QUANTUM OCTONIONS

Murray Bremner

In memoriam Magdy Assem 1954–1996

Abstract. This paper constructs a quantum deformation of the complex Cayley algebra. The method uses the representation theory of $U_q(sl(2))$, the quantized enveloping algebra of the simple complex Lie algebra $sl(2)$. The paper begins by constructing a quantum deformation of the complex quaternion algebra, since this simpler case illustrates all of the necessary steps. As intermediate results, deformations are constructed of $sl(2)$ and the 7-dimensional simple Malcev algebra.

Introduction

Throughout this paper $U_q = U_q(sl(2))$ denotes the quantized enveloping algebra of the simple complex Lie algebra $sl(2)$; details may be found in [K]. Let $V(n)_q$ denote the simple U_q-module with highest weight q^n. The quantum Clebsch-Gordan theorem shows that there exist unique U_q-module homomorphisms from $V(n)_q \otimes V(n)_q$ to $V(n)_q$ and to $V(0)_q \cong \mathbb{C}$, for any positive even integer n.

The case $n = 2$ has been studied in [B]; the maps

$$V(2)_q \otimes V(2)_q \to V(2)_q \quad \text{and} \quad V(2)_q \otimes V(2)_q \to V(0)_q, \tag{1}$$

give q-deformations of the Lie bracket and the Killing form on the adjoint module for $sl(2)$. In §1 of this paper we recall how the complex quaternion algebra may be recovered from these two homomorphisms in the classical ($q = 1$) case. After recalling in §2 some basic information about quantum groups, in §3 we quantize §1 and obtain the quantum analogue of the quaternions; see table (44).

The primary purpose of this paper is to study the case $n = 6$. In this case the maps

$$V(6)_q \otimes V(6)_q \to V(6)_q \quad \text{and} \quad V(6)_q \otimes V(6)_q \to V(0)_q, \tag{2}$$

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give q-deformations of $\mathcal{M}$, the 7-dimensional simple non-Lie Malcev algebra [S1], and the symmetric $sl(2)$-invariant bilinear form on $\mathcal{M}$. In §4 of this paper we recall how the complex octonion algebra may be recovered from these two homomorphisms in the $q = 1$ case. In §5 we quantize §4 and obtain the quantum analogue of the octonions; see table (78).

While this paper was being refereed, I received the preprint "A quantum octonion algebra" by G. Benkart and J. M. Pérez-Izquierdo, which constructs a quantum analogue of the split octonions using the representation theory of $U_q(\mathrm{D}_4)$.

1. Quaternions and $sl(2)$

Let $\mathcal{H}$ denote the complex quaternion algebra. The multiplication table for $\mathcal{H}$ is

(3)

	1	x_1	x_2	x_3
1	1	x_1	x_2	x_3
x_1	x_1	-1	x_3	$-x_2$
x_2	x_2	$-x_3$	-1	x_1
x_3	x_3	x_2	$-x_1$	-1

We define an anticommutative product $[x, y]$ on $\mathcal{H}$ by $[x, y] = xy - yx$, and denote the resulting algebra by $\mathcal{H}^-$. The scalars form an ideal in $\mathcal{H}^-$, and the quotient algebra $\mathcal{H}^-/\mathbb{C}1$ is isomorphic to the simple complex Lie algebra $sl(2)$. Setting $X_i = \frac{1}{2}x_i$ the structure constants for $\mathcal{H}^-/\mathbb{C}1$ are

(4)

	X_1	X_2	X_3
X_1	0	X_3	$-X_2$
X_2	$-X_3$	0	X_1
X_3	X_2	$-X_1$	0

The Lie algebra $sl(2)$ has basis E, H, F and commutation relations

(5)

	E	H	F
E	0	$-2E$	H
H	$2E$	0	$-2F$
F	$-H$	$2F$	0

An isomorphism between $\mathcal{H}^-/\mathbb{C}1$ and $sl(2)$ is given by

(6) $$E = X_2 + iX_3, \quad H = 2iX_1, \quad F = -X_2 + iX_3;$$

the inverse is

$$X_1 = -\frac{i}{2}H, \quad X_2 = \frac{1}{2}(E-F), \quad X_3 = -\frac{i}{2}(E+F). \tag{7}$$

The Killing form on $sl(2)$ is

$$\begin{array}{c|ccc} & E & H & F \\ \hline E & 0 & 0 & 4 \\ H & 0 & 8 & 0 \\ F & 4 & 0 & 0 \end{array} \tag{8}$$

The quaternion algebra $\mathcal{H}$ can be recovered from $sl(2)$ and the Killing form. Before explaining this we recall some of the representation theory of $sl(2)$. (References are [H], Section 7, and [FH], Lecture 11.)

Let $V(n)$ denote the (unique) simple highest weight $sl(2)$-module with highest weight n, where n is any nonnegative integer. This module has basis $v_0, \ldots, v_n$, and the $sl(2)$-action is

$$Ev_i = (n+1-i)v_{i-1}, \quad Hv_i = (n-2i)v_i, \quad Fv_i = (i+1)v_{i+1}. \tag{9}$$

The Clebsch-Gordan theorem gives the direct sum decomposition

$$V(m) \otimes V(n) \cong V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(m-n), \tag{10}$$

whenever $m \geq n$. A highest weight vector of weight $m+n-2p$ in $V(m) \otimes V(n)$ is

$$w_0^{m+n-2p} = \sum_{i=0}^{p} (-1)^i \frac{\binom{m-p+i}{i}}{\binom{n}{i}} v_i^{(m)} \otimes v_{p-i}^{(n)}, \tag{11}$$

where $v_i^{(m)}$ and $v_j^{(n)}$ are the basis vectors for $V(m)$ and $V(n)$ respectively. We also have

$$S^2 V(n) \cong V(2n) \oplus V(2n-4) \oplus \cdots, \quad \Lambda^2 V(n) \cong V(2n-2) \oplus V(2n-6) \oplus \cdots, \tag{12}$$

where S^2 and Λ^2 denote the symmetric and exterior squares. On $V(n)$ we introduce the invariant scalar product

$$(v_i, v_j) = \delta_{ij} \binom{n}{i}. \tag{13}$$

This extends to the tensor product $V(m)\otimes V(n)$ in the usual way:

$$(v_i\otimes v_j, v_k\otimes v_\ell) = (v_i, v_k)(v_j, v_\ell). \tag{14}$$

With respect to this scalar product the basis vectors $v_i\otimes v_j$ are orthogonal, as are the submodules on the right side of (10).

From (12) we see that there are unique (up to scalar) $sl(2)$-module homomorphisms $K\colon S^2V(2) \to V(0)$ and $L\colon \Lambda^2V(2) \to V(2)$. Since $V(2)$ is the adjoint module and $V(0)$ is 1-dimensional, it is clear that K is the Killing form and L is the Lie commutator on $sl(2)$. Formula (11) gives a highest weight vector w_0 for the $V(2)$ summand in $V(2)\otimes V(2)$, and (9) shows how to compute w_1 and w_2 (recall that F acts as $F\otimes 1 + 1\otimes F$):

$$w_0 = v_0\otimes v_1 - v_1\otimes v_0, \quad w_1 = 2v_0\otimes v_2 - 2v_2\otimes v_0, \quad w_2 = v_1\otimes v_2 - v_2\otimes v_1. \tag{15}$$

Formulas (13–14) show how to compute the orthogonal projection of $v_i\otimes v_j$ on $V(2)$; identifying v_i and w_i the results are

$$\begin{array}{c|ccc} & v_0 & v_1 & v_2 \\ \hline v_0 & 0 & \frac{1}{2}v_0 & \frac{1}{4}v_1 \\ v_1 & -\frac{1}{2}v_0 & 0 & \frac{1}{2}v_2 \\ v_2 & -\frac{1}{4}v_1 & -\frac{1}{2}v_2 & 0 \end{array} \tag{16}$$

Now set $a_1 = -4$, and choose any $a_0, a_2 \in \mathbb{C}$ such that $a_0a_2 = -16$. Then using the basis $V_i = a_iv_i$ we obtain the matrix representing L:

$$\begin{array}{c|ccc} & V_0 & V_1 & V_2 \\ \hline V_0 & 0 & -2V_0 & V_1 \\ V_1 & 2V_0 & 0 & -2V_2 \\ V_2 & -V_1 & 2V_2 & 0 \end{array} \tag{17}$$

which is the same as (5) with V_0, V_1, V_2 replacing E, H, F.

Similarly we find a basis vector for the $V(0)$ summand of $V(2)\otimes V(2)$:

$$z = v_0\otimes v_2 - \tfrac{1}{2}v_1\otimes v_1 + v_2\otimes v_0. \tag{18}$$

The projection of $v_i\otimes v_j$ on $V(0)$ is

$$\begin{array}{c|ccc} & v_0 & v_1 & v_2 \\ \hline v_0 & 0 & 0 & \frac{1}{3}z \\ v_1 & 0 & -\frac{2}{3}z & 0 \\ v_2 & \frac{1}{3}z & 0 & 0 \end{array} \tag{19}$$

Defining $Z = -\frac{4}{3}z$ we get the matrix representing K

(20)

	V_0	V_1	V_2
V_0	0	0	$4Z$
V_1	0	$8Z$	0
V_2	$4Z$	0	0

which is the same as (8), if we identify Z with 1 (which amounts to choosing an isomorphism $V(0) \cong \mathbb{C}$).

Given constants $\kappa, \lambda \in \mathbb{C}$ we can combine K and L to obtain an $sl(2)$-invariant (not necessarily associative) algebra structure on $V(0) \oplus V(2)$:

$$(a + x)(b + y) = \big(ab + \kappa K(x, y)\big) + \big(ay + bx + \lambda L(x, y)\big). \tag{21}$$

Here $a, b \in \mathbb{C} \cong V(0)$, and $x, y \in V(2)$. To determine κ and λ we do the following calculations:

$$\begin{aligned} x_1^2 &= 4X_1^2 = 4\Big(-\frac{i}{2}H\Big)^2 = -V_1^2 = -\Big(\kappa K(V_1, V_1) + \lambda L(V_1, V_1)\Big) = -8\kappa, \\ x_1 x_2 &= 4X_1 X_2 = 4\Big(-\frac{i}{2}H\Big)\Big(\frac{1}{2}(E - F)\Big) = i\Big(V_1 V_2 - V_1 V_0\Big) \\ &= i\Big(\kappa K(V_1, V_2) + \lambda L(V_1, V_2) - \kappa K(V_1, V_0) - \lambda L(V_1, V_0)\Big) \\ &= -2i\lambda(V_0 + V_2) = -2i\lambda(E + F) = -2i\lambda(2iX_3) = 4\lambda X_3 = 2\lambda x_3. \end{aligned}$$

This shows that (21) makes $V(0) \oplus V(2)$ into an algebra isomorphic to $\mathcal{H}$ if we set $\kappa = \frac{1}{8}$ and $\lambda = \frac{1}{2}$.

2. The Quantum Group $U_q(sl(2))$

Before constructing the quantum analogue of the quaternions, we need to recall some basic results about the quantized enveloping algebra $U_q = U_q(sl(2))$; here q is a complex number, $q \neq 0$, $q^2 \neq 1$. We refer to [K] for details.

Introduce the following notation:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]! = [n][n-1]\cdots[1], \qquad \begin{bmatrix} n \\ i \end{bmatrix} = \frac{[n]!}{[i]![n-i]!}. \tag{22}$$

These quantities are called respectively q-integers, q-factorials, and q-binomial coefficients. The notation $[n]_{q^k}$ denotes the result of replacing q by q^k in $[n]$.

As an associative algebra with 1, U_q has generators E, F, K, K^{-1}, and relations $KK^{-1} = K^{-1}K = 1$ together with

$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2} F, \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \tag{23}$$

U_q becomes a bialgebra once we define the coproduct Δ; since this is an algebra homomorphism it suffices to give it for the generators:

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \quad \Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1. \tag{24}$$

With the antipode S defined by

$$S(K^{\pm 1}) = K^{\mp 1}, \qquad S(E) = -EK^{-1}, \qquad S(F) = -KF, \tag{25}$$

U_q becomes a Hopf algebra.

For the representation theory of U_q we assume that q is not a root of unity. Let V_q denote any finite dimensional U_q-module. We call a vector $v \in V_q$, $v \neq 0$, a highest weight vector if $Ev = 0$ and $Kv = \lambda v$ for some scalar λ. We call V_q a highest weight module if it is generated by a highest weight vector. Any simple finite dimensional U_q-module is a highest weight module with highest weight $\lambda = \epsilon q^n$ where $\epsilon = \pm 1$. Any two simple finite dimensional highest weight modules with the same highest weight are isomorphic; we denote such a U_q-module by $V_q(\epsilon, n)$, and we have $\dim V_q(\epsilon, n) = n + 1$.

From now on we consider only the modules $V(n)_q = V_q(1, n)$. Let v_0 be a highest weight vector in $V(n)_q$. Then $V(n)_q$ has basis v_i $(0 \leq i \leq n)$, and the U_q-action is

$$Kv_i = q^{n-2i} v_i, \qquad Ev_i = [n - i + 1] v_{i-1}, \qquad Fv_i = [i + 1] v_{i+1}. \tag{26}$$

The quantum Clebsch-Gordan Theorem ([K], VII.7) states that

$$V(m)_q \otimes V(n)_q \cong V(m+n)_q \oplus V(m+n-2)_q \oplus \cdots \oplus V(m-n)_q, \tag{27}$$

whenever $m \geq n$. This isomorphism is the same as that which holds in the non-quantum case (10); what is different is the expression we obtain for a highest weight vector $w^{(m+n-2p)}$ $(0 \leq p \leq n)$ of weight q^{m+n-2p} in $V(m)_q \otimes V(n)_q$:

$$w_0^{(m+n-2p)} = \sum_{i=0}^{p} (-1)^i \frac{[m-p+i]![n-i]!}{[m-p]![n]!} q^{-i(m-2p+i+1)} v_i^{(m)} \otimes v_{p-i}^{(n)}, \tag{28}$$

where $v_i^{(m)}$ and $v_j^{(n)}$ are the basis vectors for $V(m)_q$ and $V(n)_q$.

On $V(n)_q$ we introduce the U_q-invariant scalar product

$$(v_i, v_j) = \delta_{ij} q^{-i(n-i-1)} \begin{bmatrix} n \\ i \end{bmatrix}. \tag{29}$$

This extends to the tensor product $V(m)_q \otimes V(n)_q$ in the usual way:

$$(v_i \otimes v_j, v_k \otimes v_\ell) = (v_i, v_k)(v_j, v_\ell). \tag{30}$$

With respect to this scalar product the basis vectors $v_i^{(m)} \otimes v_j^{(n)}$ are orthogonal, as are the submodules on the right side of (27).

3. Quantizing the Quaternions

If we quantize formula (21) we see that the quantum quaternion algebra can be defined by the formula

$$(a + x)(b + y) = \big(ab + \kappa_q K_q(x, y)\big) + \big(ay + bx + \lambda_q L_q(x, y)\big). \tag{31}$$

Here $a, b \in \mathbb{C} \cong V(0)_q$ and $x, y \in V(2)_q$; furthermore

$$K_q \colon V(2)_q \otimes V(2)_q \to V(0)_q \quad \text{and} \quad L_q \colon V(2)_q \otimes V(2)_q \to V(2)_q, \tag{32}$$

are the homomorphisms corresponding to the Killing form and the Lie bracket on the adjoint $sl(2)$-module $V(2)$. (Recall that for any simple complex Lie algebra $\mathfrak{g}$, every finite dimensional $U_q(\mathfrak{g})$-module is semisimple; see Theorem 10.1.14 of [CP], Theorem 5.17 of [Jan], or Theorem 6.2.2 of [L]. Furthermore, the $U_q(\mathfrak{g})$-module V_q has the same formal character as the $U(\mathfrak{g})$-module V, given by the classical Weyl character formula; see

Corollary 10.1.15 of [CP], Theorem 5.15 of [Jan], or Theorem 33.1.3 of [L]. These results imply that the decomposition of any tensor product of $U_q(\mathfrak{g})$-modules is the same as the decomposition of the corresponding tensor product of $U(\mathfrak{g})$-modules.)

We can compute K_q and L_q explicitly using the quantum Clebsch-Gordan theorem. It is not so clear how to quantize the scalars κ and λ; any choice of rational functions κ_q and λ_q in $\mathbb{C}(q)$ satisfying $\kappa_1 = \frac{1}{8}$ and $\lambda_1 = \frac{1}{2}$ will preserve the U_q-invariance of the resulting algebra structure on $V(2)_q \oplus V(0)_q$. More about this later; see (42–43).

We first recall the formulas for K_q and L_q presented in [B]. Using formulas (28) and (24) we find that a basis of the $V(2)_q$ summand of $V(2)_q \otimes V(2)_q$ is

$$\begin{aligned} w_0 &= v_0 \otimes v_1 - \frac{1}{q^2} v_1 \otimes v_0, \\ w_1 &= \frac{q^2+1}{q^3} v_0 \otimes v_2 + \frac{q^2-1}{q^2} v_1 \otimes v_1 - \frac{q^2+1}{q^3} v_2 \otimes v_0, \\ w_2 &= v_1 \otimes v_2 - \frac{1}{q^2} v_2 \otimes v_1. \end{aligned} \tag{33}$$

The square lengths of these vectors can be computed using formulas (29–30):

$$(w_0, w_0) = \frac{(q^4+1)[2]}{q^4}, \qquad (w_1, w_1) = \frac{(q^4+1)[2]^2}{q^4}, \qquad (w_2, w_2) = \frac{(q^4+1)[2]}{q^2} \tag{34}$$

The homomorphism L_q is the projection from $V(2)_q \otimes V(2)_q$ onto the $V(2)_q$ summand. The usual formula for orthogonal projections gives the matrix for L_q:

$$\begin{array}{cccc} & v_0 & v_1 & v_2 \\ v_0 & 0 & \frac{q^4}{q^4+1} w_0 & \frac{q^4}{[2](q^4+1)} w_1 \\ v_1 & \frac{-q^2}{q^4+1} w_0 & \frac{q^2(q^2-1)}{q^4+1} w_1 & \frac{q^4}{q^4+1} w_2 \\ v_2 & \frac{-q^4}{[2](q^4+1)} w_1 & \frac{-q^2}{q^4+1} w_2 & 0 \end{array} \tag{35}$$

The homomorphism K_q is the projection from $V(2)_q \otimes V(2)_q$ onto the $V(0)_q$ summand. A basis vector for this summand, and its square length, are

$$z = v_0 \otimes v_2 - \frac{1}{[2]} v_1 \otimes v_1 + \frac{1}{q^2} v_2 \otimes v_0, \qquad (z, z) = [3], \tag{36}$$

and so K_q is given by the matrix

$$\begin{array}{cccc} & v_0 & v_1 & v_2 \\ v_0 & 0 & 0 & \frac{q^2}{[3]} z \\ v_1 & 0 & \frac{-[2]}{[3]} z & 0 \\ v_2 & \frac{1}{[3]} z & 0 & 0 \end{array} \tag{37}$$

Now set

$$a_0 = \frac{(q^4+1)[2]}{q^4}, \qquad a_1 = a_2 = \frac{-(q^4+1)[2]}{q^3}, \tag{38}$$

and define new basis vectors $V_i = a_i v_i$. Then if we identify v_i and w_i, the matrix for L_q becomes

$$\begin{array}{cccc} & V_0 & V_1 & V_2 \\ V_0 & 0 & -q[2]\,V_0 & V_1 \\ V_1 & \frac{1}{q}[2]\,V_0 & \left(\frac{1}{q^2}-q^2\right)V_1 & -q[2]\,V_2 \\ V_2 & -V_1 & \frac{1}{q}[2]\,V_2 & 0 \end{array} \tag{39}$$

Notice that the entries of this matrix lie in the Laurent polynomial ring $\mathbb{Z}[q,q^{-1}]$, and that the matrix is invariant under the operation $q \to \frac{1}{q}$ followed by the negative transpose. When $q=1$ this matrix specializes to table (5), if we identify V_0, V_1, V_2 with E, H, F. The same specialization can be obtained by choosing any $a_0, a_1, a_2 \in \mathbb{C}(q)$ so that $q=1$ gives $a_1=-4$ and $a_0 a_2 = -16$.

We further define $Z = bz$ where

$$b = \frac{-(q^4+1)^2}{q^6[3]}. \tag{40}$$

This is different from the corresponding result in [B], because here we use the scaling factor b instead of $-\frac{4}{3}$; that is, we are quantizing the constant $-\frac{4}{3}$. The matrix for K_q then becomes

$$\begin{array}{cccc} & V_0 & V_1 & V_2 \\ V_0 & 0 & 0 & q[2]^2\,Z \\ V_1 & 0 & [2]^3\,Z & 0 \\ V_2 & \frac{1}{q}[2]^2\,Z & 0 & 0 \end{array} \tag{41}$$

Again, the entries are in $\mathbb{Z}[q,q^{-1}]$. When $q=1$ this matrix specializes to table (8), if we set $Z=1$; this amounts to choosing an isomorphism $V(0)_q \cong \mathbb{C}$. The same specialization can be obtained by choosing any $b \in \mathbb{C}(q)$ so that $q=1$ gives $b = -\frac{4}{3}$.

The next step is to determine the quantization of the constants $\kappa = \frac{1}{8}$ and $\lambda = \frac{1}{2}$. We consider bases of $V(0)_q \oplus V(2)_q$ corresponding to the previously introduced bases of

$V(0) \oplus V(2)$. For example, the basis Z, x_{1q}, x_{2q}, x_{3q} of $V(0)_q \oplus V(2)_q$ corresponds to the basis 1, x_1, x_2, x_3 of $V(0) \oplus V(2)$. The relation $X_i = \frac{1}{2} x_i$ becomes $X_{iq} = \frac{1}{[2]} x_{iq}$. We have

$$
\begin{aligned}
x_{1q}^2 &= [2]^2 X_{1q}^2 \quad \text{since } x_{1q} = [2] X_{1q} \\
&= -\frac{1}{4}[2]^2 H_q^2 \quad \text{using formula (6) also for the } q\text{-basis} \\
&= -\frac{1}{4}[2]^2 V_1^2 \quad \text{identifying } H_q \text{ with } V_1 \\
&= -\frac{1}{4}[2]^2 \kappa_q K_q(V_1, V_1) - \frac{1}{4}[2]^2 \lambda_q L_q(V_1, V_1) \quad \text{using formula (31)} \\
&= -\frac{1}{4}[2]^5 \kappa_q Z - \frac{1}{4}[2]^2 \lambda_q \left(\frac{1}{q^2} - q^2 \right) V_1 \quad \text{using formulas (41) and (39)} \\
&= -\frac{1}{4}[2]^5 \kappa_q Z + 2i \left(q - \frac{1}{q} \right) \lambda_q x_{1q}.
\end{aligned}
$$

This suggests that the natural quantization of the constant $\kappa = \frac{1}{8}$ is

$$
\kappa_q = \frac{1}{[2]^3}. \tag{42}
$$

Similarly we compute

$$
\begin{aligned}
x_{1q} x_{2q} &= [2]^2 X_{1q} X_{2q} = \frac{1}{4}[2]^2 i H(F - E) = \frac{1}{4}[2]^2 i (V_1 V_2 - V_1 V_0) \\
&= \frac{1}{4}[2]^3 i \lambda_q (-q) V_2 - \frac{1}{4}[2]^3 i \lambda_q \frac{1}{q} V_0 \\
&= -\frac{1}{4}[2]^3 i \lambda_q \left(q V_2 + \frac{1}{q} V_0 \right) = -\frac{1}{4}[2]^3 i \lambda_q \left(q F + \frac{1}{q} E \right) \\
&= -\frac{1}{4}[2]^3 i \lambda_q \left(q(-X_{2q} + i X_{3q}) + \frac{1}{q}(X_{2q} + i X_{3q}) \right) \\
&= \frac{1}{4}[2]^3 i \lambda_q \left(q - \frac{1}{q} \right) X_{2q} + \frac{1}{4}[2]^4 \lambda_q X_{3q} \\
&= \frac{1}{4}[2]^2 i \lambda_q \left(q - \frac{1}{q} \right) x_{2q} + \frac{1}{4}[2]^3 \lambda_q x_{3q}
\end{aligned}
$$

which suggests that the natural quantization of $\lambda = \frac{1}{2}$ is

$$
\lambda_q = \frac{1}{[2]}. \tag{43}
$$

Using these values for κ_q and λ_q, formula (31) gives this quantization of table (3):

(44)

	1	x_{1q}	x_{2q}	x_{3q}
1	1	x_{1q}	x_{2q}	x_{3q}
x_{1q}	x_{1q}	$-\frac{1}{4}[2]^2+\frac{1}{2}i\theta[2]x_{1q}$	$\frac{1}{4}i\theta[2]x_{2q}+\frac{1}{4}[2]^2x_{3q}$	$-\frac{1}{4}[2]^2x_{2q}+\frac{1}{4}i\theta[2]x_{3q}$
x_{2q}	x_{2q}	$\frac{1}{4}i\theta[2]x_{2q}-\frac{1}{4}[2]^2x_{3q}$	$-\frac{1}{4}[2]^2$	$-\frac{1}{4}i\theta[2]+x_{1q}$
x_{3q}	x_{3q}	$\frac{1}{4}[2]^2x_{2q}+\frac{1}{4}i\theta[2]x_{3q}$	$\frac{1}{4}i\theta[2]-x_{1q}$	$-\frac{1}{4}[2]^2$

Here we have used the notation

$$\theta = q - \frac{1}{q}.$$

Matrix (44) is the multiplication table for the quantum quaternions $\mathcal{H}_q$, which specializes to table (3) when $q = 1$. Note that all the coefficients in (44) are Laurent polynomials in q with coefficients in $\frac{1}{4}\mathbb{Z}$.

4. Cayley and Malcev Algebras

Let $\mathcal{C}$ denote the complex Cayley algebra (i.e. the octonion algebra). The multiplication table for $\mathcal{C}$ is

(45)

	1	x_1	x_2	x_3	x_4	x_5	x_6	x_7
1	1	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	x_1	-1	x_3	$-x_2$	x_5	$-x_4$	$-x_7$	x_6
x_2	x_2	$-x_3$	-1	x_1	x_6	x_7	$-x_4$	$-x_5$
x_3	x_3	x_2	$-x_1$	-1	x_7	$-x_6$	x_5	$-x_4$
x_4	x_4	$-x_5$	$-x_6$	$-x_7$	-1	x_1	x_2	x_3
x_5	x_5	x_4	$-x_7$	x_6	$-x_1$	-1	$-x_3$	x_2
x_6	x_6	x_7	x_4	$-x_5$	$-x_2$	x_3	-1	$-x_1$
x_7	x_7	$-x_6$	x_5	x_4	$-x_3$	$-x_2$	x_1	-1

(This is taken from [Jac], section 7.6, with $c_1 = c_2 = c_3 = -1$.) We define an anticommutative product $[x, y]$ on $\mathcal{C}$ by $[x, y] = xy - yx$, and denote the resulting algebra by $\mathcal{C}^-$. The scalars form an ideal in $\mathcal{C}^-$, and the quotient algebra $\mathcal{M} = \mathcal{C}^-/\mathbb{C}1$ is a simple 7-dimensional anticommutative algebra. Setting $X_i = \frac{1}{2}x_i$ the multiplication table for $\mathcal{M}$

is

$$
(46) \qquad
\begin{array}{c|ccccccc}
 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 \\
\hline
X_1 & 0 & X_3 & -X_2 & X_5 & -X_4 & -X_7 & X_6 \\
X_2 & -X_3 & 0 & X_1 & X_6 & X_7 & -X_4 & -X_5 \\
X_3 & X_2 & -X_1 & 0 & X_7 & -X_6 & X_5 & -X_4 \\
X_4 & -X_5 & -X_6 & -X_7 & 0 & X_1 & X_2 & X_3 \\
X_5 & X_4 & -X_7 & X_6 & -X_1 & 0 & -X_3 & X_2 \\
X_6 & X_7 & X_4 & -X_5 & -X_2 & X_3 & 0 & -X_1 \\
X_7 & -X_6 & X_5 & X_4 & -X_3 & -X_2 & X_1 & 0
\end{array}
$$

This algebra satisfies the Malcev identity

$$
(47) \qquad [[x,y,z],x] = [x,y,[x,z]] \quad \text{where} \quad [x,y,z] = [[x,y],z] + [[y,z],x] + [[z,x],y].
$$

Anticommutative algebras which satisfy this identity are called Malcev algebras [S1]. Since any Lie algebra satisfies $[x,y,z]=0$, it is clear that any Lie algebra is a Malcev algebra. It has been shown [S2] that over the complex numbers any simple finite dimensional Malcev algebra is either a Lie algebra or isomorphic to $\mathcal{M}$.

The characteristic polynomial of left multiplication by the general element

$$
(48) \qquad X = \sum_{i=1}^{7} c_i X_i \quad \text{is} \quad p_X(t) = t\left(t^2 + \sum_{i=1}^{7} c_i^2\right)^3.
$$

This shows that if we take $\mathbb{R}$ as the base field, the multiplication table above defines a non-split real form of $\mathcal{M}$, since no non-trivial left multiplication is diagonalizable.

In the remainder of this paper $\mathcal{M}$ will be called "the Malcev algebra" for short; strictly speaking $\mathcal{M}$ should be called "the simple finite dimensional non-Lie Malcev algebra".

A different way to define $\mathcal{M}$ is by using the representation theory of the simple Lie algebra $sl(2)$. (See [O], Chapter 6, for a similar approach to $\mathcal{M}$ using the terminology of theoretical physics.) In the case $m = n = 6$, formula (12) gives the $sl(2)$-module isomorphisms

$$
(49) \qquad S^2V(6) \cong V(12) \oplus V(8) \oplus V(4) \oplus V(0), \quad \Lambda^2 V(6) \cong V(10) \oplus V(6) \oplus V(2).
$$

The first isomorphism gives an $sl(2)$-module homomorphism $K\colon S^2V(6) \to V(0)$, that is, an $sl(2)$-invariant symmetric bilinear form on $V(6)$. The second isomorphism gives an $sl(2)$-module homomorphism $L\colon \Lambda^2(V(6)) \to V(6)$, that is, an $sl(2)$-invariant anticommutative algebra structure on $V(6)$. The structure constants of this algebra can easily be computed. From (11) we get a highest weight vector w_0 of weight 6 in $V(6)\otimes V(6)$, and the other basis vectors w_k $(1 \le k \le 6)$ are obtained by using (9):

$$\begin{aligned}
w_0 &= v_0\otimes v_3 - \tfrac{2}{3}v_1\otimes v_2 + \tfrac{2}{3}v_2\otimes v_1 - v_3\otimes v_0,\\
w_1 &= 4v_0\otimes v_4 - v_1\otimes v_3 + v_3\otimes v_1 - 4v_4\otimes v_0,\\
w_2 &= 10v_0\otimes v_5 - v_2\otimes v_3 + v_3\otimes v_2 - 10v_5\otimes v_0,\\
w_3 &= 20v_0\otimes v_6 + \tfrac{10}{3}v_1\otimes v_5 - \tfrac{4}{3}v_2\otimes v_4 + \tfrac{4}{3}v_4\otimes v_2 - \tfrac{10}{3}v_5\otimes v_1 + 20v_6\otimes v_0,\\
w_4 &= 10v_1\otimes v_6 - v_3\otimes v_4 + v_4\otimes v_3 - 10v_6\otimes v_1,\\
w_5 &= 4v_2\otimes v_6 - v_3\otimes v_5 + v_5\otimes v_3 - 4v_6\otimes v_2,\\
w_6 &= v_3\otimes v_6 - \tfrac{2}{3}v_4\otimes v_5 + \tfrac{2}{3}v_5\otimes v_4 - v_6\otimes v_3.
\end{aligned} \tag{50}$$

The square lengths of the w_k are computed using (13–14):

(51)

k	0	1	2	3	4	5	6
(w_k, w_k)	120	720	1800	2400	1800	720	120

The structure constants S_{ij}, giving the component of $v_i\otimes v_j$ in the $V(6)$ summand of $V(6)\otimes V(6)$, are defined by the equation $v_i\otimes v_j \to S_{ij}w_{i+j-3}$ and given by the orthogonal projection formula

$$S_{ij} = \frac{C_{ij}(v_i, v_i)(v_j, v_j)}{(w_{i+j-3}, w_{i+j-3})}, \tag{52}$$

where C_{ij} is the coefficient of $v_i\otimes v_j$ in w_{i+j-3}. The matrix with entries S_{ij} is

$$\begin{pmatrix}
0 & 0 & 0 & 1/6 & 1/12 & 1/30 & 1/120\\
0 & 0 & -1/2 & -1/6 & 0 & 1/20 & 1/30\\
0 & 1/2 & 0 & -1/6 & -1/8 & 0 & 1/12\\
-1/6 & 1/6 & 1/6 & 0 & -1/6 & -1/6 & 1/6\\
-1/12 & 0 & 1/8 & 1/6 & 0 & -1/2 & 0\\
-1/30 & -1/20 & 0 & 1/6 & 1/2 & 0 & 0\\
-1/120 & -1/30 & -1/12 & -1/6 & 0 & 0 & 0
\end{pmatrix} \tag{53}$$

Now choose any non-zero scalars a_0 and a_1, set

$$a_2 = -2(2!)\frac{a_0}{a_1}, \quad a_3 = 2(3!), \quad a_4 = (4!)\frac{a_1}{a_0}, \quad a_5 = 2(5!)\frac{1}{a_1}, \quad a_6 = -2(6!)\frac{1}{a_0}, \tag{54}$$

and define $V_i = a_i w_i$ for $0 \le i \le 6$. The structure constants for $V(6)$ with respect to the V_i basis are

$$V_i V_j = T_{ij} V_{i+j-3} \quad \text{where} \quad T_{ij} = \frac{a_i a_j S_{ij}}{a_{i+j-3}}, \tag{55}$$

with $a_k = 1$ and $V_k = 0$ unless $0 \le k \le 6$. This gives the multiplication table:

(56)

	V_0	V_1	V_2	V_3	V_4	V_5	V_6
V_0	0	0	0	$2V_0$	$2V_1$	$-2V_2$	$-V_3$
V_1	0	0	$2V_0$	$-2V_1$	0	V_3	$-2V_4$
V_2	0	$-2V_0$	0	$-2V_2$	V_3	0	$2V_5$
V_3	$-2V_0$	$2V_1$	$2V_2$	0	$-2V_4$	$-2V_5$	$2V_6$
V_4	$-2V_1$	0	$-V_3$	$2V_4$	0	$2V_6$	0
V_5	$2V_2$	$-V_3$	0	$2V_5$	$-2V_6$	0	0
V_6	V_3	$2V_4$	$-2V_5$	$-2V_6$	0	0	0

The triples

$$(H, E, F) = \quad (V_3, V_2, V_4), \quad (V_3, V_1, V_5), \quad (V_3, V_6, V_0) \tag{57}$$

span subalgebras of $\mathcal{M}$ isomorphic to $sl(2)$.

If we restrict the scalars to $\mathbb{R}$, we obtain a split real form of $\mathcal{M}$. The characteristic polynomial of left multiplication by the general element

$$V = \sum_{i=0}^{6} c_i V_i \quad \text{is} \quad p_V(t) = t \left(t^2 - 4 \sum_{i=0}^{3} c_i c_{6-i} \right)^3, \tag{58}$$

so left multiplication by V is diagonalizable over $\mathbb{R}$ if and only if $c_0 c_6 + c_1 c_5 + c_2 c_4 + c_3^2 \ge 0$.

The two bases X_i $(1 \le i \le 7)$ and V_i $(0 \le i \le 6)$ span algebras which are isomorphic over $\mathbb{C}$. If we define the scalars b_{ij} by

$$V_i = \sum_{j=1}^{7} b_{ij} X_j, \quad \text{for } 0 \le i \le 6, \tag{59}$$

then the change of basis matrix $B = (b_{ij})$ and its inverse are

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & -i \\ 0 & 0 & 0 & 1 & i & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -i \end{pmatrix}, \tag{60}$$

$$B^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -i & 0 & -i & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -i & 0 & 0 & 0 & -i & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 & 0 & 0 & i \end{pmatrix}. \tag{61}$$

(Here $i = \sqrt{-1}$.)

The full derivation algebra of $\mathcal{M} = V(6)$ is a simple Lie algebra of type G_2. As a G_2-module $V(6)$ has highest weight ω_2 (where α_2 is the short simple root) and we have

$$S^2(V(\omega_2)_7) \cong V(2\omega_2)_{27} \oplus V(0)_1, \qquad \Lambda^2(V(\omega_2)_7) \cong V(\omega_1)_{14} \oplus V(\omega_2)_7; \tag{62}$$

the subscripts indicate the dimensions of the modules. These decompositions were computed using SimpLie [SL]. For more information on derivation algebras of nonassociative algebras constructed using representations of $sl(2)$ see [D]; however that paper uses the classical terminology of binary forms rather than the modern terminology of $sl(2)$-modules.

To compute the homomorphism $K: V(6)\otimes V(6) \to V(0) \cong \mathbb{C}$ we first use (11) to find a basis vector for the $V(0)$ summand of $V(6)\otimes V(6)$:

$$z = v_0\otimes v_6 - \tfrac{1}{6}v_1\otimes v_5 + \tfrac{1}{15}v_2\otimes v_4 - \tfrac{1}{20}v_3\otimes v_3 + \tfrac{1}{15}v_2\otimes v_4 - \tfrac{1}{6}v_1\otimes v_5 + v_0\otimes v_6. \tag{63}$$

Since $(z,z) = 7$ we find that

(64)

i	0	1	2	3	4	5	6
$K(v_i, v_{6-i})$	$\frac{1}{7}z$	$-\frac{6}{7}z$	$\frac{15}{7}z$	$-\frac{20}{7}z$	$\frac{15}{7}z$	$-\frac{6}{7}z$	$\frac{1}{7}z$

and $K(v_i, v_j) = 0$ unless $i + j = 6$. If we set $Z = -\frac{360}{7}z$, use the basis V_i, and choose the isomorphism $V(0) \cong \mathbb{C}$ which identifies Z with 1, we obtain

(65)

i	0	1	2	3	4	5	6
$K(V_i, V_{6-i})$	4	4	4	8	4	4	4

The same results can be obtained by computing the generalized Killing form $\frac{1}{3}\mathrm{tr}(\mathrm{ad}_{V_i}\mathrm{ad}_{V_j})$.

Now we make $V(0) \oplus V(6)$ into an algebra isomorphic to $\mathcal{C}$ using formula (21) (where now $x, y \in V(6)$) and setting $\kappa = \frac{1}{8}$ and $\lambda = \frac{1}{2}$. (As in the calculations following (21), these values are obtained from the equations $x_1^2 = -1$ and $x_1 x_2 = x_3$.)

5. Quantizing the Octonions

We now reinterpret formula (31) with $x, y \in V(6)_q$. We need to compute explicitly the homomorphisms $K_q\colon V(6)_q \otimes V(6)_q \to V(0)_q$ and $L_q\colon V(6)_q \otimes V(6)_q \to V(6)_q$.

We first determine a basis w_k $(0 \le k \le 6)$ for the summand $V(6)_q$ in $V(6)_q \otimes V(6)_q$. We write C_{ij} for the coefficient of $v_i \otimes v_j$ in w_{i+j-3}. For the highest weight vector w_0 formula (28) gives

$$C_{03} = 1, \qquad C_{12} = \frac{-(q^4+1)}{q^8+q^4+1}, \qquad C_{21} = \frac{q^4+1}{q^4(q^8+q^4+1)}, \qquad C_{30} = \frac{-1}{q^{12}}. \tag{66}$$

We can determine the other C_{ij} using formulas (24) and (26). For example,

$$\begin{aligned} w_1 &= F w_0 = (K^{-1} \otimes F + F \otimes 1) w_0 \\ &= C_{03}(K^{-1} v_0 \otimes F v_3 + F v_0 \otimes v_3) + C_{12}(K^{-1} v_1 \otimes F v_2 + F v_1 \otimes v_2) + \\ &\qquad C_{21}(K^{-1} v_2 \otimes F v_1 + F v_2 \otimes v_1) + C_{30}(K^{-1} v_3 \otimes F v_0 + F v_3 \otimes v_0) \\ &= C_{03}(q^{-6} v_0 \otimes [4] v_4 + v_1 \otimes v_3) + C_{12}(q^{-4} v_1 \otimes [3] v_3 + [2] v_2 \otimes v_2) + \\ &\qquad C_{21}(q^{-2} v_2 \otimes [2] v_2 + [3] v_3 \otimes v_1) + C_{30}(v_3 \otimes v_1 + [4] v_4 \otimes v_0) \\ &= C_{04} v_0 \otimes v_4 + C_{13} v_1 \otimes v_3 + C_{22} v_2 \otimes v_2 + C_{31} v_3 \otimes v_1 + C_{40} v_4 \otimes v_0, \end{aligned}$$

where the coefficients are

$$\begin{aligned} C_{04} &= C_{03} q^{-6} [4] = \frac{(q^2+1)(q^4+1)}{q^9}, \\ C_{13} &= C_{03} + C_{12} q^{-4} [3] = \frac{q^{10} - q^8 + q^6 - q^4 - 1}{q^6(q^4 - q^2 + 1)}, \\ C_{22} &= C_{12}[2] + C_{21} q^{-2} [2] = \frac{-(q^8 - 1)}{q^7(q^4 - q^2 + 1)}, \\ C_{31} &= C_{21}[3] + C_{30} = \frac{q^{10} + q^6 - q^4 + q^2 - 1}{q^{12}(q^4 - q^2 + 1)}, \\ C_{40} &= C_{30}[4] = \frac{-(q^2+1)(q^4+1)}{q^{15}}. \end{aligned}$$

For the basis vectors w_2 and w_3 the coefficients are

$$C_{05} = \frac{C_{04} q^{-6} [5]}{[2]} = \frac{(q^4+1)(q^8+q^6+q^4+q^2+1)}{q^{18}}$$

$$C_{14} = \frac{C_{04} + C_{13}q^{-4}[4]}{[2]} = \frac{(q^2-1)(q^4+1)(q^8+q^6+q^4+q^2+1)}{q^{12}(q^4-q^2+1)}$$

$$C_{23} = \frac{C_{13}[2] + C_{22}q^{-2}[3]}{[2]} = \frac{q^{14}-q^{12}-q^8-q^6+1}{q^{10}(q^4-q^2+1)}$$

$$C_{32} = \frac{C_{22}[3] + C_{31}[2]}{[2]} = \frac{-C_{23}}{q^2}, \qquad C_{41} = \frac{C_{31}[4] + C_{40}q^2}{[2]} = \frac{C_{14}}{q^2}$$

$$C_{50} = \frac{C_{40}[5]}{[2]} = -C_{05}$$

$$C_{06} = \frac{C_{05}q^{-6}[6]}{[3]} = \frac{(q^2+1)(q^4+1)(q^4-q^2+1)(q^8+q^6+q^4+q^2+1)}{q^{27}}$$

$$C_{15} = \frac{C_{05} + C_{14}q^{-4}[5]}{[3]} = \frac{(q^4+1)(q^8+q^6+q^4+q^2+1)(q^{10}+q^6-q^4+q^2-1)}{q^{18}(q^8+q^4+1)}$$

$$C_{24} = \frac{C_{14}[2] + C_{23}q^{-2}[4]}{[3]} = \frac{(q^2+1)(q^4+1)(q^{14}-q^8-q^6-q^2+1)}{q^{13}(q^8+q^4+1)}$$

$$C_{33} = \frac{C_{23}[3] + C_{32}[3]}{[3]} = \frac{(q^4-1)(q^4+q^2+1)(q^8-3q^6+3q^4-3q^2+1)}{q^{12}(q^4-q^2+1)}$$

$$C_{42} = \frac{C_{32}[4] + C_{41}q^2[2]}{[3]} = \frac{-(q^2+1)(q^4+1)(q^{14}-q^{12}-q^8-q^6+1)}{q^{13}(q^8+q^4+1)}$$

$$C_{51} = \frac{C_{41}[5] + C_{50}q^4}{[3]} = \frac{(q^4+1)(q^8+q^6+q^4+q^2+1)(q^{10}-q^8+q^6-q^4-1)}{q^{16}(q^8+q^4+1)}$$

$$C_{60} = \frac{C_{50}[6]}{[3]} = -q^6 C_{06}$$

For w_4, w_5 and w_6 we have the relation

$$C_{ij} = C_{6-j,6-i} \text{ for } 7 \le i+j \le 9. \tag{67}$$

Since the basis $v_i \otimes v_j$ of $V(6) \otimes V(6)$ is orthogonal, it is easy to work out the component of $v_i \otimes v_j$ along w_k. Weight considerations show that this component is zero unless $k = i+j-3$, so the projection of $v_i \otimes v_j$ onto the submodule spanned by the w_k is a multiple S'_{ij} of w_{i+j-3}. We have

$$S'_{ij} = \frac{(v_i \otimes v_j, w_{i+j-3})}{(w_{i+j-3}, w_{i+j-3})} = \frac{C_{ij}(v_i,v_i)(v_j,v_j)}{(w_{i+j-3}, w_{i+j-3})} = \frac{C_{ij}q^{-i(5-i)}\begin{bmatrix}6\\i\end{bmatrix} q^{-j(5-j)}\begin{bmatrix}6\\j\end{bmatrix}}{(w_{i+j-3}, w_{i+j-3})}, \tag{68}$$

where the square lengths are

$$(w_{i+j-3}, w_{i+j-3}) = \sum_{\substack{r+s=i+j\\0\le r,s\le 6}} C_{rs}^2 (v_r,v_r)(v_s,v_s) = \sum_{\substack{r+s=i+j\\0\le r,s\le 6}} C_{rs}^2 q^{-r(5-r)}\begin{bmatrix}6\\r\end{bmatrix} q^{-s(5-s)}\begin{bmatrix}6\\s\end{bmatrix}. \tag{69}$$

If we identify v_i and w_i, the coefficients S'_{ij} are the structure constants of the nonassociative algebra structure on $V(6)_q$.

Now set

$$I_q = \frac{q^{28} - q^{26} + q^{24} + q^{20} + q^{16} + q^{12} + q^8 + q^4 - q^2 + 1}{q^8(q^4 + q^2 + 1)(q^2 + 1)}. \tag{70}$$

Note that when $q = 1$ we have $I_q = 1$. Define

$$\alpha_0 = (q^2+1)(q^4+q^3+q^2+q+1)^3, \qquad \alpha_1 = (q^2+1)(q^4+q^3+q^2+q+1)^2, \tag{71}$$

and the following q-analogues of the expressions in formula (54):

$$\begin{aligned} a_0 &= \frac{I_q\alpha_0}{q}, \quad a_1 = \frac{I_q\alpha_1}{q^3}, \quad a_2 = \frac{-[2][2]!I_q\alpha_0}{q^5\alpha_1}, \quad a_3 = \frac{[2][3]!I_q}{q^9}, \\ a_4 &= \frac{[4]!I_q\alpha_1}{q^{14}\alpha_0}, \quad a_5 = \frac{[2][5]!I_q}{q^{19}\alpha_1}, \quad a_6 = \frac{-[2][6]!I_q}{q^{26}\alpha_0}, \end{aligned} \tag{72}$$

Introduce scaled basis vectors $V_i = a_i w_i$ for $0 \le i \le 6$. With respect to the V_i the structure constants S_{ij} are as follows:

$$S_{00} = 0, \quad S_{01} = 0, \quad S_{02} = 0, \quad S_{03} = q^9 + q^3, \quad S_{04} = q^5 + q, \quad S_{05} = -q^2 - \frac{1}{q^2},$$
$$S_{06} = -\frac{1}{q} + \frac{1}{q^3} - \frac{1}{q^5},$$
$$S_{10} = 0, \quad S_{11} = 0, \quad S_{12} = q^5 + \frac{1}{q}, \quad S_{13} = q^9 - q - \frac{1}{q} - \frac{1}{q^3}, \quad S_{14} = -q^6 - q^2 + 1 + \frac{1}{q^4},$$
$$S_{15} = q^4 + 1 - \frac{1}{q^2} + \frac{1}{q^4} - \frac{1}{q^6}, \quad S_{16} = -q^3 - \frac{1}{q^3},$$
$$S_{20} = 0, \quad S_{21} = -q - \frac{1}{q^5}, \quad S_{22} = -q^6 - q^4 + \frac{1}{q^4} + \frac{1}{q^6},$$
$$S_{23} = q^9 - q^5 - q^3 - 2q - \frac{1}{q} + \frac{1}{q^5} + \frac{1}{q^7}, \quad S_{24} = -q^7 + q + \frac{1}{q} + \frac{1}{q^5} - \frac{1}{q^7},$$
$$S_{25} = -q^7 - q^5 + \frac{1}{q^3} + \frac{1}{q^5}, \quad S_{26} = q^6 + 1,$$
$$S_{30} = -\frac{1}{q^3} - \frac{1}{q^9}, \quad S_{31} = q^3 + q + \frac{1}{q} - \frac{1}{q^9}, \quad S_{32} = -q^7 - q^5 + q + \frac{2}{q} + \frac{1}{q^3} + \frac{1}{q^5} - \frac{1}{q^9},$$
$$S_{33} = q^9 - q^7 - 2q^5 - q^3 - q + \frac{1}{q} + \frac{1}{q^3} + \frac{2}{q^5} + \frac{1}{q^7} - \frac{1}{q^9},$$
$$S_{34} = q^9 - q^5 - q^3 - 2q - \frac{1}{q} + \frac{1}{q^5} + \frac{1}{q^7}, \quad S_{35} = q^9 - q - \frac{1}{q} - \frac{1}{q^3}, \quad S_{36} = q^9 + q^3,$$

$$
\begin{aligned}
&S_{40} = -\frac{1}{q} - \frac{1}{q^5}, \quad S_{41} = -q^4 - 1 + \frac{1}{q^2} + \frac{1}{q^6}, \quad S_{42} = q^7 - q^5 - q - \frac{1}{q} + \frac{1}{q^7},\\
&\quad S_{43} = -q^7 - q^5 + q + \frac{2}{q} + \frac{1}{q^3} + \frac{1}{q^5} - \frac{1}{q^9}, \quad S_{44} = -q^5 - q + \frac{1}{q} + \frac{1}{q^5}, \quad S_{45} = q^4 + 1,\\
&\quad S_{46} = 0,\\
&S_{50} = q^2 + \frac{1}{q^2}, \quad S_{51} = q^6 - q^4 + q^2 - 1 - \frac{1}{q^4}, \quad S_{52} = -q^5 - q^3 + \frac{1}{q^5} + \frac{1}{q^7},\\
&\quad S_{53} = q^3 + q + \frac{1}{q} - \frac{1}{q^9}, \quad S_{54} = -1 - \frac{1}{q^4}, \quad S_{55} = 0, \quad S_{56} = 0,\\
&S_{60} = q^5 - q^3 + q, \quad S_{61} = q^3 + \frac{1}{q^3}, \quad S_{62} = -1 - \frac{1}{q^6}, \quad S_{63} = -\frac{1}{q^3} - \frac{1}{q^9}, \quad S_{64} = 0,\\
&\quad S_{65} = 0, \quad S_{66} = 0.
\end{aligned}
$$

These rational functions lie in the Laurent polynomial ring $\mathbb{Z}[q, q^{-1}]$ and satisfy the symmetry relation $S_{ji}(q) = -S_{ij}(q^{-1})$. They define a q-deformation of the Malcev algebra $\mathcal{M}$, and specialize when $q = 1$ to the structure constants displayed in table (56).

We also need to compute K_q explicitly. Formulas (28–30) give a basis vector for the $V(0)_q$ summand of $V(6)_q \otimes V(6)_q$ and its square length:

$$
z = \sum_{i=0}^{6} (-1)^i \, \frac{q^{i(5-i)}}{\left[{6 \atop i} \right]} \, v_i \otimes v_{6-i}, \qquad (z, z) = [7]. \tag{73}
$$

Projecting $V(6)_q \otimes V(6)_q$ onto $V(0)_q \cong \mathbb{C}z$ gives

$$
K_q(v_i, v_{6-i}) = \frac{(-1)^i q^{(i-1)(i-6)} \left[{6 \atop i} \right]}{[7]} \, z, \tag{74}
$$

and $K_q(v_i, v_j) = 0$ unless $i + j = 6$. Now scale the basis for $V(0)_q$, by defining

$$
Z = \frac{-(q^4 - q^2 + 1)(q^8 - q^6 + q^4 - q^2 + 1)[2]_{q^2}[2]_{q^4}^2[3]_{q^3}^2[5]_{q^2}}{q^{24}[7]} \, z. \tag{75}
$$

In terms of the basis V_i, and choosing the isomorphism $V(0)_q \cong \mathbb{C}$ which identifies Z with 1, we have

(76)

i	0	1	2	3	4	5	6
$K_q(V_i, V_{6-i})$	$q^3[2]^2$	$q^2[2]^2$	$q[2]^2$	$[2]^3$	$\frac{1}{q}[2]^2$	$\frac{1}{q^2}[2]^2$	$\frac{1}{q^3}[2]^2$

This provides a q-deformation of table (65), the symmetric $sl(2)$-invariant bilinear form on $\mathcal{M}$.

Computing x_{1q}^2 and $x_{1q}x_{2q}$ using K_q and L_q suggests that the natural choices for κ_q and λ_q are

$$\kappa_q = \frac{1}{[2]^3}, \qquad \lambda_q = \frac{1}{[2]}. \tag{77}$$

The calculations omitted here are very similar to those preceding (42) and (43). When $q = 1$ we have $\kappa_1 = \frac{1}{8}$ and $\lambda_1 = \frac{1}{2}$.

Using these values for κ_q and λ_q we compute the multiplication table for the quantum octonions. This table specializes to table (45) when $q = 1$.

(78)

$$
\begin{aligned}
x_{1q}x_{1q} &= -\frac{1}{4}[2]^2 - \frac{1}{2}i\frac{q^{18} - q^{16} - 2q^{14} - q^{12} - q^{10} + q^8 + q^6 + 2q^4 + q^2 - 1}{q^9}x_{1q} \\
x_{1q}x_{2q} &= -\frac{1}{4}i\frac{q^{18} - q^{16} - 2q^{14} - q^{12} - q^{10} + q^8 + q^6 + 2q^4 + q^2 - 1}{q^9}x_{2q} \\
&\quad - \frac{1}{4}\frac{q^{18} + q^{16} - q^{12} - 3q^{10} - 3q^8 - q^6 + q^2 + 1}{q^9}x_{3q} \\
x_{1q}x_{3q} &= \frac{1}{4}\frac{q^{18} + q^{16} - q^{12} - 3q^{10} - 3q^8 - q^6 + q^2 + 1}{q^9}x_{2q} \\
&\quad - \frac{1}{4}i\frac{q^{18} - q^{16} - 2q^{14} - q^{12} - q^{10} + q^8 + q^6 + 2q^4 + q^2 - 1}{q^9}x_{3q} \\
x_{1q}x_{4q} &= -\frac{1}{4}i\frac{q^{18} + q^{12} - q^6 - 1}{q^9}x_{4q} - \frac{1}{4}\frac{q^{18} - q^{12} - 2q^{10} - 2q^8 - q^6 + 1}{q^9}x_{5q} \\
x_{1q}x_{5q} &= \frac{1}{4}\frac{q^{18} - q^{12} - 2q^{10} - 2q^8 - q^6 + 1}{q^9}x_{4q} - \frac{1}{4}i\frac{q^{18} + q^{12} - q^6 - 1}{q^9}x_{5q} \\
x_{1q}x_{6q} &= -\frac{1}{4}i\frac{q^{18} + q^{12} - q^6 - 1}{q^9}x_{6q} - \frac{1}{4}\frac{q^{18} + q^{12} + q^6 + 1}{q^9}x_{7q} \\
x_{1q}x_{7q} &= \frac{1}{4}\frac{q^{18} + q^{12} + q^6 + 1}{q^9}x_{6q} - \frac{1}{4}i\frac{q^{18} + q^{12} - q^6 - 1}{q^9}x_{7q} \\
x_{2q}x_{1q} &= -\frac{1}{4}i\frac{q^{18} - q^{16} - 2q^{14} - q^{12} - q^{10} + q^8 + q^6 + 2q^4 + q^2 - 1}{q^9}x_{2q} \\
&\quad + \frac{1}{4}\frac{q^{18} + q^{16} - q^{12} - 3q^{10} - 3q^8 - q^6 + q^2 + 1}{q^9}x_{3q} \\
x_{2q}x_{2q} &= -\frac{1}{4}[2]^2 + \frac{1}{2}i\frac{q^{10} - 1}{q^5}x_{1q} - \frac{1}{4}\frac{q^{12} - q^{11} + q^{10} - q^7 + q^5 - q^2 + q - 1}{q^6}x_{4q} \\
&\quad - \frac{1}{4}i\frac{q^{12} + q^{11} + q^{10} + q^7 - q^5 - q^2 - q - 1}{q^6}x_{5q} \\
x_{2q}x_{3q} &= -\frac{1}{4}i[2]\theta - \frac{1}{2}\frac{2q^{14} - q^{12} - 2q^8 - 2q^6 - q^2 + 2}{q^7}x_{1q}
\end{aligned}
$$

$$+\frac{1}{4}i\frac{q^{12}+q^{11}+q^{10}+q^{7}-q^{5}-q^{2}-q-1}{q^{6}}x_{4q}$$
$$-\frac{1}{4}\frac{q^{12}-q^{11}+q^{10}-q^{7}+q^{5}-q^{2}+q-1}{q^{6}}x_{5q}$$

$$x_{2q}x_{4q}=-\frac{1}{4}\frac{q^{13}+q^{11}-q^{10}-q^{6}+q^{4}-q^{3}-q+1}{q^{6}}x_{2q}$$
$$+\frac{1}{4}i\frac{q^{13}+q^{11}+q^{10}+q^{6}-q^{4}-q^{3}-q-1}{q^{6}}x_{3q}+\frac{1}{4}\frac{q^{9}+q^{6}+q^{5}+1}{q^{5}}x_{6q}$$
$$-\frac{1}{4}i\frac{q^{9}-q^{6}+q^{5}-1}{q^{5}}x_{7q}$$

$$x_{2q}x_{5q}=-\frac{1}{4}i\frac{q^{13}+q^{11}+q^{10}+q^{6}-q^{4}-q^{3}-q-1}{q^{6}}x_{2q}$$
$$-\frac{1}{4}\frac{q^{13}+q^{11}-q^{10}-q^{6}+q^{4}-q^{3}-q+1}{q^{6}}x_{3q}+\frac{1}{4}i\frac{q^{9}-q^{6}+q^{5}-1}{q^{5}}x_{6q}$$
$$+\frac{1}{4}\frac{q^{9}+q^{6}+q^{5}+1}{q^{5}}x_{7q}$$

$$x_{2q}x_{6q}=-\frac{1}{4}\frac{q^{11}+q^{5}+q^{4}+1}{q^{5}}x_{4q}+\frac{1}{4}i\frac{q^{11}+q^{5}-q^{4}-1}{q^{5}}x_{5q}$$

$$x_{2q}x_{7q}=-\frac{1}{4}i\frac{q^{11}+q^{5}-q^{4}-1}{q^{5}}x_{4q}-\frac{1}{4}\frac{q^{11}+q^{5}+q^{4}+1}{q^{5}}x_{5q}$$

$$x_{3q}x_{1q}=-\frac{1}{4}\frac{q^{18}+q^{16}-q^{12}-3q^{10}-3q^{8}-q^{6}+q^{2}+1}{q^{9}}x_{2q}$$
$$-\frac{1}{4}i\frac{q^{18}-q^{16}-2q^{14}-q^{12}-q^{10}+q^{8}+q^{6}+2q^{4}+q^{2}-1}{q^{9}}x_{3q}$$

$$x_{3q}x_{2q}=\frac{1}{4}i[2]\theta+\frac{1}{2}\frac{2q^{14}-q^{12}-2q^{8}-2q^{6}-q^{2}+2}{q^{7}}x_{1q}$$
$$+\frac{1}{4}i\frac{q^{12}+q^{11}+q^{10}+q^{7}-q^{5}-q^{2}-q-1}{q^{6}}x_{4q}$$
$$-\frac{1}{4}\frac{q^{12}-q^{11}+q^{10}-q^{7}+q^{5}-q^{2}+q-1}{q^{6}}x_{5q}$$

$$x_{3q}x_{3q}=-\frac{1}{4}[2]^{2}+\frac{1}{2}i\frac{q^{10}-1}{q^{5}}x_{1q}$$
$$+\frac{1}{4}\frac{q^{12}-q^{11}+q^{10}-q^{7}+q^{5}-q^{2}+q-1}{q^{6}}x_{4q}$$
$$+\frac{1}{4}i\frac{q^{12}+q^{11}+q^{10}+q^{7}-q^{5}-q^{2}-q-1}{q^{6}}x_{5q}$$

$$x_{3q}x_{4q}=\frac{1}{4}i\frac{q^{13}+q^{11}+q^{10}+q^{6}-q^{4}-q^{3}-q-1}{q^{6}}x_{2q}$$

$$
\begin{aligned}
&+\frac{1}{4}\frac{q^{13}+q^{11}-q^{10}-q^{6}+q^{4}-q^{3}-q+1}{q^{6}}x_{3q}+\frac{1}{4}i\frac{q^{9}-q^{6}+q^{5}-1}{q^{5}}x_{6q}\\
&+\frac{1}{4}\frac{q^{9}+q^{6}+q^{5}+1}{q^{5}}x_{7q}\\
x_{3q}x_{5q} &= -\frac{1}{4}\frac{q^{13}+q^{11}-q^{10}-q^{6}+q^{4}-q^{3}-q+1}{q^{6}}x_{2q}\\
&+\frac{1}{4}i\frac{q^{13}+q^{11}+q^{10}+q^{6}-q^{4}-q^{3}-q-1}{q^{6}}x_{3q}-\frac{1}{4}\frac{q^{9}+q^{6}+q^{5}+1}{q^{5}}x_{6q}\\
&+\frac{1}{4}i\frac{q^{9}-q^{6}+q^{5}-1}{q^{5}}x_{7q}\\
x_{3q}x_{6q} &= \frac{1}{4}i\frac{q^{11}+q^{5}-q^{4}-1}{q^{5}}x_{4q}+\frac{1}{4}\frac{q^{11}+q^{5}+q^{4}+1}{q^{5}}x_{5q}\\
x_{3q}x_{7q} &= -\frac{1}{4}\frac{q^{11}+q^{5}+q^{4}+1}{q^{5}}x_{4q}+\frac{1}{4}i\frac{q^{11}+q^{5}-q^{4}-1}{q^{5}}x_{5q}\\
x_{4q}x_{1q} &= -\frac{1}{4}i\frac{q^{18}+q^{12}-q^{6}-1}{q^{9}}x_{4q}+\frac{1}{4}\frac{q^{18}-q^{12}-2q^{10}-2q^{8}-q^{6}+1}{q^{9}}x_{5q}\\
x_{4q}x_{2q} &= \frac{1}{4}\frac{q^{13}-q^{12}-q^{10}+q^{9}-q^{7}-q^{3}+q^{2}+1}{q^{7}}x_{2q}\\
&+\frac{1}{4}i\frac{q^{13}+q^{12}+q^{10}+q^{9}-q^{7}-q^{3}-q^{2}-1}{q^{7}}x_{3q}-\frac{1}{4}\frac{q^{9}+q^{4}+q^{3}+1}{q^{4}}x_{6q}\\
&-\frac{1}{4}i\frac{q^{9}-q^{4}+q^{3}-1}{q^{4}}x_{7q}\\
x_{4q}x_{3q} &= \frac{1}{4}i\frac{q^{13}+q^{12}+q^{10}+q^{9}-q^{7}-q^{3}-q^{2}-1}{q^{7}}x_{2q}\\
&-\frac{1}{4}\frac{q^{13}-q^{12}-q^{10}+q^{9}-q^{7}-q^{3}+q^{2}+1}{q^{7}}x_{3q}+\frac{1}{4}i\frac{q^{9}-q^{4}+q^{3}-1}{q^{4}}x_{6q}\\
&-\frac{1}{4}\frac{q^{9}+q^{4}+q^{3}+1}{q^{4}}x_{7q}\\
x_{4q}x_{4q} &= -\frac{1}{4}[2][2]_{q^2}-\frac{1}{2}i\frac{q^{12}+q^{8}-q^{4}-1}{q^{6}}x_{1q}\\
x_{4q}x_{5q} &= -\frac{1}{4}i[2]\theta_{q^2}-\frac{1}{2}\frac{q^{12}-2q^{10}+q^{8}-2q^{6}+q^{4}-2q^{2}+1}{q^{6}}x_{1q}\\
x_{4q}x_{6q} &= \frac{1}{4}\frac{q^{6}+q^{5}+q+1}{q^{3}}x_{2q}-\frac{1}{4}i\frac{q^{6}-q^{5}-q+1}{q^{3}}x_{3q}\\
x_{4q}x_{7q} &= \frac{1}{4}i\frac{q^{6}-q^{5}-q+1}{q^{3}}x_{2q}+\frac{1}{4}\frac{q^{6}+q^{5}+q+1}{q^{3}}x_{3q}\\
x_{5q}x_{1q} &= -\frac{1}{4}\frac{q^{18}-q^{12}-2q^{10}-2q^{8}-q^{6}+1}{q^{9}}x_{4q}-\frac{1}{4}i\frac{q^{18}+q^{12}-q^{6}-1}{q^{9}}x_{5q}
\end{aligned}
$$

$$x_{5q}x_{2q} = -\frac{1}{4}i\frac{q^{13}+q^{12}+q^{10}+q^9-q^7-q^3-q^2-1}{q^7}x_{2q} + \frac{1}{4}\frac{q^{13}-q^{12}-q^{10}+q^9-q^7-q^3+q^2+1}{q^7}x_{3q} + \frac{1}{4}i\frac{q^9-q^4+q^3-1}{q^4}x_{6q} - \frac{1}{4}\frac{q^9+q^4+q^3+1}{q^4}x_{7q}$$

$$x_{5q}x_{3q} = \frac{1}{4}\frac{q^{13}-q^{12}-q^{10}+q^9-q^7-q^3+q^2+1}{q^7}x_{2q} + \frac{1}{4}i\frac{q^{13}+q^{12}+q^{10}+q^9-q^7-q^3-q^2-1}{q^7}x_{3q} + \frac{1}{4}\frac{q^9+q^4+q^3+1}{q^4}x_{6q} + \frac{1}{4}i\frac{q^9-q^4+q^3-1}{q^4}x_{7q}$$

$$x_{5q}x_{4q} = \frac{1}{4}i[2]\theta_{q^2} + \frac{1}{2}\frac{q^{12}-2q^{10}+q^8-2q^6+q^4-2q^2+1}{q^6}x_{1q}$$

$$x_{5q}x_{5q} = -\frac{1}{4}[2][2]_{q^2} - \frac{1}{2}i\frac{q^{12}+q^8-q^4-1}{q^6}x_{1q}$$

$$x_{5q}x_{6q} = -\frac{1}{4}i\frac{q^6-q^5-q+1}{q^3}x_{2q} - \frac{1}{4}\frac{q^6+q^5+q+1}{q^3}x_{3q}$$

$$x_{5q}x_{7q} = \frac{1}{4}\frac{q^6+q^5+q+1}{q^3}x_{2q} - \frac{1}{4}i\frac{q^6-q^5-q+1}{q^3}x_{3q}$$

$$x_{6q}x_{1q} = -\frac{1}{4}i\frac{q^{18}+q^{12}-q^6-1}{q^9}x_{6q} + \frac{1}{4}\frac{q^{18}+q^{12}+q^6+1}{q^9}x_{7q}$$

$$x_{6q}x_{2q} = \frac{1}{4}\frac{q^{11}+q^7+q^6+1}{q^6}x_{4q} + \frac{1}{4}i\frac{q^{11}+q^7-q^6-1}{q^6}x_{5q}$$

$$x_{6q}x_{3q} = \frac{1}{4}i\frac{q^{11}+q^7-q^6-1}{q^6}x_{4q} - \frac{1}{4}\frac{q^{11}+q^7+q^6+1}{q^6}x_{5q}$$

$$x_{6q}x_{4q} = -\frac{1}{4}\frac{q^6+q^5+q+1}{q^3}x_{2q} + \frac{1}{4}i\frac{q^6-q^5-q+1}{q^3}x_{3q}$$

$$x_{6q}x_{5q} = \frac{1}{4}i\frac{q^6-q^5-q+1}{q^3}x_{2q} + \frac{1}{4}\frac{q^6+q^5+q+1}{q^3}x_{3q}$$

$$x_{6q}x_{6q} = -\frac{1}{4}[2][2]_{q^3} - \frac{1}{2}i\frac{q^{10}-q^8+q^6-q^4+q^2-1}{q^5}x_{1q}$$

$$x_{6q}x_{7q} = -\frac{1}{4}i[2]\theta_{q^3} - \frac{1}{2}\frac{q^{10}-q^8+q^6+q^4-q^2+1}{q^5}x_{1q}$$

$$x_{7q}x_{1q} = -\frac{1}{4}\frac{q^{18}+q^{12}+q^6+1}{q^9}x_{6q} - \frac{1}{4}i\frac{q^{18}+q^{12}-q^6-1}{q^9}x_{7q}$$

$$x_{7q}x_{2q} = -\frac{1}{4}i\frac{q^{11}+q^7-q^6-1}{q^6}x_{4q} + \frac{1}{4}\frac{q^{11}+q^7+q^6+1}{q^6}x_{5q}$$

$$x_{7q}x_{3q} = \frac{1}{4}\frac{q^{11}+q^7+q^6+1}{q^6}x_{4q} + \frac{1}{4}i\frac{q^{11}+q^7-q^6-1}{q^6}x_{5q}$$
$$x_{7q}x_{4q} = -\frac{1}{4}i\frac{q^6-q^5-q+1}{q^3}x_{2q} - \frac{1}{4}\frac{q^6+q^5+q+1}{q^3}x_{3q}$$
$$x_{7q}x_{5q} = -\frac{1}{4}\frac{q^6+q^5+q+1}{q^3}x_{2q} + \frac{1}{4}i\frac{q^6-q^5-q+1}{q^3}x_{3q}$$
$$x_{7q}x_{6q} = \frac{1}{4}i[2]\theta_{q^3} + \frac{1}{2}\frac{q^{10}-q^8+q^6+q^4-q^2+1}{q^5}x_{1q}$$
$$x_{7q}x_{7q} = -\frac{1}{4}[2][2]_{q^3} - \frac{1}{2}i\frac{q^{10}-q^8+q^6-q^4+q^2-1}{q^5}x_{1q}$$

In this table we use the notation

$$\theta_{q^n} = q^n - \frac{1}{q^n}.$$

It is easy to check, by considering the eigenvalues of the diagonalizable element (V_1 in $\mathcal{H}_q$ and V_3 in $\mathcal{C}_q$), that there is no subalgebra of $\mathcal{C}_q$ isomorphic to $\mathcal{H}_q$ for all q; this is possible only for special values of q.

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References

[B] M. Bremner, *Quantum deformations of simple Lie algebras*, Canad. Math. Bull. **40** (1997), 143-148.
[CP] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge, 1994.
[D] J. Dixmier, *Certaines algèbres nonassociatives simples définies par la transvection des formes binaires*, J. Reine Angew. Math. **346** (1984), 110-128.
[FH] W. Fulton and J. Harris, *Representation Theory: A First Course*, Springer-Verlag, New York, 1991.
[H] J. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 1972.
[Jac] N. Jacobson, *Basic Algebra I*, W.H. Freeman, San Francisco, 1974.
[Jan] J. C. Jantzen, *Lectures on Quantum Groups*, American Mathematical Society, Providence, 1995.

[K] C. Kassel, *Quantum Groups*, Springer-Verlag, New York, 1995.
[L] G. Lusztig, *Introduction to Quantum Groups*, Birkhäuser, Boston, 1993.
[O] S. Okubo, *Introduction to Octonion and Other Non-Associative Algebras in Physics*, Cambridge University Press, 1995.
[S1] A.A. Sagle, *Malcev algebras*, Trans. Amer. Math. Soc. **101** (1961), 426-458.
[S2] A.A. Sagle, *Simple Malcev algebras over fields of characteristic zero*, Pacific J. Math. **12** (1962), 1057-1078.
[SL] *SimpLie, Macintosh Software for Representations of Simple Lie Algebras*, Centre de Recherches Mathématiques, Université de Montréal.

Department of Mathematics and Statistics, University of Saskatchewan, McLean Hall, 106 Wiggins Road, Saskatoon, SK, S7N 5E6, Canada
E-mail address: bremner@math.usask.ca