Kuhlmann, M. Marshall and myself) has built up at the U of S since 1997, this Centre has strong relations to other research groups nationally and internationally. One of its main goals is the training of young researchers, providing information and consultation for their research programs and the opportunity to meet established and renowned researchers in their area.

- Maintainance and further development of the Valuation Theory Home Page.

- Organization of the Seventh Annual Colloquiumfest and the Second International Valuation Theory Conference and Workshop.