Construction of exact plasma equilibrium solutions with different geometries.

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Abstract

Infinite families of exact isotropic and anisotropic plasma equilibria with and without dynamics can be constructed in different geometries, using the representation of the static MHD equilibrium system in coordinates connected with magnetic surfaces. A sample equilibrium anisotropic model of earth magnetosheath plasma is given.

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The isotropic incompressible Magnetohydrodynamics (MHD) equations [1] and Chew-Golberger-Low (CGL) anisotropic plasma equations [2] are widely used in thermonuclear fusion, astrophysics and geophysics research [1]-[8].

For many applications, equilibrium and quasi-equilibrium plasma configurations are of particular interest. The search for exact equilibrium solutions to MHD and CGL systems is going during last decades. Several types and families of exact solutions were found, primarily in the isotropic MHD framework, using mainly reductions by symmetry groups (as Grad-Shafranov and JFKO equations for axial and helical symmetry, e.g. [3], [9]-[12]) and Euler potentials (e.g. [13, 14]). A CGL analog of the Grad-Shafranov equation is known [15] but has not been widely used due to its complexity.

Recently, infinite groups of symmetries of MHD and CGL equilibrium systems [16]-[19] and an infinite-parameter set of transformations from MHD to CGL equilibria [18] have been found. These symmetries and transformations alter physical parameters of plasma equilibria, but preserve solution topology. For effective analytical modeling of physical phenomena, a greater diversity of exact solutions with different geometries is required, but only a limited number is available.

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The main purpose of this letter is to report a rather general method of construction of exact plasma equilibria having various types of magnetic field geometry and different physical properties. Combined with infinite symmetries and transformations mentioned above, new solutions provide a tool for modeling a variety of plasma phenomena with exact solutions in MHD and CGL frameworks. The suggested method also allows the construction of exact time-independent Euler and irrotational Navier-Stokes fluid flows.

The method illustrates the importance of knowledge and explicit use of the topological properties in the analysis of physical PDE systems with several spatial variables.

The isotropic incompressible Magnetohydrodynamics (MHD) equilibrium equations are

$$\rho \mathbf{V} \times \text{curl } \mathbf{V} - \frac{1}{\mu} \mathbf{B} \times \text{curl } \mathbf{B} - \text{grad } P - \rho \text{grad } \frac{\mathbf{V}^2}{2} = 0, \quad (1)$$

$$\text{div}(\rho \mathbf{V}) = 0, \quad \text{curl}(\mathbf{V} \times \mathbf{B}) = 0, \quad \text{div } \mathbf{B} = 0, \quad \text{div } \mathbf{V} = 0. \quad (2)$$

They describe plasmas maintained isotropic by frequent collisions. In strongly magnetized or rarified plasmas, the plasma pressure P of the MHD model is replaced by a 3×3 pressure tensor, giving rise to the Chew-Golberger-Low (CGL) equilibrium model [2, 18]:

$$\begin{aligned} \rho \mathbf{V} \times \text{curl } \mathbf{V} - \left(\frac{1}{\mu} - \tau \right) \mathbf{B} \times \text{curl } \mathbf{B} &= \text{grad } p_{\perp} + \rho \text{grad } \frac{\mathbf{V}^2}{2} + \\ &+ \tau \text{grad } \frac{\mathbf{B}^2}{2} + \mathbf{B}(\mathbf{B} \cdot \text{grad } \tau), \end{aligned} \quad (3)$$

$$\text{div}(\rho \mathbf{V}) = 0, \quad \text{curl}(\mathbf{V} \times \mathbf{B}) = 0, \quad \text{div } \mathbf{B} = 0, \quad (4)$$

$p_{\parallel}$ and $p_{\perp}$ are pressure components along the magnetic field and in the transverse direction. The MHD equilibrium system requires one equation of state; the CGL system needs two. An incompressibility condition $\text{div } \mathbf{V} = 0$ is often imposed as one of such equations.

A general MHD or CGL equilibrium possesses a family of nested 2D magnetic surfaces, to which velocity $\mathbf{V}$ and magnetic field $\mathbf{B}$ are tangent. It is also the case for all static ($\mathbf{V} = 0$) MHD equilibrium reductions except Beltrami-type solutions [16, 18, 20]. In many important cases, an *orthogonal* coordinate system can be constructed, with one of the coordinates constant on magnetic surfaces [21]. In such coordinates, the static plasma equilibrium system takes a particular form, and exact solutions are often found.

We rewrite static MHD equilibrium system

$$\mathbf{J} \times \mathbf{B} = \text{grad } P, \quad \text{div } \mathbf{B} = 0, \quad \text{curl } \mathbf{B} = \mu \mathbf{J}, \quad (5)$$

(which is equivalent to the equilibrium Euler equation of incompressible fluid motion,) in coordinates connected with magnetic surfaces. Suppose a triply-orthogonal coordinate system (u_i) , $i = 1, 2, 3$, is given, such that the coordinate u_3 enumerates magnetic surfaces. Then the pressure is $P(\mathbf{r}) = P(u_3)$, and the magnetic field does not have a u_3 - component: $\mathbf{B} = B^i \mathbf{e}_i$, $i = 1, 2$. (From here on, summation in the repeated indices from 1 to 2 is assumed.) Using the form of linear differential operators in orthogonal coordinates [22], we find from (5) that there exists a potential $\Phi = \Phi(u_1, u_2, u_3)$, through which the magnetic field is expressed:

$$\mathbf{B} = \frac{1}{\sqrt{g_{ii}}} \frac{\partial \Phi}{\partial u_i} \mathbf{e}_i \equiv \text{grad}_{(u_1, u_2)} \Phi, \quad (6)$$

where (u_1, u_2) means that only u_1 - and u_2 - parts of the operator are used. Here $g_{ij} = g_{ii}^2 \delta_{ij}$, $i, j = 1, 2, 3$, are the components of the metric tensor. The electric current density is

$$\mathbf{J} = \frac{1}{\mu} \text{curl } \mathbf{B} = \frac{1}{\mu} \left(-\frac{1}{\sqrt{g_{22}g_{33}}} \frac{\partial^2 \Phi}{\partial u_2 \partial u_3} \mathbf{e}_1 + \frac{1}{\sqrt{g_{11}g_{33}}} \frac{\partial^2 \Phi}{\partial u_1 \partial u_3} \mathbf{e}_2 \right). \quad (7)$$

In coordinates (u_i) , u_1 - and u_2 - projections of the first (vector) equation of (5) vanish identically, and the solenoidality condition $\text{div } \mathbf{B} = 0$ and the u_3 - projection of the first equation are

$$\frac{\partial}{\partial u_i} \left(\frac{g}{g_{ii}} \frac{\partial \Phi}{\partial u_i} \right) = 0, \quad g = \sqrt{g_{11}g_{22}g_{33}}; \quad (8)$$

$$\frac{1}{g_{ii}} \frac{\partial \Phi}{\partial u_i} \frac{\partial^2 \Phi}{\partial u_i \partial u_3} = -\mu P'(u_3). \quad (9)$$

Hence the system of four equations (5), in orthogonal coordinates (u_i) , rewrites as two equations for two unknown functions $\Phi(u_1, u_2, u_3)$, $P(u_3)$. The equation (8) is a (u_1, u_2) -part of the Laplace equation $\Delta \Phi = 0$, and the equation (9) may be written as

$$(\text{grad}_{(u_1, u_2)} \Phi \cdot \text{grad}_{(u_1, u_2)} (\partial \Phi / \partial u_3)) = -\mu P'(u_3).$$

In coordinates with $g_{11} = g_{11}(u_1, u_2)$, $g_{22} = g_{22}(u_1, u_2)$, the equation (9) has a direct interpretation. It simplifies to $\partial / \partial u_3 (\mathbf{B}^2 / 2\mu + P) = 0$, meaning that the gradient of total plasma energy in the direction transverse to magnetic surfaces vanishes. The total energy of such configurations is finite if and only if the plasma domain is bounded in the direction transverse to magnetic surfaces. For example, for cylindrically-symmetric solutions of the Grad-Shafranov equation, with domains unbounded in cylindrical radius r and zero polar

component of the magnetic field, in every layer $c_1 < z < c_2$, the total energy is infinite (though the magnetic energy may be finite.) The same is true for the solutions obtained in e.g. [3].

We now list several classes of orthogonal coordinates in which explicit solutions of equations (8), (9) are readily found.

Now we look for exact solutions of the system (8), (9) in different coordinates. We include into consideration force-free ($\text{curl } \mathbf{B} = \alpha(\mathbf{r})\mathbf{B}$) and even "vacuum" magnetic fields ($\text{curl } \mathbf{B} = 0$, $P = \text{const}$). These fields are of interest for physical modeling because they can serve as *starting solutions* in infinite symmetries (15) and transformations (16) to produce families of physically non-trivial MHD and CGL equilibria, as shown below.

Case 1. First we consider cases when $\Phi(u_1, u_2, u_3)$ essentially depends on the magnetic surface coordinate u_3 . Thus the plasma electric current density (7) is nonzero.

The analysis of equations (8), (9) shows it has a force-free plasma equilibrium solution

$$\Phi_1(u_1, u_2, u_3) = C_1(u_3)\nu(u_1) + C_2(u_3)\lambda(u_2), \quad P_1 = \text{const} \quad (10)$$

when the metric tensor components satisfy [23]

$$g_{11}/g_{22} = \prod_{i=1}^3 a_i^2(u_i), \quad g_{33} = \mathcal{F}^2(u_3, \lambda(u_2) - \nu(u_1)C_2(u_3)/a_3^2(u_3)C_1(u_3)), \quad (11)$$

with $(C_1^2(u_3))' + a_3^2(u_3)(C_2^2(u_3))' = 0$, $\nu'(u_1) = a_1(u_1)$, $\lambda'(u_2) = 1/a_2(u_2)$. The expression (10) defines a force-free equilibrium $\text{curl } \mathbf{B} = \alpha(\mathbf{r})\mathbf{B}$ with the coefficient $\alpha(\mathbf{r}) = \alpha(u_3) = C_1'(u_3)/\sqrt{g_{33}}C_2(u_3)$. Here $a_i(u_i)$, $\mathcal{F}$, $C_k(u_3)$ are arbitrary sufficiently smooth functions.

Moreover, if the metric tensor components of orthogonal coordinates (u_i) satisfy more restricted relations $g_{11}/g_{22} = a_1^2(u_1)a_2^2(u_2)$, $g_{33} = \mathcal{F}^2(u_3)$, then the previous solution is readily generalized: any linear combination of

$$\Phi_2(u_1, u_2, u_3) = \int t(k)e^{\tau n(k)\nu} [D_1(u_3) \cos(n(k)\lambda) + D_2(u_3) \sin(n(k)\lambda)] dk, \quad (12)$$

$$\Phi_3(u_1, u_2, u_3) = \int t(k)e^{\tau n(k)\lambda} [D_3(u_3) \cos(n(k)\nu) + D_4(u_3) \sin(n(k)\nu)] dk, \quad (13)$$

defines a force-free plasma equilibrium with magnetic field (6) and the same coefficient $\alpha(\mathbf{r}) = C_1'(u_3)/\sqrt{g_{33}}C_2(u_3)$.

These solutions are defined for any $D_k(u_3)$ satisfying $d/du_3(D_1^2(u_3)+D_2^2(u_3)) = 0$, $d/du_3(D_3^2(u_3)+D_4^2(u_3)) = 0$. Here $\nu(u_1)$ and $\lambda(u_2)$ are same as above; $\tau = \pm 1$, $n(k)$ is an arbitrary function, and $t(k)$ is an arbitrary generalized function. ($n(k), t(k)$ must be chosen so that the integrals converge.)

Case 2. When the unknown function $\Phi(u_1, u_2, u_3)$ is independent of the magnetic surface variable u_3 , the equation (9) vanishes, provided $P = \text{const}$. The remaining equation (8) is the $(u_1 - u_2)$ -part of a 3D Laplace equation. Therefore in any coordinates where the full 3D Laplace equation $\Delta\phi(u_1, u_2, u_3) = 0$ admits a 2-dimensional solution (dependent only on u_1, u_2), there exists a gradient ("vacuum") magnetic field configuration

$$\text{div } \mathbf{B} = 0, \quad \text{curl } \mathbf{B} = 0 \quad (14)$$

corresponding to this solution, and this magnetic field is tangent to surfaces $u_3 = \text{const}$.

Many coordinate systems that admit geometrically nontrivial two-dimensional solutions of the Laplace equation are known [22]. Conditions on metric necessary and sufficient for separability of Laplace equation and existence of two-dimensional solutions are available in [24]. New orthogonal coordinates where Laplace equation admits 2D solution can also be found by methods involving a conformal transformations of the complex plane (see [22].)

Any force-free or "vacuum" plasma equilibrium solution $\{\mathbf{B}_0, P_0\}$ defined by (10), (12)-(13) or (14) can be transformed into physically non-trivial infinite families [16] of dynamic MHD equilibria

$$\begin{aligned} \mathbf{B}_1 &= m(\mathbf{r})\mathbf{B}_0, \quad \mathbf{V}_1 = \frac{n(\mathbf{r})}{a(\mathbf{r})\sqrt{\mu\rho}} \mathbf{B}_0, \\ \rho_1 &= a^2(\mathbf{r})\rho, \quad P_1 = CP_0 + \frac{C\mathbf{B}_0^2 - \mathbf{B}_1^2}{2\mu}, \quad m^2(\mathbf{r}) - n^2(\mathbf{r}) = C = \text{const} \end{aligned} \quad (15)$$

with the same set of magnetic field lines (functions $a(\mathbf{r})$, $m(\mathbf{r})$, $\rho(\mathbf{r})$ are arbitrary functions constant on magnetic field lines.)

In a similar way, the original force-free or "vacuum" equilibrium, or its dynamic isotropic extension (15), can be further transformed into an infinite family of anisotropic CGL equilibrium solutions. For a given incompressible MHD equilibrium $\{\mathbf{V}, \mathbf{B}, P, \rho\}$ with density ρ constant on magnetic field lines and streamlines, the formulas

$$\begin{aligned} \mathbf{B}_2 &= f(\mathbf{r})\mathbf{B}, \quad \mathbf{V}_2 = g(\mathbf{r})\mathbf{V}, \quad \rho_2 = C_0\rho\mu/g^2(\mathbf{r}), \\ p_{\perp 2} &= C_0\mu P + C_1 + (C_0 - f^2(\mathbf{r})) \mathbf{B}^2/2\mu, \quad p_{\parallel 2} = C_0\mu P + C_1 - (C_0 - f^2(\mathbf{r})) \mathbf{B}^2/2\mu \end{aligned} \quad (16)$$

define exact CGL plasma equilibria [18]. $f(\mathbf{r})$ and $g(\mathbf{r})$ are arbitrary functions constant on the magnetic field lines and streamlines; $C_0, C_1 = \text{const.}$

The transformations (15) and (16) are infinite-dimensional Abelian Lie groups of symmetries of MHD and CGL equilibrium equations (1)-(2) and (3)-(4) respectively; the group structure is $G_m = A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2$, $G_c = A_m \oplus A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$, where R^+ is a multiplicative group of positive numbers, and A_m is an additive Abelian group of smooth functions in $\mathbb{R}^3$ that are constant on magnetic field lines and plasma streamlines. The numbers of connected components of G_m and G_c are 8 and 16 [18]. These symmetry groups can be found using the classical Lie group analysis procedure [19].

A sequential application of symmetries (15) and transformations (16) produces exact anisotropic equilibria with different relations between pressure tensor components. The resulting solutions, under the proper choice of transformation parameters, are free from fire-hose and mirror instabilities [18].

A different type of nontrivial dynamic MHD solutions arising from (14) is Chandrasekhar-type MHD equilibria $\mathbf{V} = \pm \mathbf{B}/\sqrt{\mu\rho}$, $P = C_0 - \mathbf{B}^2/2\mu$, which are invariants of Bogoyavlenskij symmetries (15) [17, 25].

Under certain conditions on the metric, the equilibrium "vacuum" magnetic fields (14) can be extended with a Killing component in the u_3 -direction.

In particular, direct verification shows that if $\Delta\phi(u_1, u_2) = 0$ and $g_{11} = g_{11}(u_1, u_2) = g_{22}$, $g_{33} = g_{33}(u_3)$, then not only the magnetic field (6) with pressure $P(u_3) = \text{const}$ solves the Plasma Equilibrium equations (5), but so does the extended magnetic field

$$\mathbf{B}_K^{(1)} = \sum_{i=1}^2 \frac{1}{\sqrt{g_{ii}}} \frac{\partial \phi}{\partial u_i} \mathbf{e}_i + K(u_1, u_2) \mathbf{e}_3 \quad (17)$$

with pressure $P = C - K^2(u_1, u_2)/2$. This equilibrium is neither "vacuum" nor force-free. The function $K(u_1, u_2)$ is a harmonic conjugate of the solution $\phi(u_1, u_2)$: $\Delta K(u_1, u_2) = 0$, $\text{grad } \phi(u_1, u_2) \cdot \text{grad } K(u_1, u_2) = 0$.

Also, when $g_{11} = g_{11}(u_1, u_2)$, $g_{22} = g_{22}(u_1, u_2)$, $g_{33} = a^2(u_3)\mathcal{F}^2(u_1, u_2)$, not only the magnetic field (6) solves the Plasma Equilibrium equations (5), but so does the extended magnetic field

$$\mathbf{B}_K^{(2)} = \sum_{i=1}^2 \frac{1}{\sqrt{g_{ii}}} \frac{\partial \phi}{\partial u_i} \mathbf{e}_i + \frac{D}{\sqrt{g_{33}}} \mathbf{e}_3; \quad D = \text{const.} \quad (18)$$

For this magnetic field, $\text{div } \mathbf{B}_K^{(2)} = 0$, $\text{curl } \mathbf{B}_K^{(2)} = 0$, it corresponds to an equilibrium with $P = \text{const}$.

Example 1. Non-symmetric cylindrical plasma equilibria with current sheets. In cartesian coordinates (x, y, z) , we conformally transform the plane (x, y) by a complex transformation $Z' = a \cosh Z$: $x = a \cosh u_1 \cos u_2$, $y = a \sinh u_1 \sin u_2$, $z = u_3$. For the coordinates (u_i) , $g_{11} = g_{22} = (\partial x / \partial u_1)^2 + (\partial y / \partial u_1)^2$, $g_{33} = 1$. This metric satisfies necessary conditions for solutions (12), (13) to exist. We pick a solution (12) with $n(k) = k$, $t(k) = 1/(0.1k + 1)$, $D_2(u_3) = 0$ extended with a linear term: $\Phi(u_1, u_2) = \int_1^2 t(k)(\sinh(ku_1) \cos(ku_2) - 3u_2)dk$. Its harmonic conjugate is $K(u_1, u_2) = \int_1^2 t(k)(\cosh(ku_1) \sin(ku_2) + 3u_1)dk$. The magnetic field (6) is parallel to the (u_1, u_2) - plane and tangent to levels of $K(u_1, u_2) = \text{const}$. When extended with a Killing component in u_3 - direction (17), it gives rise to a plasma equilibrium

$$\mathbf{B} = \sum_{i=1}^2 \frac{1}{\sqrt{g_{ii}}} \frac{\partial \phi}{\partial u_i} \mathbf{e}_i + K(u_1, u_2) \mathbf{e}_3, \quad \mathbf{J} = 1/\mu \text{ curl } \mathbf{B}, \quad P = C - K^2(u_1, u_2)/2. \quad (19)$$

This magnetic field is also tangent to a family of cylinders of non-symmetric section $K(u_1, u_2) = \text{const}$ along z -axis, but becomes *non-planar* and has a nonzero pressure and current density. Cross-sections of several magnetic surfaces are shown on Fig. 1. The configuration is well-defined in the region between any two cylinders (for example, $K = 0.8$ and $K = 1.15$.) The plasma domain can be restricted to this region, with $\mathbf{B} = 0$ outside, by the usual procedure of introducing a boundary surface current $\mathbf{i}_b(\mathbf{r}_1) = \mu^{-1} \mathbf{B}(\mathbf{r}) \times \mathbf{n}_{out}(\mathbf{r}_1)$, where $\mathbf{r}_1$ is a point on the boundary of the plasma domain, and $\mathbf{n}_{out}$ is an outward normal.

Subsequent application of Bogoyavlenskij symmetries (15) and/or MHD→CGL transformations (16) maps this equilibrium into families of, respectively, MHD and CGL plasma equilibria. The latter have the same magnetic field lines as the original solutions, but different values of physical parameters (magnetic field, velocity, current, density and pressure(s)). These parameters are controlled by the choice of values of the arbitrary functions on each magnetic field line. These solutions model hollow cylindrical jet-like plasma structures bounded by current sheets and stretched in z -direction.

Example 2. Model of Earth magnetosheath plasma. In this example, we construct a family of generally non-symmetric plasma equilibria with ellipsoidal magnetic surfaces, modeling the equilibrium solar wind flow in the Earth magnetosheath near the stagnation point, past the bow shock.

In prolate spheroidal system of coordinates $(u_1, u_2, u_3) = (\theta, \phi, \eta)$, surfaces $\eta = \text{const}$ define prolate spheroids [22]. The 3-dimensional Laplace equation in this system is separable and admits a family of solutions $\Phi = H(\eta)T(\theta)$, $H(\eta) = A_1\mathcal{P}_p(\cosh \eta) + B_1\mathcal{L}_p(\cosh \eta)$, $T(\theta) = A_2\mathcal{P}_p(\cos \theta) + B_2\mathcal{L}_p(\cos \theta)$, where $\mathcal{P}_p(z)$, $\mathcal{L}_p(z)$ are Legendre wave functions of first and second kind.

The necessary boundary conditions for the magnetic field $\mathbf{B}$ (6) of the solar wind are: (i) $\lim_{|\mathbf{r}| \rightarrow \infty} \mathbf{B} = M_0 \mathbf{e}_z$, (ii) $\mathbf{B} \cdot \mathbf{e}_\eta|_{\eta=\eta_0} = 0$ (the magnetic field is tangent to the magnetopause surface $\eta = \eta_0$ near the stagnation point, and is homogeneous along z -axis far from the magnetopause.) A particular solution satisfying these properties is given by $\Phi_0 = H(\eta)T(\theta) = A_1 \cosh(\eta) \cos(\theta) + B_1 Q_1(\cosh(\eta)) \cos(\theta)$, where $Q_1(x)$ is a Legendre function of the second kind, and

$$B_1 = (\cosh^2(\eta_0) - 1) \ln \frac{\cosh(\eta_0) + 1}{\cosh(\eta_0) - 1} - 2 \cosh(\eta_0).$$

The resulting magnetic field (6) $\mathbf{B}_0 = \text{grad } \Phi_0(\eta, \theta)$ is axially symmetric and well-defined in the domain $\eta > \eta_0$. The field lines in the vicinity of the magnetopause are shown on Fig.2. (The constants used are $a = 1$, $M_0 = 1$, $\eta_0 = 0.3$, $A_1 = 1$; a is the parameter of coordinate system.)

After the application of the symmetries (15) to the "vacuum" configuration $P_0 = \text{const}$, $\mathbf{B}_0$, one gets a family of *isotropic dynamic configurations*

$$\begin{aligned} \mathbf{B}_1 &= m(\mathbf{r})\mathbf{B}_0, \quad \mathbf{V}_1 = \frac{n(\mathbf{r})}{a(\mathbf{r})\sqrt{\mu\rho_0(\mathbf{r})}}\mathbf{B}_0, \quad \rho_1 = a^2(\mathbf{r})\rho_0(\mathbf{r}), \\ P_1 &= CP_0 - n^2(\mathbf{r})\mathbf{B}_0^2/(2\mu), \quad m^2(\mathbf{r}) - n^2(\mathbf{r}) = C = \text{const}, \quad P_0 = \text{const}. \end{aligned} \quad (20)$$

Here $a(\mathbf{r})$, $m(\mathbf{r})$, $\rho_0(\mathbf{r})$ are arbitrary functions constant on magnetic field lines and streamlines (which coincide, as $\mathbf{V}_1 \parallel \mathbf{B}_1$).

To construct an *anisotropic* model, the transformations (16) are used. The resulting anisotropic equilibrium $(\mathbf{B}_2, \mathbf{V}_2, p_{\parallel 2}, p_{\perp 2}, \rho_2)$ is defined by (P_0 was set to 0):

$$\begin{aligned} \mathbf{B}_2 &= f(\mathbf{r})m(\mathbf{r})\mathbf{B}_0, \quad \mathbf{V}_2 = g(\mathbf{r})\frac{n(\mathbf{r})}{a(\mathbf{r})\sqrt{\mu\rho_0(\mathbf{r})}}\mathbf{B}_0, \quad \rho_2 = C_0 a^2(\mathbf{r})\rho_0(\mathbf{r})\mu/g(\mathbf{r})^2, \\ p_{\perp 2} &= C_1 + (\mu C_0 C - f^2(\mathbf{r})m^2(\mathbf{r})) \mathbf{B}_0^2/2\mu, \\ p_{\parallel 2} &= C_1 + (f^2(\mathbf{r})m^2(\mathbf{r}) - \mu C_0(C + 2n^2(\mathbf{r}))) \mathbf{B}_0^2/2\mu. \end{aligned} \quad (21)$$

Depending on the choice of values of the arbitrary functions $a(\mathbf{r})$, $m(\mathbf{r})$, $\rho_0(\mathbf{r})$, $f(\mathbf{r})$, $g(\mathbf{r})$ on each magnetic field line, the configuration can be made axially-symmetric or non-symmetric.

This equilibrium is free from fire-hose instability when $C_0 > 0$. The solution is made free from mirror instability by the proper choice of the range of $f(\mathbf{r})$ [18].

The general relation between the pressure components of anisotropic pressure tensor is

$$p_{\perp 2}/p_{\parallel 2} = 1 + 2\tau \mathbf{B}_2^2/2p_{\parallel 2}, \quad \tau = C_0/f^2(\mathbf{r}) - 1/\mu > 0; \quad (22)$$

its particular form is determined by the choice of arbitrary functions $f(\mathbf{r})$, $m(\mathbf{r})$. The relation (22) and the incompressibility condition $\text{div } \mathbf{V} = 0$ can play the role of two equations of state that close the general CGL equilibrium system. Numerical modeling of anisotropic magnetosheath plasma using the CGL approximation and similar equations of state has been performed in [8].

The suggested approach will be used for the construction of fully 3-dimensional static and dynamic plasma equilibrium models in MHD and CGL frameworks. Solutions with and without geometrical symmetries, for plasma domains of different shapes, are available. For anisotropic plasmas, model solutions with different relations between pressure components can be chosen.

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- [21] For such a triply-orthogonal coordinate system to exist, the family of magnetic surfaces has to be the family of Lamé (in the terminology of Darboux.) For a smooth family of surfaces defined by $(x, y, z) = \text{const}$ to be a family of Lamé, the function $w(x, y, z)$ must satisfy a certain PDE of order 3 [26].) Many examples of families of Lamé are known; they include sets of parallel surfaces; sets of surfaces of revolution; Ribaucour surfaces, and other families.
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- [23] Relations of the type (11) between metric components coefficients are not unnatural. Spherical, cylindrical and many other orthogonal coordinates systems satisfy these relations. New triples of orthogonal coordinates with necessary relations between metric components can be obtained using conformal mappings of the complex plane and a subsequent translation or rotation (see [22, 24].)
- [24] P. Moon and D. E. Spencer, *Field Theory for Engineers* (D. Van Nostrand Company, 1961.)
- [25] S. Chandrasekhar, *Proc. Natl. Acad. Sci. USA* **42**, 273 (1956).
- [26] L.P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces* (Dover, New York, 1960.)

FIG.1: Non-circular cylindrical magnetic surfaces (levels $K(u_1, u_2) = 0.8, 0.9, 1, 1.1, 1.15.$)
 Elliptic coordinates (u_1, u_2) shown in grey.

FIG.2: Magnetic field lines in the Earth magnetosheath model.