On the semigroup algebra of binary relations*

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Abstract

The semigroup of binary relations on $\{1,\ldots,n\}$ with the relative product is isomorphic to the semigroup B_n of $n{\times}n$ matrices over $\{0,1\}$ with the Boolean matrix product. Over any field F, we prove that there is an ideal K_n in FB_n of dimension $(2^n-1)^2$, and we construct an explicit isomorphism of K_n with the matrix algebra $M_{2^n-1}(F)$.

Let $Z_n = \{1,\ldots,n\}$ $(n \geq 1)$; then $P(Z_n^2)$, the power set of the ordered pairs, is the set of binary relations on Z_n. For $X, Y \in P(Z_n^2)$ the relative product is $XY = \{\, (i,k) \,|\, \exists j \in Z_n \text{ with } (i,j) \in X \text{ and } (j,k) \in Y \,\}$. We identify $X \in P(Z_n^2)$ with the Boolean matrix $X = (x_{ij})$ where $x_{ij} = 1$ if $(i,j) \in X$ and $x_{ij} = 0$ if $(i,j) \notin X$; the relative product coincides with the Boolean matrix product.

Definition 1. The set B_n of all $X = (x_{ij})$ with the Boolean matrix product is the **semigroup of Boolean matrices**. Over any field F, the vector space FB_n with the bilinear extension of the Boolean matrix product is the **semigroup algebra of Boolean matrices**.

Clearly $|B_n| = 2^{n^2}$. The subset of permutation matrices in B_n is a subgroup isomorphic to the symmetric group S_n. For the theory of B_n as an abstract semigroup see Schwarz [5]. Recent monographs on related topics are Maddux [4] and Jespers and Okniński [2].

Kim and Roush [3] studied linear representations of B_n, and showed that the ideal K_n (Definition 7 below) is isomorphic to the matrix algebra $M_{2^n-1}(F)$ (Corollary 21 below)

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without however giving an explicit isomorphism. The purpose of the present paper is to provide an explicit isomorphism by constructing standard matrix units in K_n.

Definition 2. We write O for the Boolean zero matrix and J for the Boolean matrix in which every entry is 1.

Definition 3. The **row matrix** $R(P)$ of $P \subseteq Z_n$ is the Boolean matrix with $R(P)_{ij} = 1$ if $i \in P$ and $R(P)_{ij} = 0$ if $i \notin P$. The **column matrix** $C(Q)$ of $Q \subseteq Z_n$ is the Boolean matrix with $C(Q)_{ij} = 1$ if $j \in Q$ and $C(Q)_{ij} = 0$ if $j \notin Q$. The **rectangle** of $P, Q \subseteq Z_n$ is the Boolean matrix $\Box(P,Q) = R(P)C(Q)$. (The term "rectangle" comes from relation algebras; see Maddux [4]. Rectangles are called "cross-vectors" by Kim and Roush [3].)

Lemma 4. *We have $R(P) = O$ (respectively $C(Q) = O$) if and only if $P = \emptyset$ (respectively $Q = \emptyset$), and $\Box(P,Q) = O$ if and only if either $P = \emptyset$ or $Q = \emptyset$. Hence there are 2^n-1 distinct nonzero row (column) matrices, and $(2^n-1)^2$ distinct nonzero rectangles.*

Proof. Obvious. □

Lemma 5. *If $X, Y \in B_n$ and $P, Q \subseteq Z_n$ then $XR(P) = R(U)$ and $C(Q)Y = C(V)$ where*

$$k \in U \iff \sum_{j \in P} x_{kj} = 1, \qquad \ell \in V \iff \sum_{i \in Q} y_{i\ell} = 1 \qquad \textit{(Boolean sums).}$$

Proof. Straightforward. □

Lemma 6. *The elements $\Box(P,Q) - O$ with $\Box(P,Q) \neq O$ (equivalently, $P \neq \emptyset$ and $Q \neq \emptyset$) form a basis of an ideal in FB_n.*

Proof. For $X, Y \in B_n$ we have $X(\Box(P,Q) - O)Y = XR(P)C(Q)Y - XOY = R(U)C(V) - O = \Box(U,V) - O$ with U, V as in Lemma 5. Replacing X and Y by linear combinations of Boolean matrices, we see that the elements $\Box(P,Q) - O$ span an ideal in FB_n. Distinct ordered pairs (P,Q) with $P \neq \emptyset$ and $Q \neq \emptyset$ give distinct nonzero rectangles $\Box(P,Q)$ in B_n, and distinct nonzero rectangles are linearly independent in FB_n; hence the elements $\Box(P,Q) - O$ with $\Box(P,Q) \neq O$ form a basis of this ideal. □

Definition 7. We write K_n for the ideal of FB_n with the basis of Lemma 6.

Lemma 8. *We have $C(Q)R(P) = O$ if $P \cap Q = \emptyset$ and $C(Q)R(P) = J$ if $P \cap Q \neq \emptyset$.*

Proof. Straightforward. □

Lemma 9. *We have $XO = OX = O$ for $X \in B_n$. We have $R(P)J = R(P)$ and $JC(Q) = C(Q)$ for $P, Q \subseteq Z_n$.*

Proof. Obvious. □

Definition 10. We fix an ordering $P_1, \ldots, P_{2^n-1}$ of the nonempty subsets of Z_n. For $i, j = 1, \ldots, 2^n-1$ we define $R_{ij} = \Box(P_i, P_j)$ and $\epsilon_{ij} = 0 \in F$ if $P_i \cap P_j = \emptyset$, $\epsilon_{ij} = 1 \in F$ if $P_i \cap P_j \neq \emptyset$. We write $\mathcal{E}$ for the matrix whose ij-entry is ϵ_{ij}.

Remark 11. We regard $\mathcal{E}$ as a matrix over F since we will need $\mathcal{E}^{-1}$ in Theorem 20. (As a Boolean matrix $\mathcal{E}$ represents the intersection relation on the nonempty subsets of Z_n.)

Lemma 12. *We have* $R_{ij}R_{k\ell} = (1-\epsilon_{jk})O + \epsilon_{jk}R_{i\ell}$ *for any* $i,j,k,\ell = 1,\dots,2^n-1$.

Proof. We have

$$\begin{aligned} R_{ij}R_{k\ell} &= \square(P_i,P_j)\square(P_k,P_\ell) = R(P_i)C(P_j)R(P_k)C(P_\ell) = R(P_i)[(1-\epsilon_{jk})O + \epsilon_{jk}J]C(P_\ell) \\ &= (1-\epsilon_{jk})R(P_i)OC(P_\ell) + \epsilon_{jk}R(P_i)JC(P_\ell) = (1-\epsilon_{jk})O + \epsilon_{jk}R(P_i)C(P_\ell) \\ &= (1-\epsilon_{jk})O + \epsilon_{jk}\square(P_i,P_\ell) = (1-\epsilon_{jk})O + \epsilon_{jk}R_{i\ell}, \end{aligned}$$

as required. ◻

Definition 13. We set $X_{ij} = R_{ij} - O$ for any $i,j = 1,\dots,2^n-1$.

Lemma 14. *We have* $X_{ij}X_{k\ell} = \epsilon_{jk}X_{i\ell}$ *for any* $i,j,k,\ell = 1,\dots,2^n-1$.

Proof. We have

$$\begin{aligned} X_{ij}X_{k\ell} &= \big(R_{ij}-O\big)\big(R_{k\ell}-O\big) = R_{ij}R_{k\ell} - R_{ij}O - OR_{k\ell} + O^2 = R_{ij}R_{k\ell} - O - O + O \\ &= R_{ij}R_{k\ell} - O = (1-\epsilon_{jk})O + \epsilon_{jk}R_{i\ell} - O = -\epsilon_{jk}O + \epsilon_{jk}R_{i\ell} = \epsilon_{jk}\big(R_{i\ell}-O\big) = \epsilon_{jk}X_{i\ell}, \end{aligned}$$

as required. ◻

Remark 15. The elements X_{ij} are very similar to the standard matrix units. This is analogous to the natural representation of the symmetric group; see Clifton [1].

Lemma 16. *For fixed ℓ, the elements $X_{k\ell}$ ($k = 1,\dots,2^n-1$) form a basis of a left ideal in K_n of dimension 2^n-1. For fixed i, the elements X_{ij} ($j = 1,\dots,2^n-1$) form a basis of a right ideal in K_n of dimension 2^n-1.*

Proof. Immediate. ◻

Definition 17. We set $\overline{\epsilon}_{ij} = 0$ if $P_i \cup P_j \neq Z_n$ and $\overline{\epsilon}_{ij} = -(-1)^{n+|P_i|+|P_j|}$ if $P_i \cup P_j = Z_n$, for $i,j = 1,\dots,2^n-1$.

Lemma 18. *The matrix $\mathcal{E}$ is invertible over F: we have $\mathcal{E}^{-1} = (\overline{\epsilon}_{ij})$.*

Proof. The ij-entry of the matrix product $(\epsilon_{ik})(\overline{\epsilon}_{kj})$ is

$$\sum_{k=1}^{2^n-1} \epsilon_{ik}\overline{\epsilon}_{kj} = -(-1)^{n+|P_j|} \sum_{P_i\cap P_k\neq\emptyset,\, P_k\cup P_j = Z_n} (-1)^{|P_k|}$$

We must show that this reduces to δ_{ij}.

Case 1: Suppose $P_i = P_j$. Then $P_k \cup P_j = Z_n$ implies $\overline{P_i} \subseteq P_k$, and $P_i \cap P_k \neq \emptyset$ implies $P_k = \overline{P_i} \cup Q$ for $\emptyset \neq Q \subseteq P_i$. The sum reduces to

$$-(-1)^{n+|P_i|} \sum_{\emptyset\neq Q\subseteq P_i} (-1)^{|\overline{P_i}|+|Q|} = -(-1)^{n+|P_i|} \sum_{\emptyset\neq Q\subseteq P_i} (-1)^{n-|P_i|+|Q|} = -\sum_{\emptyset\neq Q\subseteq P_i} (-1)^{|Q|}$$

$$= -\sum_{\ell=1}^{|P_i|}(-1)^\ell\binom{|P_i|}{\ell} = -(-1) = 1,$$

since $|P_i| \neq 0$.

Case 2: Suppose $P_i \setminus P_j \neq \emptyset$. From $P_k \cup P_j = Z_n$ we get $\overline{P_j} \subseteq P_k$, and this implies $P_i \cap P_k \neq \emptyset$. Thus $P_k = \overline{P_j} \cup Q$ for $Q \subseteq P_j$ (Q may be empty). The sum reduces to

$$-(-1)^{n+|P_j|}\sum_{Q\subseteq P_j}(-1)^{|\overline{P_j}|+|Q|} = -\sum_{Q\subseteq P_j}(-1)^{|Q|} = -\sum_{\ell=0}^{|P_j|}(-1)^\ell\binom{|P_j|}{\ell} = 0,$$

since $|P_j| \neq 0$. (The same argument applies if $P_j \setminus P_i \neq \emptyset$. Since P_i and P_j are nonempty, this includes the case that $P_i \cap P_j = \emptyset$.)

Case 3: Suppose $P_i \subset P_j$ ($P_i \neq P_j$). As before $\overline{P_j} \subseteq P_k$ and so $P_k = \overline{P_j} \cup Q \cup R$ (disjoint union) where $Q \subseteq P_j \setminus P_i$ (Q may be empty) and $\emptyset \neq R \subseteq P_i$ ($R = P_i \cap P_k$). The sum reduces to

$$\begin{aligned}
&-(-1)^{n+|P_j|}\sum_{Q\subseteq P_j\setminus P_i}\sum_{\emptyset\neq R\subseteq P_i}(-1)^{|\overline{P_j}|+|Q|+|R|} = -\sum_{Q\subseteq P_j\setminus P_i}\sum_{\emptyset\neq R\subseteq P_i}(-1)^{|Q|+|R|} \\
&= -\sum_{\emptyset\neq R\subseteq P_i}(-1)^{|R|}\sum_{Q\subseteq P_j\setminus P_i}(-1)^{|Q|} = -\sum_{\emptyset\neq R\subseteq P_i}(-1)^{|R|}\sum_{\ell=0}^{|P_j\setminus P_i|}(-1)^\ell\binom{|P_j\setminus P_i|}{\ell} = 0,
\end{aligned}$$

since $|P_j \setminus P_i| \neq 0$. This completes the proof. □

Definition 19. For $p, q = 1, \ldots, 2^n{-}1$ we define $(2^n{-}1)\times(2^n{-}1)$ matrices by $(a^{pq}_{ij}) = E_{pq}\mathcal{E}^{-1}$ where E_{pq} is the matrix unit; thus $a^{pq}_{ij} = \delta_{pi}\overline{\epsilon}_{qj}$. We define elements of K_n by

$$Y_{pq} = \sum_{i=1}^{2^n-1}\sum_{j=1}^{2^n-1} a^{pq}_{ij}X_{ij} \quad (p, q = 1, \ldots, 2^n{-}1).$$

Theorem 20. *The elements Y_{pq} satisfy the matrix unit equations $Y_{pq}Y_{rs} = \delta_{qr}Y_{ps}$.*

Proof. We have

$$\begin{aligned}
Y_{pq}Y_{rs} &= \Big(\sum_i\sum_j a^{pq}_{ij}X_{ij}\Big)\Big(\sum_k\sum_\ell a^{rs}_{k\ell}X_{k\ell}\Big) = \sum_i\sum_j\sum_k\sum_\ell a^{pq}_{ij}a^{rs}_{k\ell}X_{ij}X_{k\ell} \\
&= \sum_i\sum_j\sum_k\sum_\ell a^{pq}_{ij}a^{rs}_{k\ell}\epsilon_{jk}X_{i\ell} = \sum_i\sum_j\sum_k\sum_\ell \delta_{pi}\overline{\epsilon}_{qj}\delta_{rk}\overline{\epsilon}_{s\ell}\epsilon_{jk}X_{i\ell} \\
&= \sum_i\sum_k\sum_\ell \delta_{pi}\Big(\sum_j\overline{\epsilon}_{qj}\epsilon_{jk}\Big)\delta_{rk}\overline{\epsilon}_{s\ell}X_{i\ell} = \sum_i\sum_k\sum_\ell\delta_{pi}\delta_{qk}\delta_{rk}\overline{\epsilon}_{s\ell}X_{i\ell} \\
&= \sum_i\sum_\ell\delta_{pi}\Big(\sum_k\delta_{qk}\delta_{rk}\Big)\overline{\epsilon}_{s\ell}X_{i\ell} = \sum_i\sum_\ell\delta_{pi}\delta_{qr}\overline{\epsilon}_{s\ell}X_{i\ell} \\
&= \delta_{qr}\sum_i\sum_\ell\delta_{pi}\overline{\epsilon}_{s\ell}X_{i\ell} = \delta_{qr}\sum_i\sum_\ell a^{ps}_{i\ell}X_{i\ell} = \delta_{qr}Y_{ps},
\end{aligned}$$

as required. □

Corollary 21. *The ideal K_n is isomorphic to the matrix algebra $M_{2^n-1}(F)$. (In particular, the elements Y_{pq} are linearly independent over F.)*

Proof. Immediate. □

Corollary 22. *The element of K_n corresponding to the identity matrix is*

$$I_n = \sum_{i=1}^{2^n-1} \sum_{j=1}^{2^n-1} \overline{\epsilon}_{ij} X_{ij}.$$

Proof. The diagonal matrix units Y_{pp} have coefficients $E_{pp}\mathcal{E}^{-1}$ and their sum is $\mathcal{E}^{-1}$. □

Example 23. For $n = 2$, the algebra FB_2 has dimension 16 and the ideal $K_2 \cong M_3(F)$ has dimension 9. We fix this ordering of the subsets: $\{1\}$, $\{2\}$, $\{1, 2\}$; then

$$\mathcal{E} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathcal{E}^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

The ideal K_n has these matrix units:

$$Y_{11} = -\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, Y_{12} = -\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, Y_{13} = -\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
$$Y_{21} = -\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, Y_{22} = -\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, Y_{23} = -\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$
$$Y_{31} = -\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, Y_{32} = -\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, Y_{33} = -\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Example 24. If $\operatorname{char} F \neq 2$ then FB_2 is semisimple and decomposes as an orthogonal direct sum of full matrix algebras: $FB_2 \cong M_3(F) \oplus M_2(F) \oplus F \oplus F \oplus F$. The 2×2 summand has these matrix units:

$$D_{11} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad D_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$
$$D_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad D_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The three copies of F have the following basis elements:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$
$$C = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Example 25. If $\operatorname{char} F = 2$ then FB_2 is not semisimple. Since the coefficients of the basis elements A, D_{ij}, Y_{ij} in Examples 23 and 24 are ± 1, the products of these elements will satisfy the same matrix unit relations in characteristic 2. Thus FB_2 still has a semisimple subalgebra isomorphic to $M_3(F) \oplus M_2(F) \oplus F$. However, in place of B and C we must take the basis elements

$$B' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$C' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Since $B^2 = B$ we have $(2B)^2 = 2(2B)$ and so $(B')^2 = 0$ in characteristic 2, and similarly for C; and we still have $BC = CB = 0$. It follows that when char $F = 2$ the radical of FB_2 has dimension 2 and basis $\{B', C'\}$.

Remark 26. The most important open problem is to generalize Examples 24 and 25 to $n \geq 3$: that is, to determine the complete decomposition of FB_n in all characteristics.

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