GENERALIZED LIE-YAMAGUTI STRUCTURES ON THE $\mathfrak{sl}(3,\mathbb{C})$-MODULES $V(a,a)$

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Abstract. The Lie algebra $L = \mathfrak{sl}(3,\mathbb{C})$ has an irreducible representation $V = V(a,a)$ of dimension $(a+1)^3$ for all $a \geq 0$. The projections $\alpha\colon \Lambda^2 V \to L$ and $\beta\colon \Lambda^2 V \to V$ define a binary-ternary algebra structure on V by $[x,y] = \beta(x \wedge y)$ and $[x,y,z] = \alpha(x \wedge y)\,.\,z$. We use computer algebra to study the polynomial identities in degree ≤ 7 satisfied by this structure for $a = 2,3,4$. For $V(2,2)$ the identities are exactly the defining identities of Lie-Yamaguti algebras. For $V(3,3)$ and $V(4,4)$ one of the defining identities of Lie-Yamaguti algebras does not hold, and we obtain a natural generalization of Lie-Yamaguti algebras.

1. Introduction

A Lie triple system is a vector space V over a field F together with a trilinear map $(-,-,-)\colon V \times V \times V \to V$ such that

$$(a,a,b) = 0,$$
$$(a,b,c) + (b,c,a) + (c,a,b) = 0,$$
$$(a,b,(c,d,e)) - ((a,b,c),d,e) - (c,(a,b,d),e) - (c,d,(a,b,d)) = 0,$$

for all $a,b,c \in V$; see Lister [6]. If L is a Lie algebra with bracket $[a,b]$ then any subspace of L closed under the operation $[[a,b],c]$ is a Lie triple system. The following generalization was introduced by Yamaguti [9], who used the name "general Lie triple system"; the current terminology was introduced by Kinyon and Weinstein [5].

Definition 1.1. A *Lie-Yamaguti algebra* is a vector space V over a field F together with a bilinear map $[-,-]\colon V \times V \to V$ and a trilinear map $(-,-,-)\colon V \times V \times V \to V$ such that

$$[a,a] = 0, \tag{1}$$
$$(a,a,b) = 0, \tag{2}$$
$$[[a,b],c] + [[b,c],a] + [[c,a],b] + (a,b,c) + (b,c,a) + (c,a,b) = 0, \tag{3}$$
$$([a,b],c,d) + ([b,c],a,d) + ([c,a],b,d) = 0, \tag{4}$$
$$[(a,b,c),d] - [(a,b,d),c] - (a,b,[c,d]) = 0, \tag{5}$$
$$((a,b,c),d,e) - ((a,b,d),c,e) - (a,b,(c,d,e)) + (c,d,(a,b,e)) = 0, \tag{6}$$

for all $a,b,c,d,e \in V$.

If we assume that $[a,b] = 0$ for all $a,b \in V$ then the definition of a Lie-Yamaguti algebra reduces to that of a Lie triple system. For recent work on Lie triple systems and Lie-Yamaguti algebras, see Benito et al. [1, 2] and Martín [7]. (A Bol algebra is another similar but distinct generalization of Lie triple system; see Pérez-Izquierdo [8].)

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$V(a,b)$	$\dim \mathrm{Hom}_L(\Lambda^2 V, L)$	$\dim \mathrm{Hom}_L(\Lambda^2 V, V)$	$\dim V(a,b)$
$V(1,1)$	1	1	8
$V(2,2)$	1	1	27
$V(3,3)$	1	2	64
$V(4,1)$	0	1	35
$V(4,4)$	1	2	125
$V(5,2)$	0	1	81
$V(5,5)$	1	3	216
$V(6,3)$	0	2	154
$V(6,6)$	1	3	343
$V(7,4)$	0	2	260
$V(7,7)$	1	4	512
$V(8,5)$	0	3	405
$V(8,8)$	1	4	729
$V(9,3)$	0	1	280
$V(9,6)$	0	3	595
$V(9,9)$	1	5	1000

TABLE 1. Multiplicities of $L = V(1,1)$ and $V = V(a,b)$ in $\Lambda^2 V$

Lie-Yamaguti algebras arise in the following situation. Let M be a Lie algebra, and suppose that $M = L \oplus V$ is the direct sum of subspaces for which $[L,L] \subseteq L$ (so L is a Lie subalgebra) and $[L,V] \subseteq V$ (so V is an L-module for the restriction of the adjoint representation). Let $\pi_L, \pi_V \colon M \to M$ be the projections of M onto L and V respectively. On the subspace V we define a bilinear operation $[a,b]$ and a trilinear operation (a,b,c) as follows; here $[a,b]_M$ denotes the Lie bracket in M:

$$[a,b] = \pi_V(\,[a,b]_M\,), \qquad (a,b,c) = [\,\pi_L([a,b]_M),\, c\,]_M, \tag{7}$$

for all $a,b,c \in V$. Then V is a Lie-Yamaguti algebra with respect to these operations.

In order to motivate the generalized Lie-Yamaguti structures studied in this paper, we consider the following situation. Let L be a Lie algebra and let V be an L-module. Let $\Lambda^2 V$ be the exterior square of V, and assume that there exist nonzero L-module maps $\alpha \colon \Lambda^2 V \to L$ and $\beta \colon \Lambda^2 V \to V$. On V we define a bilinear operation $[a,b]$ and a trilinear operation (a,b,c) as follows:

$$[a,b] = \beta(a,b), \qquad (a,b,c) = \alpha(a,b) \cdot c, \tag{8}$$

where the dot denotes the action of L on V. It is obvious that this structure satisfies (1) and (2); and it follows from the L-module structure of V that it also satisfies (5) and (6).

In this paper we consider these structures over the Lie algebra $L = \mathfrak{sl}(3,\mathbb{C})$. In order to find suitable L-modules V, we use the computer algebra system `LiE` [4] to decompose the exterior squares of the simple $\mathfrak{sl}(3,\mathbb{C})$-modules $V = V(a,b)$ with highest weight (a,b); software restrictions require $a,b \leq 9$, and by symmetry of the Dynkin diagram we may assume $a \geq b$. Table 1 gives the multiplicities of the adjoint module $L = V(1,1)$ and the module $V = V(a,b)$ in the exterior square $\Lambda^2 V$ in all cases (with the indicated restrictions on a,b) for which either multiplicity is positive, together with the dimension of $V(a,b)$. Both multiplicities are positive if and only if $a = b \geq 1$; the L-modules $V(a,a)$ will be the focus of

	α_2		$\alpha_1{+}\alpha_2$	
$-\alpha_1$		0		α_1
	$-\alpha_1{-}\alpha_2$		$-\alpha_2$	

TABLE 2. Root system of $L = \mathfrak{sl}(3,\mathbb{C})$

$$X_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad X_3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Y_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad Y_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

TABLE 3. Standard ordered basis of $L = \mathfrak{sl}(3,\mathbb{C})$

this paper. (If $a > b > 0$ and $a{-}b$ is a multiple of 3, then the multiplicity of L is 0 but the multiplicity of V is positive; these modules have no ternary structure of the required form, but they have binary structures which we intend to study in a future publication.)

2. THE LIE ALGEBRA $L = \mathfrak{sl}(3,\mathbb{C})$ AND ITS SIMPLE MODULES $V = V(a,a)$

The simple Lie algebra $L = \mathfrak{sl}(3,\mathbb{C})$ has dimension 8 and consists of the 3×3 matrices of trace 0 with the operation $[x,y] = xy - yx$. Table 2 gives the root system and Table 3 the standard basis. The simple L-module with highest weight (a,b) will be denoted $V(a,b)$. The adjoint representation is isomorphic to $V(1,1)$; any nonzero multiple of X_3 is a highest weight vector.

Lemma 2.1. *The L-module $V(a,b)$ has dimension*

$$\dim V(a,b) = \frac{1}{2}(a+1)(b+1)(a+b+2).$$

In particular,

$$\dim V(a,a) = (a+1)^3.$$

Proof. A straightforward application of the Weyl dimension formula. □

As an abstract L-module $\Lambda^2 V(a,a)$ has dimension

$$\binom{(a+1)^3}{2} = \frac{1}{2}\, a(a+1)^3(a^2+3a+3),$$

which grows very rapidly. We are interested only in the summands $L = V(1,1)$ and $V = V(a,a)$, and so we use the following construction which gives a realization of the maps $\alpha\colon \Lambda^2 V \to L$ and $\beta\colon \Lambda^2 \to V$ by projections of the commutator in the Lie algebra $\mathfrak{gl}(A)$ for

$$A = \binom{a+2}{2},$$

$$\begin{array}{llllll} X_1 \,.\, x = 0, & X_1 \,.\, y = x, & X_1 \,.\, z = 0, & X_2 \,.\, x = 0, & X_2 \,.\, y = 0, & X_2 \,.\, z = y, \\ X_3 \,.\, x = 0, & X_3 \,.\, y = 0, & X_3 \,.\, z = x, & H_1 \,.\, x = x, & H_1 \,.\, y = -y, & H_1 \,.\, z = 0, \\ H_2 \,.\, x = 0, & H_2 \,.\, y = y, & H_2 \,.\, z = -z, & Y_1 \,.\, x = y, & Y_1 \,.\, y = 0, & Y_1 \,.\, z = 0, \\ Y_2 \,.\, x = 0, & Y_2 \,.\, y = z, & Y_2 \,.\, z = 0, & Y_3 \,.\, x = z, & Y_3 \,.\, y = 0, & Y_3 \,.\, z = 0. \end{array}$$

TABLE 4. Action of $L = \mathfrak{sl}(3, \mathbb{C})$ on x, y, z

$$\begin{array}{llll} X_1 = x\dfrac{\partial}{\partial y}, & X_2 = y\dfrac{\partial}{\partial z}, & X_3 = x\dfrac{\partial}{\partial z}, & H_1 = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}, \\ H_2 = y\dfrac{\partial}{\partial y} - z\dfrac{\partial}{\partial z}, & Y_1 = y\dfrac{\partial}{\partial x}, & Y_2 = z\dfrac{\partial}{\partial y}, & Y_3 = z\dfrac{\partial}{\partial x}. \end{array}$$

TABLE 5. Action of $L = \mathfrak{sl}(3, \mathbb{C})$ in terms of differential operators

which grows much more slowly.

The standard basis of L in Table 3 corresponds to the action of L on the natural 3-dimensional module $V(1,0)$. If we write x, y, z for the standard basis of $\mathbb{C}^3$ then we obtain Table 4; expressing this action in terms of differential operators gives Table 5.

It is well-known that $S^a V(1,0) \approx V(a,0)$ for all $a \geq 1$, and hence

$$\dim V(a,0) = \binom{a+2}{2}.$$

A basis of $V(a,0)$ consists of the monomials appearing in the expansion of $(x+y+z)^a$; that is, the monomials $x^i y^j z^k$ with $i+j+k = a$ in lexicographical order. The action of L is given by applying the differential operators in Table 5. In this way we obtain an embedding $\rho\colon L \hookrightarrow \mathfrak{gl}(A)$, and $\mathfrak{gl}(A)$ becomes an L-module by $x \cdot y = [\rho(x), y]$. The identity matrix is a highest weight vector of weight $(0,0)$ and the matrix $\rho(X_3)$ is a highest weight vector of weight $(1,1)$. The derivation rule implies that the divided power

$$\frac{\rho(X_3)^\ell}{\ell!},$$

is a highest weight vector of weight (ℓ,ℓ) for $0 \leq \ell \leq a$. The well-known equation

$$\binom{a+2}{2}^2 = \sum_{\ell=0}^{a} (\ell+1)^3,$$

implies the direct sum decomposition of L-modules,

$$\mathfrak{gl}(A) \approx \bigoplus_{\ell=0}^{a} V(\ell,\ell). \tag{9}$$

A more abstract point of view is as follows. For any vector space V ($\dim_F V < \infty$) we have

$$V \otimes_F V^* \approx \mathrm{End}_F(V).$$

Since $V(a,0)^* \approx V(0,a)$ this implies

$$V(a,0) \otimes_F V(0,a) \approx \mathrm{End}_F(V(a,0)).$$

		$(-2,4)$		$(0,3)$		$(2,2)$		
	$(-3,3)$		$(-1,2)$		$(1,1)$		$(3,0)$	
$(-4,2)$		$(-2,1)$		$(0,0)$		$(2,-1)$		$(4,-2)$
	$(-3,0)$		$(-1,-1)$		$(1,-2)$		$(3,-3)$	
		$(-2,-2)$		$(0,-3)$		$(2,-4)$		

TABLE 6. Weight system of $V = V(2,2)$

$$
\begin{array}{llll}
M_1 = \xi, & M_2 = Y_1\xi, & M_3 = Y_2\xi, & M_4 = Y_1Y_2\xi, \\
M_5 = Y_3\xi, & M_6 = Y_1^{(2)}\xi, & M_7 = Y_2^{(2)}\xi, & M_8 = Y_1Y_3\xi, \\
M_9 = Y_1^{(2)}Y_2\xi, & M_{10} = Y_1Y_2^{(2)}\xi, & M_{11} = Y_2Y_3\xi, & M_{12} = Y_1Y_2Y_3\xi, \\
M_{13} = Y_1^{(2)}Y_2^{(2)}\xi, & M_{14} = Y_3^{(3)}\xi, & M_{15} = Y_1^{(2)}Y_3\xi, & M_{16} = Y_2^{(2)}Y_3\xi, \\
M_{17} = Y_1Y_3^{(2)}\xi, & M_{18} = Y_1^{(2)}Y_2Y_3\xi, & M_{19} = Y_1Y_2^{(2)}Y_3\xi, & M_{20} = Y_2Y_3^{(2)}\xi, \\
M_{21} = Y_1Y_2Y_3^{(2)}\xi, & M_{22} = Y_3^{(3)}\xi, & M_{23} = Y_1^{(2)}Y_3^{(2)}\xi, & M_{24} = Y_2^{(2)}Y_3^{(2)}\xi, \\
M_{25} = Y_1Y_3^{(3)}\xi, & M_{26} = Y_2Y_3^{(3)}\xi, & M_{27} = Y_3^{(4)}\xi. &
\end{array}
$$

TABLE 7. Standard basis of $V = V(2,2)$

$$
\rho(X_1) = \begin{bmatrix} . & 1 & . & . & . & . \\ . & . & . & 2 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix} \quad
\rho(X_2) = \begin{bmatrix} . & . & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 2 \\ . & . & . & . & . & . \end{bmatrix} \quad
\rho(X_3) = \begin{bmatrix} . & . & 1 & . & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}
$$

$$
\rho(H_1) = \begin{bmatrix} 2 & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & -2 & . & . \\ . & . & . & . & -1 & . \\ . & . & . & . & . & . \end{bmatrix} \quad
\rho(H_2) = \begin{bmatrix} . & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & -1 & . & . & . \\ . & . & . & 2 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & -2 \end{bmatrix}
$$

$$
\rho(Y_1) = \begin{bmatrix} . & . & . & . & . & . \\ 2 & . & . & . & . & . \\ . & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & . & . & . \end{bmatrix} \quad
\rho(Y_2) = \begin{bmatrix} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 2 & . & . \\ . & . & . & . & 1 & . \end{bmatrix} \quad
\rho(Y_3) = \begin{bmatrix} . & . & . & . & . & . \\ . & . & . & . & . & . \\ 2 & . & . & . & . & . \\ . & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \end{bmatrix}
$$

TABLE 8. The representation ρ of $L = V(1,1)$ by 6×6 matrices

Fixing an ordered basis of $V(a,0)$ we obtain

$$V(a,0) \otimes_F V(0,a) \approx F^A \otimes_F (F^A)^* \approx \mathrm{Mat}_A(F) \approx \mathfrak{gl}_A(F).$$

For $a = 0$ we obtain the trivial 1-dimensional module, and for $a = 1$ we obtain the adjoint module. Therefore we start by considering the case $a = 2$.

$$\rho(M_1)=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_2)=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_3)=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\rho(M_4)=\begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_5)=\begin{bmatrix} \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_6)=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\rho(M_7)=\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_8)=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_9)=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\rho(M_{10})=\begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_{11})=\begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_{12})=\begin{bmatrix} -2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\rho(M_{13})=\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_{14})=\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \quad \rho(M_{15})=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\rho(M_{16})=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_{17})=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_{18})=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\rho(M_{19})=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_{20})=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \end{bmatrix} \quad \rho(M_{21})=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\rho(M_{22})=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_{23})=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_{24})=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

$$\rho(M_{25})=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_{26})=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \rho(M_{27})=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

TABLE 9. The representation ρ of $V = V(2,2)$ by 6×6 matrices

3. The L-module $V = V(2,2)$

The simple L-module $V = V(2,2)$ has dimension 27. Table 6 gives the weight system; the 12 outer weights have multiplicity 1, the 6 middle weights have multiplicity 2, and the central weight $(0,0)$ has multiplicity 3. If ξ is a highest weight vector in V, then a basis for V is given by the elements (computed by GAP) in Table 7 where $Y^{(m)} = Y^m/m!$ (divided powers). Since $a = 2$ we have $A = 6$; the construction of the previous section gives the representations of L and V by 6×6 matrices displayed in Tables 8 and 9.

Definition 3.1. We have two ordered bases of $\mathfrak{gl}(6,\mathbb{C})$: the *basis of matrix units*,

$$E_{11},\ \dots,\ E_{16},\ E_{21},\ \dots,\ E_{26},\ \dots,\ E_{61},\ \dots,\ E_{66};$$

and the *basis of weight vectors*:

$$I_6,\ \rho(X_1),\ \rho(X_2),\ \rho(X_3),\ \rho(H_1),\ \rho(H_2),\ \rho(Y_1),\ \rho(Y_2),\ \rho(Y_3),\ \rho(M_1),\ \rho(M_2),\ \dots,\ \rho(M_{27}).$$

In the *change of basis matrix* K the columns contain the coefficients of the weight vectors as linear combinations of the matrix units.

We use this representation of $L = V(1,1)$ and $V = V(2,2)$ to efficiently compute the maps α and β. If $A, B \in V$ are expressed as row vectors with respect to the basis of Table 9,

$$A = [a_1, a_2, \dots, a_{27}], \qquad B = [b_1, b_2, \dots, b_{27}],$$

then we define A' and B' by inserting nine zeros at the front:

$$A' = [0, 0, \dots, 0, a_1, a_2, \dots, a_{27}], \qquad B' = [0, 0, \dots, 0, b_1, b_2, \dots, b_{27}].$$

We transpose A' and B', left-multiply by the change of basis matrix K, and transpose again. We obtain $A'' = A'K^t$ and $B'' = B'K^t$, which express A and B as row vectors with respect to the basis of matrix units. We write $[A'']$ and $[B'']$ for the corresponding 6×6 matrices, and compute the commutator $[C''] = [\,[A''], [B'']\,]$. We now reverse the process, obtaining the row vector C'' in the basis of matrix units, and then the row vector $C' = C''(K^{-1})^t$ in the basis of weight vectors. Components 2–9, respectively 10–36, of C' give the coordinates of $\alpha(A,B)$, respectively $\beta(A,B)$. The structure constants for the projection α are given in Table 11; only the nonzero results are displayed. (The structure constants for the projection β are omitted.)

In this case the direct sum decomposition (9) has the following form:

$$\mathfrak{gl}(6) \approx V(0,0) \oplus V(1,1) \oplus V(2,2),$$

and from this we obtain

$$\mathfrak{sl}(6) \approx V(1,1) \oplus V(2,2).$$

It follows that $V(2,2)$ is a Lie-Yamaguti algebra with respect to the decomposition $M = L \oplus V$ where $M = \mathfrak{sl}(6)$, $L = \mathfrak{sl}(3)$ and $V = V(2,2)$. We will see below that the polynomial identities in degree ≤ 5 satisfied by the resulting binary-ternary structure on $V(2,2)$ are exactly the defining identities of Lie-Yamaguti algebras.

4. Nonassociative polynomials

Before describing our computational methods for studying polynomial identities, we recall some definitions.

[1].[2] = 2[1],	[1].[4] = 3[3],	[1].[5] = −[3],	[1].[6] = [2]
[1].[8] = −[4]+[5],	[1].[9] = 2[4],	[1].[10] = 4[7],	[1].[11] = −2[7]
[1].[12] = −2[10]+2[11],	[1].[13] = 3[10],	[1].[14] = −[11],	[1].[15] = −[9]
[1].[17] = −[12],	[1].[18] = [12]−2[13],	[1].[19] = 3[16],	[1].[20] = −2[16]
[1].[21] = −2[19]+[20],	[1].[22] = −[20],	[1].[23] = −[17]−[18],	[1].[25] = −[21]−[22]
[1].[26] = −2[24],	[1].[27] = −[26],	[2].[3] = 2[1],	[2].[4] = 2[2]
[2].[5] = [2],	[2].[7] = [3],	[2].[8] = 2[6],	[2].[9] = 2[6]
[2].[10] = [4],	[2].[11] = [4]+2[5],	[2].[12] = 2[8]+2[9],	[2].[13] = [9]
[2].[14] = [8],	[2].[16] = [10]+[11],	[2].[17] = 2[15],	[2].[18] = −[15]
[2].[19] = [12]+2[13],	[2].[20] = [12]+2[14],	[2].[21] = 2[17]+2[18],	[2].[22] = [17]
[2].[24] = [19]+[20],	[2].[25] = 2[23],	[2].[26] = [21]+2[22],	[2].[27] = [25]
[3].[4] = −2[1],	[3].[5] = 4[1],	[3].[8] = 3[2],	[3].[9] = −2[2]
[3].[10] = −[3],	[3].[11] = 3[3],	[3].[12] = 2[4]−2[5],	[3].[13] = −[4]
[3].[14] = 3[5],	[3].[15] = 2[6],	[3].[16] = 2[7],	[3].[17] = 2[8]
[3].[18] = −2[8]+[9],	[3].[19] = [10]−[11],	[3].[20] = 2[11],	[3].[21] = [12]−2[14]
[3].[22] = 2[14],	[3].[23] = [15],	[3].[24] = [16],	[3].[25] = [17]
[3].[26] = [20],	[3].[27] = [22],	[4].[1] = 2[1],	[4].[3] = 3[3]
[4].[4] = [4],	[4].[5] = [5],	[4].[6] = −2[6],	[4].[7] = 4[7]
[4].[8] = −[8],	[4].[9] = −[9],	[4].[10] = 2[10],	[4].[11] = 2[11]
[4].[15] = −3[15],	[4].[16] = 3[16],	[4].[17] = −2[17],	[4].[18] = −2[18]
[4].[19] = [19],	[4].[20] = [20],	[4].[21] = −[21],	[4].[22] = −[22]
[4].[23] = −4[23],	[4].[24] = 2[24],	[4].[25] = −3[25],	[4].[27] = −2[27]
[5].[1] = 2[1],	[5].[2] = 3[2],	[5].[4] = [4],	[5].[5] = [5]
[5].[6] = 4[6],	[5].[7] = −2[7],	[5].[8] = 2[8],	[5].[9] = 2[9]
[5].[10] = −[10],	[5].[11] = −[11],	[5].[15] = 3[15],	[5].[16] = −3[16]
[5].[17] = [17],	[5].[18] = [18],	[5].[19] = −2[19],	[5].[20] = −2[20]
[5].[21] = −[21],	[5].[22] = −[22],	[5].[23] = 2[23],	[5].[24] = −4[24]
[5].[26] = −3[26],	[5].[27] = −2[27],	[6].[1] = [2],	[6].[2] = 2[6]
[6].[3] = [4],	[6].[4] = 2[9],	[6].[5] = [8],	[6].[7] = [10]
[6].[8] = 2[15],	[6].[9] = −3[15],	[6].[10] = 2[13],	[6].[11] = [12]
[6].[12] = 2[18],	[6].[13] = −3[17]−3[18],	[6].[14] = [17],	[6].[16] = [19]
[6].[17] = 2[23],	[6].[18] = −6[23],	[6].[19] = −2[21]−2[22],	[6].[20] = [21]
[6].[21] = −4[25],	[6].[22] = [25],	[6].[24] = −[26],	[6].[26] = −2[27]
[7].[1] = [3],	[7].[2] = [4]+[5],	[7].[3] = 2[7],	[7].[4] = 2[10]+[11]
[7].[5] = [11],	[7].[6] = [8]+[9],	[7].[8] = [12]+2[14],	[7].[9] = [12]+2[13]
[7].[10] = [16],	[7].[11] = 2[16],	[7].[12] = 2[19]+2[20],	[7].[13] = [19]
[7].[14] = [20],	[7].[15] = 2[17]+[18],	[7].[17] = [21]+3[22],	[7].[18] = −2[22]
[7].[19] = 2[24],	[7].[20] = 2[24],	[7].[21] = [26],	[7].[22] = [26]
[7].[23] = [25],	[7].[25] = 2[27],	[8].[1] = [5],	[8].[2] = [8]
[8].[3] = [11],	[8].[4] = [12],	[8].[5] = 2[14],	[8].[6] = [15]
[8].[7] = [16],	[8].[8] = 2[17],	[8].[9] = [18],	[8].[10] = [19]
[8].[11] = 2[20],	[8].[12] = 2[21],	[8].[13] = −[21]−[22],	[8].[14] = 3[22]
[8].[15] = 2[23],	[8].[16] = 2[24],	[8].[17] = 3[25],	[8].[18] = −4[25]
[8].[19] = −2[26],	[8].[20] = 3[26],	[8].[21] = −6[27],	[8].[22] = 4[27]

TABLE 10. Action of standard basis of L on standard basis of V

Definition 4.1. Let $[-,-]$ denote an anticommutative binary operation. By a *binary association type* in degree n we mean a valid placement of $n{-}1$ pairs of brackets in $x = x_1 x_2 \cdots x_n$. Each type has a unique factorization $x = [y,z]$ where y and z have degree $< n$ and $\deg(y) \ge \deg(z)$. The *ordering on the association types* is inductive: if it has been defined on types of degree $< n$, then in degree n type $[y,z]$ precedes type $[y',z']$ if either (i) $\deg(y) > \deg(y')$, or (ii) $\deg(y) = \deg(y')$ and y precedes y', or (iii) $y = y'$ and $\overline{z}$ precedes $\overline{z}'$, where the bar replaces $x_{p+1} \cdots x_{p+q}$ by $x_1 \cdots x_q$ for $p = \deg(y)$ and $q = \deg(z)$.

Example 4.2. Here are the binary association types in degree ≤ 7; the commas within monomials are omitted:

$$x_1; \qquad [x_1x_2]; \qquad [[x_1x_2]x_3]; \qquad [[[x_1x_2]x_3]x_4],\ [[x_1x_2][x_3x_4]];$$

$$[[[[x_1x_2]x_3]x_4]x_5],\ [[[x_1x_2][x_3x_4]]x_5],\ [[[x_1x_2]x_3][x_4x_5]];$$

$$[[[[[x_1x_2]x_3]x_4]x_5]x_6],\ [[[[x_1x_2][x_3x_4]]x_5]x_6],\ [[[[x_1x_2]x_3][x_4x_5]]x_6],\ [[[[x_1x_2]x_3]x_4][x_5x_6]],$$

[1].[21] = 6[3],	[1].[22] = −4[3],	[1].[25] = −2[2],	[1].[26] = 2[1]
[1].[27] = 2[4]+2[5],	[2].[19] = 4[3],	[2].[20] = −6[3],	[2].[21] = −2[2]
[2].[22] = −2[2],	[2].[24] = 2[1],	[2].[26] = 2[4]+4[5],	[2].[27] = 2[6]
[3].[17] = 6[3],	[3].[18] = −8[3],	[3].[21] = −8[1],	[3].[22] = 2[1]
[3].[23] = 2[2],	[3].[25] = −4[4]−2[5],	[3].[27] = 2[7],	[4].[12] = 16[3]
[4].[13] = −6[3],	[4].[14] = −6[3],	[4].[17] = 2[2],	[4].[18] = 4[2]
[4].[19] = −12[1],	[4].[20] = 8[1],	[4].[21] = −8[4]−8[5],	[4].[22] = 2[4]+2[5]
[4].[25] = −6[6],	[4].[26] = 4[7],	[4].[27] = −2[8],	[5].[12] = −12[3]
[5].[13] = 2[3],	[5].[14] = 12[3],	[5].[17] = 6[2],	[5].[18] = −8[2]
[5].[19] = 4[1],	[5].[20] = −6[1],	[5].[21] = 6[4]+6[5],	[5].[22] = −4[4]−4[5]
[5].[25] = 2[6],	[5].[26] = 2[7],	[5].[27] = 4[8],	[6].[16] = −2[3]
[6].[19] = −2[2],	[6].[20] = −2[2],	[6].[24] = 2[5],	[6].[26] = 2[6]
[7].[15] = −2[3],	[7].[17] = −2[1],	[7].[18] = 6[1],	[7].[23] = 2[4]
[7].[25] = −2[7],	[8].[10] = −4[3],	[8].[11] = 12[3],	[8].[12] = 4[2]
[8].[13] = −4[2],	[8].[14] = 6[2],	[8].[16] = −4[1],	[8].[19] = 4[5]
[8].[20] = −6[5],	[8].[21] = 14[6],	[8].[22] = −6[6],	[8].[24] = 2[7]
[8].[26] = 6[8],	[9].[10] = 6[3],	[9].[11] = −8[3],	[9].[12] = 4[2]
[9].[13] = 6[2],	[9].[14] = −4[2],	[9].[16] = 6[1],	[9].[19] = −6[5]
[9].[20] = 4[5],	[9].[21] = −16[6],	[9].[22] = 4[6],	[9].[24] = 2[7]
[9].[26] = −4[8],	[10].[12] = −12[1],	[10].[13] = 12[1],	[10].[14] = 2[1]
[10].[15] = −2[2],	[10].[17] = −2[4],	[10].[18] = 6[4],	[10].[21] = −8[7]
[10].[22] = 2[7],	[10].[23] = 4[6],	[10].[25] = 2[8],	[11].[12] = 16[1]
[11].[13] = −6[1],	[11].[14] = −6[1],	[11].[15] = −4[2],	[11].[17] = 6[4]
[11].[18] = −8[4],	[11].[21] = 4[7],	[11].[22] = −6[7],	[11].[23] = −2[6]
[11].[25] = −6[8],	[12].[17] = 16[6],	[12].[18] = −28[6],	[12].[19] = −4[7]
[12].[20] = −4[7],	[12].[21] = −28[8],	[12].[22] = 12[8],	[13].[17] = −6[6]
[13].[18] = 18[6],	[13].[19] = −6[7],	[13].[20] = 4[7],	[13].[21] = 8[8]
[13].[22] = −2[8],	[14].[17] = −6[6],	[14].[18] = 8[6],	[14].[19] = 4[7]
[14].[20] = −6[7],	[14].[21] = 18[8],	[14].[22] = −12[8],	[15].[16] = 2[4]−2[5]
[15].[19] = 6[6],	[15].[20] = −4[6],	[15].[24] = 2[8],	[16].[17] = 4[7]
[16].[18] = −2[7],	[16].[23] = 2[8],	[17].[19] = 8[8],	[17].[20] = −12[8]
[18].[19] = −14[8],	[18].[20] = 16[8],		

TABLE 11. Projection $\alpha\colon \Lambda^2 V \to L$ (coefficients multiplied by 5)

$$[[[x_1x_2][x_3x_4]][x_5x_6]],\ [[[x_1x_2]x_3][[x_4x_5]x_6]];$$

$$[[[[[[x_1x_2]x_3]x_4]x_5]x_6]x_7],\ [[[[[x_1x_2][x_3x_4]]x_5]x_6]x_7],\ [[[[[x_1x_2]x_3][x_4x_5]]x_6]x_7],$$

$$[[[[[x_1x_2]x_3]x_4][x_5x_6]]x_7],\ [[[[x_1x_2][x_3x_4]][x_5x_6]]x_7],\ [[[[x_1x_2]x_3][[x_4x_5]x_6]]x_7],$$

$$[[[[[x_1x_2]x_3]x_4]x_5][x_6x_7]],\ [[[[x_1x_2][x_3x_4]]x_5][x_6x_7]],\ [[[[x_1x_2]x_3][x_4x_5]][x_6x_7]],$$

$$[[[[x_1x_2]x_3]x_4][[x_5x_6]x_7]],\ [[[x_1x_2][x_3x_4]][[x_5x_6]x_7]].$$

Definition 4.3. A *binary monomial* in degree n is a permutation $\pi \in S_n$ applied to a binary association type in degree n: $\pi(x_1 \cdots x_n) = x_{\pi(1)} \cdots x_{\pi(n)}$ (brackets omitted). If x and x' are two binary monomials in degree n with the same association type, then x and x' are *equivalent* if anticommutativity implies $x = \pm x'$. A binary monomial is in *normal form* if every submonomial of the form $[y, y']$, where $y = x_i \cdots$ and $y' = x_j \cdots$ (brackets omitted) have the same association type, satisfies $i < j$.

Lemma 4.4. *Every binary association type T has a number s of symmetries implied by anticommutativity. Every equivalence class in T contains 2^s monomials, exactly one of which is in normal form. Hence the number of equivalence classes in T is $n!/2^s$.*

Definition 4.5. The *number of binary monomials* in degree n is the sum over all association types T of the number of equivalence classes in T. Each equivalence class is represented by its monomial in normal form.

Example 4.6. The association types in Example 4.2 have the following numbers of symmetries: 0; 1; 1; 1, 3; 1, 3, 2; 1, 3, 2, 2, 4, 3; 1, 3, 2, 2, 4, 3, 2, 4, 3, 2, 4. In each degree

$1 \le n \le 7$ the number of monomials is 1, 1, 3, 15, 105, 945, 10395. In degree 7 the number of monomials in each type is 2520, 630, 1260, 1260, 315, 630, 1260, 315, 630, 1260, 315.

Definition 4.7. The *space P_n of binary polynomials in degree* n is the vector space over $\mathbb{Q}$ with basis consisting of the binary monomials in normal form. This space has the *structure of an S_n-module*: we apply a permutation π to the symbols in the monomial x, preserving the association type; we obtain the monomial x^π, which in general is not in normal form; we apply anticommutativity to obtain $\pi.x = \pm x'$, where x' is the unique monomial in normal form equivalent to x^π.

Definition 4.8. Let $I(x_1, x_2, \dots, x_n)$ be a binary polynomial of degree n. From I we obtain $n+1$ *lifted polynomials* in degree $n+1$; we introduce the symbol x_{n+1} and perform n internal brackets and one external bracket:

$$I([x_1 x_{n+1}], x_2, \dots, x_n), \quad \dots, \quad I(x_1, x_2, \dots, [x_n x_{n+1}]), \quad [I(x_1, x_2, \dots, x_n), x_{n+1}].$$

If I is a polynomial identity in degree n for an anticommutative algebra A, then each lifted polynomial obtained from I is an identity for A in degree $n+1$.

Definition 4.9. Let $(-,-,-)$ denote a ternary operation which is skew-symmetric in the first and second arguments. A *ternary association type* in degree n (odd) is a valid placement of $(n-1)/2$ pairs of parentheses in $x = x_1 x_2 \cdots x_n$. Each type has a unique factorization $x = (y, z, w)$ where y, z, w have degree $< n$ and $\deg(y) \ge \deg(z)$. The *ordering on the association types* is inductive: if it has been defined on types of degree $< n$, then in degree n type (y, z, w) precedes type (y', z', w') if either (i) $\deg(y) > \deg(y')$, or (ii) $\deg(y) = \deg(y')$ and y precedes y', or (iii) $y = y'$ and $\deg(z) > \deg(z')$, or (iv) $y = y'$, $\deg(z) = \deg(z')$ and z precedes z', or (v) $y = y'$, $z = z'$ and w precedes w'.

Example 4.10. The ternary association types in degrees ≤ 7 (commas omitted):

$$x_1; \qquad (x_1 x_2 x_3); \qquad ((x_1 x_2 x_3) x_4 x_5),\ (x_1 x_2 (x_3 x_4 x_5));$$
$$(((x_1 x_2 x_3) x_4 x_5) x_6 x_7),\ ((x_1 x_2 (x_3 x_4 x_5)) x_6 x_7),\ ((x_1 x_2 x_3)(x_4 x_5 x_6) x_7),\ ((x_1 x_2 x_3) x_4 (x_5 x_6 x_7)),$$
$$(x_1 x_2 ((x_3 x_4 x_5) x_6 x_7)),\ (x_1 x_2 (x_3 x_4 (x_5 x_6 x_7))).$$

Remark 4.11. Definition 4.3, Lemma 4.4 and Definition 4.5 extend in the obvious way to ternary monomials.

Example 4.12. The association types in Example 4.10 have the following numbers of symmetries: 0; 1; 1, 2; 1, 2, 3, 2, 2, 3. In each degree $n = 1, 3, 5, 7$ the number of monomials is 1, 3, 90, 7560. In degree 7 the number of monomials in each type is 2520, 1260, 630, 1260, 1260, 630.

Definition 4.13. The *space Q_n of ternary polynomials* in degree n (odd) is the vector space over $\mathbb{Q}$ with basis consisting of the ternary monomials in normal form. Definition 4.7 extends in the obvious way to ternary polynomials.

Definition 4.14. Let $I(x_1, x_2, \dots, x_n)$ be a ternary polynomial of degree n (odd). From I we obtain $n+2$ *lifted polynomials* in degree $n+2$; we introduce the symbols x_{n+1}, x_{n+2} and perform n internal brackets and two external brackets:

$$I((x_1 x_{n+1} x_{n+2}), x_2, \dots, x_n), \quad \dots, \quad I(x_1, x_2, \dots, (x_n x_{n+1} x_{n+2})),$$
$$(I(x_1, x_2, \dots, x_n), x_{n+1}, x_{n+2}), \quad (x_{n+1}, x_{n+2}, I(x_1, x_2, \dots, x_n)).$$

If I is a polynomial identity in degree n for a ternary algebra A, then each lifted polynomial obtained from I is an identity for A in degree $n+2$.

We now consider identities in which each term can be either a binary monomial, a ternary monomial, or a "mixed" monomial which combines the two operations. The following table gives the number of association types and number of monomials in degrees ≤ 7:

degree	binary	ternary	mixed	types	monomials
1	-	-	-	-	1
2	1	-	-	1	1
3	1	1	-	2	6
4	2	-	3	5	45
5	3	2	8	13	510
6	6	-	32	38	7245
7	11	6	96	113	126630

The total number of types is obtained by setting $T_1 = 1$ and then using the following recursive formula, where $\delta = 0$ if n is odd and $\delta = 1$ if n is even:

$$T_n = \sum_{i+j=n,\, i>j} T_i T_j + \delta \binom{T_{n/2}}{2} + \sum_{i+j+k=n,\, i>j} T_i T_j T_k + \delta \binom{T_{n/2}}{2} T_k.$$

Example 4.15. Degree 4 is the first degree in which mixed types occur. In the following list we use the superscript b to indicate a pure binary type; there are no pure ternary types since the degree is even:

$$[[[x_1x_2]x_3]x_4]^b, \quad [(x_1x_2x_3)x_4], \quad [[x_1x_2][x_3x_4]]^b, \quad ([x_1x_2]x_3x_4), \quad (x_1x_2[x_3x_4]).$$

These types have (respectively) 1, 1, 3, 1, 2 symmetries and 12, 12, 3, 12, 6 monomials, for a total of 45. Degree 5 is the first degree in which all three kinds occur (binary, ternary, mixed). In the following list we use a superscript t to indicate a pure ternary type:

$$[[[[x_1x_2]x_3]x_4]x_5]^b, \quad [[(x_1,x_2,x_3)x_4]x_5], \quad [[[x_1x_2][x_3x_4]]x_5]^b, \quad [([x_1x_2],x_3,x_4)x_5],$$
$$[(x_1,x_2,[x_3x_4])x_5], \quad [[[x_1x_2]x_3][x_4x_5]]^b, \quad [(x_1,x_2,x_3)[x_4x_5]], \quad ([[x_1x_2]x_3],x_4,x_5),$$
$$((x_1,x_2,x_3),x_4,x_5)^t, \quad ([x_1x_2],[x_3x_4],x_5), \quad ([x_1x_2],x_3,[x_4x_5]), \quad (x_1,x_2,[[x_3x_4]x_5]),$$
$$(x_1,x_2,(x_3,x_4,x_5))^t.$$

These types have (respectively) 1, 1, 3, 1, 2, 2, 2, 1, 1, 3, 2, 2, 2 symmetries and 60, 60, 15, 60, 30, 30, 30, 60, 60, 15, 30, 30, 30 monomials, for a total of 510.

5. Computational methods

We use two algorithms: "fill and reduce" determines a basis for the vector space of polynomial identities in degree n satisfied by the nonassociative algebra A; "module generators" applies the representation theory of the symmetric group S_n to extract a subset of the basis which generates the space of identities as an S_n-module. A modification of "fill and reduce" also takes into account the lifted identities in degree n implied by known identities in degree $< n$. For simplicity of exposition, we will describe these algorithms for the case of an algebra with a single anticommutative binary operation.

5.1. **Fill and reduce.** Let A be a nonassociative algebra of dimension d, and let q be the number of multilinear nonassociative polynomials in degree n. We allocate memory for a matrix E of size $(q+d) \times q$ consisting of an upper block of size $q \times q$ and a lower block of size $d \times q$; at first the matrix E is zero. We perform the following steps until the rank of E has stabilized; that is, the rank has not increased for some fixed number s of iterations:

(1) Generate n pseudorandom column vectors $a_1, \dots, a_n$ of dimension d, representing elements of the algebra A.
(2) For each $j = 1, \dots, q$, evaluate nonassociative monomial j on the elements $a_1, \dots, a_n$ and store the resulting column vector in rows $q{+}1$ to $q{+}d$ of column j.
(3) Compute the row canonical form of E; the lower block of E is now zero.

After the rank has stabilized, the nullspace of E contains the coefficient vectors of the linear dependence relations on the nonassociative monomials that are satisfied by many pseudorandom choices of elements of A. (It is clear that this algorithm can be easily extended to the case of homogeneous identities which are not necessarily multilinear.) We extract the canonical basis of the nullspace; these vectors represent nonassociative polynomials which are "probably" polynomial identities satisfied by A; these identities still need to be proven directly, or at least checked independently by another computation.

We usually assume that the algebra A is defined over the field $\mathbb{Q}$ of rational numbers; but in order to save computer time and memory we use modular arithmetic in the finite field $\mathbb{F}_p$ for some prime number p. We can estimate the probability of persistent errors in the fill and reduce algorithm, where an error means the row reduction produces a leading entry of some row which is nonzero in rational arithmetic but which is zero in modular arithmetic. In order for this to happen, the row reduction must produce a row whose leading entry in position (i,j) is a rational number in lowest terms which is divisible by p (that is, the numerator is a multiple of p); the probability of this is $1/p$. We can make the (i,j) entry equal to 1 using rational arithmetic, but it will be 0 using modular arithmetic. Since the algebra has dimension d, and the lower block of E has size $d \times q$, we expect to perform d operations of scalar multiplication of a row during the row reduction. Therefore the chance that no error occurs during one iteration of the algorithm is

$$\left(1 - \frac{1}{p}\right)^d.$$

If we perform s iterations of the algorithm after the rank has stabilized, then the chance of a persistent error is

$$\left(1 - \left(1 - \frac{1}{p}\right)^d\right)^s.$$

For example, if we use $p = 101$ for an algebra of dimension $d = 27$ and perform $s = 10$ iterations after the rank has stabilized, then the chance of a persistent error is $\approx 0.5268421642 \times 10^{-6}$, roughly one in two million; if $s = 100$, then the chance of a persistent error is $\approx 0.1647423478 \times 10^{-62}$, which is extremely small. There remains a small chance that when we use modular arithmetic, the rank of the matrix will be smaller than if we used rational arithmetic, and thus the nullspace will be too big. To overcome this difficulty, we compute a basis of the nullspace using modular arithmetic, and reconstruct the most probable corresponding integral vectors. We then check each of these integral basis identities using rational arithmetic by generating pseudorandom integral algebra elements and evaluating the corresponding polynomial identity. If we obtain the zero vector consistently for a large number of trials, we have obtained further confirmation of the hypothetical identity.

In order to reduce the number of identities we need to check, we first extract a set of module generators from the linear basis of the nullspace.

5.2. **Module generators.** Let $I_1, \dots, I_\ell$ be a basis for the polynomial identities in degree n satisfied by a nonassociative algebra A; that is, a basis for a certain subspace of the q-dimensional vector space of nonassociative polynomials in degree n. We allocate memory for a matrix G of size $(q+n!) \times q$ consisting of an upper block of size $q \times q$ and a lower block of size $n! \times q$. We set oldrank $\leftarrow 0$ and then perform the following steps for $k = 1, \dots, \ell$:

(1) Set $i \leftarrow 0$.
(2) For each permutation π in the symmetric group S_n do the following:
   (i) Set $i \leftarrow i + 1$.
   (ii) For each $j = 1, \dots, q$ do the following:
      - Let c_j be the coefficient in I_k corresponding to monomial m_j.
      - If $c_j \neq 0$ then apply π to m_j obtaining πm_j and replace πm_j by the standard representative $m_{\overline{j}} = \overline{\pi m_j}$ of its equivalence class.
      - Store the corresponding coefficient $\pm c_j$ in row $q+i$ and column $\overline{j}$ of the matrix G.
(3) Compute the row canonical form of G; the lower block of G is now zero.
(4) Set newrank $\leftarrow$ rank(G).
(5) If oldrank $<$ newrank then:
   (i) Record I_k as a new module generator.
   (ii) Set oldrank $\leftarrow$ newrank.

Suppose that we already know a set of lifted identities in degree n which are consequences of known identities in degree $< n$. We apply the module generators algorithm to these lifted identities; at the end of the computation the row space of G contains a basis for the subspace of identities in degree n which are consequences of identities of lower degree. We then apply the module generators algorithm to the linear basis produced by the fill and reduce algorithm: the canonical basis of the nullspace of E which gives a basis for the subspace of all identities in degree n. In this way we obtain a set of module generators for the space of all identities in degree n modulo the space of known identities in degree n; that is, a set of module generators for the new identities in degree n.

As before, we usually assume that the algebra A is defined over $\mathbb{Q}$, but to save time and memory we use arithmetic over $\mathbb{F}_p$. To justify this approach, we use the following argument. Let I be an identity with coefficients c_j with respect to the ordered basis m_j of multilinear nonassociative monomials in degree n. If $p > n$ then the group algebra $\mathbb{F}_p S_n$ is semisimple, and is isomorphic to the direct sum of matrix algebras $\mathrm{Mat}_{d(\lambda)}(\mathbb{F}_p)$ over all partitions λ of n where $d(\lambda)$ is the dimension of the corresponding irreducible representation of S_n. If there are t association types in degree n, then we consider the direct sum of t copies of $\mathbb{F}_p S_n$, one for each association type. Each of the monomials m_j consists of an underlying permutation $\overline{m_j}$ and an association type $t(j)$; the monomial m_j is represented by the matrix $R_\lambda(\overline{m_j})$ in copy $t(j)$ of $\mathbb{F}_p S_n$. By the algorithm of Clifton [3] we know that the entries of the representation matrices $R_\lambda(\overline{m_j})$ belong to $\{0, 1, -1\}$. Hence the entries of the representation matrix for the complete identity I have absolute values no greater than the sum of the absolute values of the coefficients c_j. If we choose a prime p which also satisfies

$$p > 2 \sum_j |c_j|,$$

and use symmetric representatives modulo p (that is, integers between $-p/2$ and $p/2$), then the entries of the representation matrices for the identity I will be the same whether we interpret them as integers or as elements of $\mathbb{F}_p$. We now appeal to the following result.

Lemma 5.1. *Let $T \subseteq \mathbb{Q}S_n$ be the left ideal generated by elements $I_1, \ldots, I_k \in \mathbb{Z}S_n$. Let p be a prime number such that*

$$p > \max\Big(n, 2 \sum_j |c_{1j}|, \ldots, 2 \sum_j |c_{kj}|\Big),$$

and let $\mathbb{F}_p$ be the field with p elements. Let $I'_1, \ldots, I'_k \in \mathbb{F}_p S_n$ be the elements obtained by reducing the coefficients of $I_1, \ldots, I_k$ modulo p, and let $T_p \subseteq \mathbb{F}_p S_n$ be the left ideal generated by $I'_1, \ldots, I'_k$. Then $\dim_{\mathbb{Q}} T = \dim_{\mathbb{F}_p} T_p$.

Proof. The group algebra $\mathbb{Q}S_n$ is the direct sum of simple ideals over all partitions λ of n. For each λ the corresponding simple ideal is isomorphic to the algebra of $d(\lambda) \times d(\lambda)$ matrices, where $d(\lambda)$ is the dimension of the corresponding irreducible representation of S_n. Let $G^{(\lambda)}$ be the $kd(\lambda) \times d(\lambda)$ matrix over $\mathbb{Z}$ consisting of a column of $d(\lambda) \times d(\lambda)$ blocks in which block i is the integral representation matrix $R_\lambda(I_i)$ computed according to Clifton [3]. By the semisimplicity of $\mathbb{Q}S_n$ we have

$$\dim_{\mathbb{Q}} T = \sum_\lambda d(\lambda) \operatorname{rank}_{\mathbb{Q}} G^{(\lambda)},$$

since each nonzero row of the row canonical form of $G^{(\lambda)}$ contributes one copy of the irreducible representation (that is, the minimal left ideal) corresponding to λ. Let $G_p^{(\lambda)}$ be the reduction of $G^{(\lambda)}$ modulo p; then $\mathbb{F}_p S_n$ is semisimple since $p > n$, and so the same argument shows that

$$\dim_{\mathbb{F}_p} T_p = \sum_\lambda d(\lambda) \operatorname{rank}_{\mathbb{F}_p} G_p^{(\lambda)}.$$

The isomorphism of $\mathbb{Q}S_n$ with a direct sum of matrix algebras expresses the matrix units as linear combinations of permutations in which the denominators of the coefficients are divisors of $n!$. It follows that the same group algebra elements are defined over $\mathbb{F}_p$, and that they satisfy the same matrix unit relations $E_{qr} E_{st} = \delta_{rs} E_{qt}$. Since $p > 2 \sum_j |c_j|$ and since computation of the row canonical form can be expressed as a sequence of left multiplications by elementary matrices, it follows that

$$\operatorname{rank}_{\mathbb{Q}} G^{(\lambda)} = \operatorname{rank}_{\mathbb{F}_p} G_p^{(\lambda)}.$$

This completes the proof. □

We close this section with an important special case.

Remark 5.2. Suppose that I is an identity with rational coefficients satisfied by the algebra A which has integral structure constants with respect to a given basis. We multiply I by the least common multiple of the denominators of its coefficients, obtaining an identity I' with integral coefficients; we then divide I' by the greatest common divisor of its coefficients, obtaining an identity I'' which is primitive in the sense that its coefficients are integers with no common factor. It is clear that I'' is a polynomial identity satisfied by the algebra A, and that the reduction of I'' modulo p is nonzero for any prime number p. Thus the existence of identities in characteristic 0 implies the existence of identities in characteristic p for all p, and so non-existence in characteristic p for any p implies non-existence in characteristic 0. Therefore we can verify non-existence of identities over $\mathbb{Q}$ by computation over $\mathbb{F}_p$ for any p.

6. Polynomial identities for the L-module $V(2,2)$

In this section we describe our computational results on the polynomial identities satisfied by the binary operation $[a,b]$ and the ternary operation (a,b,c) as defined by equation (8) on the module $V = V(2,2)$ over the Lie algebra $L = \mathfrak{sl}(3,\mathbb{C})$.

Theorem 6.1. *Every polynomial identity of degree ≤ 7, satisfied by the binary operation $[-,-]$ on $V(2,2)$ is a consequence of the anticommutative identity $[a,a]=0$.*

Proof. We describe the computations only for degree 7; the computations for lower degrees are similar but simpler. In fact, it suffices to consider only degree 7, since existence of identities in degree n implies existence of identities in degree $n+1$. We create a matrix of size 10422×10395, consisting of a 10395×10395 upper block and a 27×10395 lower block, and initialize it to zero. The columns correspond to the 10395 binary monomials in degree 7. We perform $10395/27 = 385$ iterations of the fill and reduce process, and find that after each iteration the rank of the matrix has increased by 27. After the last iteration, the rank is 10395, and so the nullspace is zero. We now appeal to Remark 5.2. □

Theorem 6.2. *Every polynomial identity of degree ≤ 5, satisfied by the ternary operation $(-,-,-)$ on $V(2,2)$ is a consequence of the ternary derivation identity*

$$[a,b,[c,d,e]] = [[a,b,c],d,e] + [c,[a,b,d],e] + [c,d,[a,b,e]].$$

Proof. For $n=3$ the matrix has size $(3+27)\times 3$; the calculations are trivial and are omitted. Every identity in degree 3 is a consequence of ternary anticommutativity. The ternary Jacobi identity $(a,b,c)+(b,c,a)+(c,a,b)=0$ does not hold, so $V(2,2)$ is not a Lie triple system.

For degree 5, there are two association types, $[[x_1x_2x_3]x_4x_5]$ and $[x_1x_2[x_3x_4x_5]]$, with 60 and 30 monomials respectively. For type 1, we have all permutations x_1,x_2,x_3,x_4,x_5 of a,b,c,d,e in which $x_1 < x_2$ (lexicographically); for type 2 we have all permutations in which $x_1 < x_2$ and $x_3 < x_4$. The expansion matrix has size $(90+27)\times 90$. The first iteration generates these 5 pseudorandom row vectors, representing elements of V, with $p = 101$:

41 10 43 37 70 37 42 83 50 78 98 81 14 96 89 94 53 82 97 20 16 59 4 11 41 97 73
34 50 40 71 90 56 43 71 83 17 85 9 86 40 92 41 55 59 77 31 53 85 15 41 4 29 31
62 96 18 64 26 24 89 1 37 77 46 80 15 75 77 51 78 46 59 52 39 55 53 19 53 98 99
53 14 73 76 85 33 16 32 6 95 88 63 70 2 43 43 66 47 3 2 83 56 21 63 24 65 21
7 13 83 74 36 39 8 67 21 46 35 61 14 15 20 39 15 1 25 26 49 99 81 87 69 58 80

We evaluate all 90 monomials using these vectors for a,b,c,d,e and store the results as column vectors in the lower block of the matrix. We reduce the matrix, and find that it has rank 27. The second iteration generates these 5 pseudorandom row vectors:

34 95 97 40 66 88 7 71 45 71 4 96 67 39 54 66 79 44 48 47 8 19 30 30 94 56 6
68 44 18 44 17 72 92 32 35 83 70 70 39 29 29 88 14 28 55 7 85 65 34 26 14 69 58
17 95 7 5 83 23 20 84 47 19 95 73 69 17 18 2 54 77 26 8 71 43 78 30 81 23 87
22 100 3 59 98 70 62 27 69 75 18 12 54 29 60 38 76 24 7 69 77 78 58 21 22 17 90
32 82 80 29 66 80 81 43 32 11 93 96 70 33 18 80 80 8 85 66 25 25 56 24 19 84 68

After the fill and reduce, the matrix has rank 54. The third iteration generates these 5 pseudorandom row vectors:

14 37 91 75 58 50 51 61 47 42 39 85 87 98 42 97 11 54 44 11 3 53 99 60 31 53 74

72 14 54 12 89 32 69 63 64 68 84 64 75 88 26 58 31 43 18 80 22 88 43 18 97 30 57

37 18 63 8 91 20 80 83 31 99 6 1 85 48 63 70 65 53 52 97 73 73 68 38 84 27 22

3 25 10 21 65 17 37 11 72 90 81 32 34 45 54 62 46 32 80 91 51 13 50 6 57 52 64

23 80 23 81 12 66 24 79 10 38 18 23 50 60 55 14 88 29 57 28 5 84 24 95 47 75 57

After the fill and reduce, the matrix has rank 60. Performing another 100 iterations does not increase the rank beyond 60. The canonical basis of the nullspace consists of 30 vectors, each with 4 nonzero coefficients in the set $\{1, -1\}$ using symmetric representatives modulo p. We sort the corresponding identities lexicographically by the first term, and normalize them so that the first coefficient is positive.

We now perform the module generators algorithm. The first nullspace vector increases the rank to 15, and the second to 30. Removing the first vector from the list and repeating the computation shows that the second vector by itself increases the rank to 30, so the second identity (the ternary derivation identity) generates the entire space of identities. □

Corollary 6.3. *Every identity in degree 7 for the ternary operation $(-,-,-)$ is a consequence of ternary anticommutativity and the ternary derivation identity.*

Proof. The proof consists of two parts: first, computing the rank of the lifted identities in degree 7; second, computing the rank of all identities in degree 7. Let $I(a,b,c,d,e) = 0$ be the ternary derivation identity:

$$I(a,b,c,d,e) = [a,b,[c,d,e]] - [[a,b,c],d,e] - [c,[a,b,d],e] - [c,d,[a,b,e]].$$

It is easy to see that $I(a,b,c,d,e)$ is skew-symmetric in a,b and c,d; hence it suffices to consider five lifted identities in degree 7:

$$\begin{array}{lll} I([a,f,g],b,c,d,e), & I(a,b,[c,f,g],d,e), & I(a,b,c,d,[e,f,g]), \\ {[I(a,b,c,d,e),f,g]}, & [f,g,I(a,b,c,d,e)]. & \end{array}$$

We create a matrix of size $(7560+5040) \times 7560$ and initialize it to zero. The columns correspond to the ternary monomials in degree 7. We perform the module generators algorithm on these lifted identities, and find that the rank of the matrix increases to 1260, 2520, 3150, 4200 and finally 4410. The submodule in degree 7 consisting of consequences of the ternary derivation identity therefore has dimension 4410.

We now construct a matrix of size $(7560+27) \times 7560$ and perform the fill and reduce algorithm. The rank increases by 27 after each iteration, until iteration 116, after which the rank is 3132. During iteration 117 the rank increases to 3150, and it does not increase during another 100 iterations. Therefore the nullspace has dimension 4410, which is the same as the dimension of the module of lifted identities. It follows that there are no new identities in degree 7. □

Theorem 6.4. *Every polynomial identity of degree ≤ 3 relating the binary and ternary operations is a consequence of the mixed Jacobi identity,*

$$[[a,b],c] + [[b,c],a] + [[c,a],b] + (a,b,c) + (b,c,a) + (c,a,b) = 0.$$

Every polynomial identity of degree $\leq$ 4 relating the binary and ternary operations is a consequence of the mixed Jacobi identity, together with these two identities:

$$([a,b],c,d) + ([b,c],a,d) + ([c,a],b,d) = 0,$$
$$(a,b,[c,d]) - [(a,b,c),d] + [(a,b,d),c] = 0.$$

Every polynomial identity of degree $\leq$ 6 is a consequence of these three identities together with binary and ternary anticommutativity and the ternary derivation identity.

Proof. Degree 3: We construct a matrix of size $(6+27) \times 6$ and perform the fill and reduce process. The rank reaches 5 after the first iteration and does not increase for another 100 iterations. The nullspace has dimension 1; a basis vector corresponds to the mixed Jacobi identity.

Degree 4: Since the mixed Jacobi identity $J(a,b,c)$ is an alternating function of its three arguments, we need only two lifted identities in degree 4: $J([a,d],b,c)$ and $[J(a,b,c),d]$. We construct a matrix of size $(45+24) \times 45$ and perform the module basis algorithm: after the first identity, the rank is 6; after the second identity, the rank is 10. If we process these two identities in the reverse order, we get ranks 4 and 10; hence both identities are necessary.

We now perform the fill and reduce algorithm with a matrix of size $(45+27) \times 45$. The rank reaches 26 after the first iteration, and does not increase for another 100 iterations; hence the nullspace has dimension 19. We compute the canonical basis of the nullspace, and sort the vectors first by number of nonzero components and then by the position of the first nonzero component. We obtain 4, 4, 2, 1, 3, 1, 1, 2, 1 identities with respectively 3, 6, 7, 8, 11, 15, 16, 17, 18 terms.

We now process these identities using the module generator algorithm, starting with the already partially filled matrix from the lifted ternary derivation identities. Identity 1 increases the rank from 10 to 16, and identity 4 increases the rank from 16 to 19. (None of the other identities increase the rank.) Identity 4 by itself increases the rank from 10 to 14, so identity 1 is not a consequence of identity 4. Identities 1 and 4 are equivalent to the two identities of degree 4 in the definition of Lie-Yamaguti algebras.

Degree 5: There are 13 association types and 510 monomials. During the fill and reduce process, the rank increases by 27 after each of the first 7 iterations; after iteration 8 the rank reaches 214, and does not increase for another 100 iterations. The nullspace therefore has dimension 296; we extract the canonical basis and sort the vectors as described in degree 4. The numbers of terms in the corresponding identities (and the number of identities with that number of terms) are as follows: 3 (24), 4 (12), 5 (1), 6 (43), 7 (10), 8 (27), 9 (1), 11 (15), 12 (12), 15 (6), 16 (14), 17 (10), 18 (10), 19 (1), 20 (4), and the remaining identities have more than 20 terms; the longest has 106 terms.

We now determine all the identities in degree 5 that are consequences of previously discovered identities. We have to consider the following cases:

- In degree 3 we have the mixed Jacobi identity, which can be lifted to degree 5 either using the binary operation to lift it from degree 3 to degree 4 and then from degree 4 to degree 5, or using the ternary operation to lift it from degree 3 to degree 5.
- In degree 4 we have two identities which can be lifted to degree 5 using the binary operation.
- In degree 5 we have the ternary derivation identity.

Let $J(a,b,c)$ denote the mixed Jacobi identity in degree 3. Since it is an alternating function of its three arguments, it suffices to consider two consequences in degree 4, namely

$J([a,d],b,c)$ and $[J(a,b,c),d]$. The identity $J([a,d],b,c)$ alternates in a,d and in b,c and hence produces three lifted identities in degree 5:

$$J([[a,e],d],b,c), \qquad J([a,d],[b,e],c), \qquad [J([a,d],b,c),e]. \tag{10}$$

The identity $[J(a,b,c),d]$ alternates in a,b,c and hence produces another three lifted identities in degree 5:

$$[J([a,e],b,c),d], \qquad [J(a,b,c),[d,e]], \qquad [[J(a,b,c),d],e], \tag{11}$$

Using the ternary operation, $J(a,b,c)$ produces another three lifted identities in degree 5:

$$J((a,d,e),b,c), \qquad (J(a,b,c),d,e), \qquad (d,e,J(a,b,c)). \tag{12}$$

Now let $K_1(a,b,c,d)$ and $K_2(a,b,c,d)$ be the two identities in degree 4. The identity $K_1(a,b,c,d)$ alternates in a,b,c and hence produces three lifted identities in degree 5:

$$K_1([a,e],b,c,d), \qquad K_1(a,b,c,[d,e]), \qquad [K_1(a,b,c,d),e]. \tag{13}$$

The identity $K_2(a,b,c,d)$ alternates in a,b and in c,d and hence produces another three lifted identities in degree 5:

$$K_2([a,e],b,c,d), \qquad K_2(a,b,[c,e],d), \qquad [K_2(a,b,c,d),e]. \tag{14}$$

Altogether we have 16 known identities in degree 5: the 15 identities of equations (10)–(14), together with the ternary derivation identity. We now apply the module generators algorithm to this list of 16 identities. The cumulative ranks are as follows:

up to 1:	30	up to 5:	85	up to 9:	155	up to 13:	235
up to 2:	40	up to 6:	105	up to 10:	185	up to 14:	255
up to 3:	75	up to 7:	135	up to 11:	195	up to 15:	280
up to 4:	75	up to 8:	155	up to 12:	215	up to 16:	296

The total rank of known identities is 296, which equals the nullspace dimension obtained from the fill and reduce algorithm. It follows that there are no new identities in degree 5: every identity in degree 5 is a consequence of the mixed Jacobi identity in degree 3, the two identities in degree 4, and the ternary derivation identity in degree 5. In the list of known identities in degree 5, identities 4 and 9 are superfluous, since they do not increase the rank. Further computations with the same algorithm show that identities 5 and 11 are also superfluous; that is, the 12 identities with numbers 1, 2, 3, 6, 7, 8, 10, 12, 13, 14, 15, 16 generate the entire 296-dimensional space of known identities.

Degree 6: To find known identities in degree 6, we can either use the ternary operation to lift an identity from degree 4, or use the binary operation to lift an identity from degree 5. In degree 4, we have two lifted forms of the mixed Jacobi identity and two new identities:

$$J([a,d],b,c), \qquad [J(a,b,c),d], \qquad K_1(a,b,c,d), \qquad K_2(a,b,c,d).$$

The first lifted form of J alternates in a,d and in b,c so it produces four lifted identities. The second lifted form of J alternates in a,b,c so it produces four lifted identities. The identity K_1 alternates in a,b so it produces five lifted identities. The identity K_2 alternates in a,b

and in c, d so it produces four lifted identities. Altogether this gives 17 identities:

$$
(*) \begin{cases}
J([(a,e,f),d],b,c),\ J([a,d],(b,e,f),c),\ (J([a,d],b,c),e,f),\ (e,f,J([a,d],b,c)), \\
[J((a,e,f),b,c),d],\ [J(a,b,c),(d,e,f)],\ ([J(a,b,c),d],e,f),\ (e,f,[J(a,b,c),d]), \\
K_1((a,e,f),b,c,d),\ K_1(a,b,(c,e,f),d),\ K_1(a,b,c,(d,e,f)),\ (K_1(a,b,c,d),e,f), \\
(e,f,K_1(a,b,c,d)), \\
K_2((a,e,f),b,c,d),\ K_2(a,b,(c,e,f),d),\ (K_2(a,b,c,d),e,f),\ (e,f,K_2(a,b,c,d)).
\end{cases}
$$

In degree 5, we have 12 identities (omitting numbers 4, 5, 9, 11 from the complete list of 16):

$$
\begin{array}{llll}
J([[a,e],d],b,c), & J([a,d],[b,e],c), & [J([a,d],b,c),e], & [[J(a,b,c),d],e], \\
J((a,d,e),b,c), & (J(a,b,c),d,e), & K_1([a,e],b,c,d), & [K_1(a,b,c,d),e], \\
K_2([a,e],b,c,d), & K_2(a,b,[c,e],d), & [K_2(a,b,c,d),e], & L(a,b,c,d,e),
\end{array}
$$

where L is the ternary derivation identity. These produce the following lifted identities:

$$
(**) \begin{cases}
J([[[a,f],e],d],b,c),\ J([[a,e],[d,f]],b,c),\ J([[a,e],d],[b,f],c),\ [J([[a,e],d],b,c),f], \\
J([[a,f],d],[b,e],c),\ J([a,d],[b,e],[c,f]),\ [J([a,d],[b,e],c),f], \\
[J([[a,f],d],b,c),e],\ [J([a,d],[b,f],c),e],\ [J([a,d],b,c),[e,f]],\ [[J([a,d],b,c),e],f], \\
[[J([a,f],b,c),d],e],\ [[J(a,b,c),[d,f]],e],\ [[J(a,b,c),d],[e,f]],\ [[[J(a,b,c),d],e],f], \\
J(([a,f],d,e),b,c),\ J((a,d,[e,f]),b,c),\ J((a,d,e),[b,f],c),\ [J((a,d,e),b,c),f], \\
(J([a,f],b,c),d,e),\ (J(a,b,c),[d,f],e),\ (J(a,b,c),d,[e,f]),\ [(J(a,b,c),d,e),f], \\
K_1([[a,f],e],b,c,d),\ K_1([a,e],[b,f],c,d),\ K_1([a,e],b,[c,f],d),\ K_1([a,e],b,c,[d,f]), \\
[K_1([a,e],b,c,d),f], \\
[K_1([a,f],b,c,d),e],\ [K_1(a,b,[c,f],d),e],\ [K_1(a,b,c,[d,f]),e],\ [K_1(a,b,c,d),[e,f]], \\
[[K_1(a,b,c,d),e],f], \\
K_2([[a,f],e],b,c,d),\ K_2([a,e],[b,f],c,d),\ K_2([a,e],b,[c,f],d),\ [K_2([a,e],b,c,d),f], \\
K_2([a,f],b,[c,e],d),\ K_2(a,b,[[c,f],e],d),\ K_2(a,b,[c,e],[d,f]),\ [K_2(a,b,[c,e],d),f], \\
[K_2([a,f],b,c,d),e],\ [K_2(a,b,[c,f],d),e],\ [K_2(a,b,c,d),[e,f]],\ [[K_2(a,b,c,d),e],f], \\
L([a,f],b,c,d,e),\ L(a,b,[c,f],d,e),\ L(a,b,c,d,[e,f]),\ [L(a,b,c,d,e),f].
\end{cases}
$$

The 66 identities labeled $(*)$ and $(**)$ generate the space of known identities in degree 6. The rest of the argument is similar to that in degree 5; there are no new identities. □

Remark 6.5. The results in this section show that the polynomial identities in degrees ≤ 5 for the binary-ternary structure on $V(2,2)$ coincide precisely with the defining identities for Lie-Yamaguti algebras.

7. Polynomial identities for the L-modules $V(3,3)$ and $V(4,4)$

The L-module $V = V(3,3)$ has dimension 64. Its exterior square contains one copy of the adjoint module L and two copies of V. We choose a particular submodule of $\Lambda^2 V$ which is isomorphic to V using the embedding described in Section 2. The module $V(3,0)$ can be identified with the symmetric cube of the natural representation $V(1,0)$. Hence a basis of $V(3,0)$ consists of these 10 cubic monomials,

$$
x^3, \quad x^2y, \quad x^2z, \quad xy^2, \quad xyz, \quad xz^2, \quad y^3, \quad y^2z, \quad yz^2, \quad z^3.
$$

From this we obtain realization in terms of 10×10 matrices of the decomposition

$$
V(3,0) \otimes V(0,3) \approx V(0,0) \oplus V(1,1) \oplus V(2,2) \oplus V(3,3).
$$

From the projections of the Lie bracket in the subspace $V(3,3)$ onto the subspaces $V(1,1)$ and $V(3,3)$ we obtain binary and ternary operations on $V(3,3)$ as we did for $V(2,2)$. Our computational methods give the following results.

Theorem 7.1. *Let $L = \mathfrak{sl}(3)$ and $V = V(3,3)$. Every polynomial identity of degree ≤ 7 for the binary operation $[-,-]$ is a consequence of the anticommutative identity $[a,a] = 0$. Every polynomial identity of degree ≤ 7 for the ternary operation $(-,-,-)$ is a consequence of the ternary derivation identity. There are no polynomial identities of degree ≤ 3 relating the binary and ternary operations; in particular, the mixed Jacobi identity does not hold, so $V(3,3)$ is not a Lie-Yamaguti algebra. Every polynomial identity of degree ≤ 4 relating the binary and ternary operations is a consequence of the two identities in degree 4 from the definition of Lie-Yamaguti algebras. Every polynomial identity of degree ≤ 6 is a consequence of the previous identities.*

The L-module $V = V(4,4)$ has dimension 125. Its exterior square contains one copy of the adjoint module L and two copies of V. We choose a particular submodule of $\Lambda^2 V$ which is isomorphic to V using the embedding described in Section 2. The module $V(4,0)$ can be identified with the fourth exterior power of the natural representation $V(1,0)$. Hence a basis of $V(3,0)$ consists of these 15 quartic monomials,

$$x^4,\ x^3y,\ x^3z,\ x^2y^2,\ x^2yz,\ x^2z^2,\ xy^3,\ xy^2z,\ xyz^2,\ xz^3,\ y^4,\ y^3z,\ y^2z^2,\ yz^3,\ z^4,$$

From this we obtain realization in terms of 15×15 matrices of the decomposition

$$V(4,0) \otimes V(0,4) \approx V(0,0) \oplus V(1,1) \oplus V(2,2) \oplus V(3,3) \oplus V(4,4).$$

From the projections of the Lie bracket in the subspace $V(4,4)$ onto the subspaces $V(1,1)$ and $V(4,4)$ we obtain binary and ternary operations on $V(4,4)$ as we did for $V(2,2)$ and $V(3,3)$. Our computational methods give the same results for $V(4,4)$ as for $V(3,3)$; that is, Theorem 7.1 also holds for $V(4,4)$.

Remark 7.2. The binary-ternary structures on $V(3,3)$ and $V(4,4)$ provide proper generalizations of Lie-Yamaguti algebras, since they satisfy all the defining identities of Lie-Yamaguti algebras except the mixed Jacobi identity.

8. Conclusion

None of the binary structures on the $\mathfrak{sl}_3(\mathbb{C})$-modules $V(a,a)$ for $a = 2,3,4$ satisfy an identity of degree ≤ 7 which is not a consequence of identity (1). This suggests that, regarded exclusively as binary algebras, these modules do not lie in any familiar variety of anticommutative algebras. On the other hand, in all three cases, the ternary structures satisfy identities (2) and (6); furthermore, in all three cases identities (4) and (5) are satisfied which relate the binary and ternary structures. However, only $V(2,2)$ satisfies identity (3), and this is the only identity in Definition 1.1 which is not "homogeneous in the operations" (in all the other identities, each term has the same number of binary operations and the same number of ternary operations). Hence only $V(2,2)$ is a Lie-Yamaguti algebra; but $V(3,3)$ and $V(4,4)$ provide examples of binary-ternary algebras which are very close to Lie-Yamaguti algebras. It seems reasonable to expect that the same identities are satisfied by the binary-ternary structures on $V(a,a)$ for all $a \geq 3$. Since these generalized Lie-Yamaguti structures arise from a very natural construction in representation theory, they seem to be an interesting topic for further research.

References

[1] P. Benito, C. Draper and A. Elduque: *On some algebras related to simple Lie triple systems.* J. Algebra 219 (1999) 234–254.

[2] P. Benito, C. Draper and A. Elduque: *Lie-Yamaguti algebras related to* $\mathfrak{g}_2$. J. Pure Appl. Algebra 202 (2005) 22–54.

[3] J. M. Clifton: *A simplification of the computation of the natural representation of the symmetric group* S_n. Proc. Amer. Math. Soc. 83 (1981) 248–250.

[4] A. Cohen et al.: *LiE: A computer algebra package for Lie group computations.* See: `http://young.sp2mi.univ-poitiers.fr/~marc/LiE/`

[5] M. Kinyon and A. Weinstein: *Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces.* Amer. J. Math. 123 (2001) 525–550.

[6] W. G. Lister: *A structure theory of Lie triple systems.* Trans. Amer. Math. Soc. 72 (1952) 217–242.

[7] F. Martín-Herce: *Álgebras de Lie-Yamaguti y sistemas algebraicos no asociativos.* Doctoral Thesis, University of La Rioja, Spain, June 2006.

[8] J. M. Pérez-Izquierdo: *An envelope for Bol algebras.* J. Algebra 284 (2005) 480–493.

[9] K. Yamaguti: *On the Lie triple system and its generalization.* J. Sci. Hiroshima Univ. Ser. A 21 (1957/1958) 155–160.

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