ON THE DEFINITION OF QUASI-JORDAN ALGEBRA

MURRAY R. BREMNER

Abstract. Velásquez and Felipe recently introduced quasi-Jordan algebras based on the product $a \triangleleft b = \frac{1}{2}(a \dashv b \,+\, b \vdash a)$ in an associative dialgebra with operations $\dashv$ and $\vdash$. We determine the polynomial identities of degree ≤ 4 satisfied by this product. In addition to right commutativity and the right quasi-Jordan identity, we obtain a new associator-derivation identity.

Loday [4, 5] defined an **(associative) dialgebra** to be a vector space with two bilinear operations $a \dashv b$ and $a \vdash b$ satisfying these polynomial identities:

$$(a \dashv b) \dashv c = a \dashv (b \dashv c), \quad (a \vdash b) \vdash c = a \vdash (b \vdash c), \quad (a \vdash b) \dashv c = a \vdash (b \dashv c),$$
$$(a \vdash b) \vdash c = (a \dashv b) \vdash c, \quad a \dashv (b \dashv c) = a \dashv (b \vdash c).$$

Let $w = \overline{a_1 \cdots a_n}$ $(a_1, \ldots, a_n \in X)$ be a dialgebra monomial over X; the bar indicates some placement of parentheses and choice of operations. If $w \in X$ then the **center** $c(w) = w$; otherwise $c(w_1 \dashv w_2) = c(w_1)$ and $c(w_1 \vdash w_2) = c(w_2)$.

Lemma 1. [5] *If $w = \overline{a_1 \cdots a_n}$ and $c(w) = a_k$ then*

$$w = (a_1 \vdash \cdots \vdash a_{k-1}) \vdash a_k \dashv (a_{k+1} \dashv \cdots \dashv a_n).$$

We write $w = a_1 \cdots a_{k-1}\widehat{a}_k a_{k+1} \cdots a_n$ for this **normal form** of w.

Lemma 2. [5] *The monomials $a_1 \cdots a_{k-1}\widehat{a}_k a_{k+1} \cdots a_n$ $(1 \leq k \leq n)$ for $a_1, \ldots, a_n \in X$ form a basis of the free dialgebra on X.*

Velásquez and Felipe [8, 9] introduced the **(right) quasi-Jordan product** in a dialgebra over a field of characteristic $\neq 2$:

$$a \triangleleft b = \tfrac{1}{2}(a \dashv b \,+\, b \vdash a).$$

We omit the operation symbol and the scalar $\frac{1}{2}$, and write $ab = a \dashv b \,+\, b \vdash a$.

Lemma 3. [8] *The quasi-Jordan product satisfies the right commutative identity:*

$$a(bc) = a(cb).$$

Expanding the left side gives a result which is symmetric in b and c:

$$a(bc) = a \dashv (b \dashv c + c \vdash b) + (b \dashv c + c \vdash b) \vdash a$$
$$= a \dashv (b \dashv c) + a \dashv (c \vdash b) + (b \dashv c) \vdash a + (c \vdash b) \vdash a = \widehat{a}bc + \widehat{a}cb + bc\widehat{a} + cb\widehat{a}.$$

Lemma 4. [8] *The quasi-Jordan product satisfies the right quasi-Jordan identity:*

$$(ba)a^2 = (ba^2)a.$$

Expanding both sides gives the same result:

$$(ba)a^2 = (b \dashv a + a \vdash b) \dashv (a \dashv a + a \vdash a) + (a \dashv a + a \vdash a) \vdash (b \dashv a + a \vdash b)$$
$$= (b \dashv a) \dashv (a \dashv a) + (b \dashv a) \dashv (a \vdash a) + (a \vdash b) \dashv (a \dashv a) + (a \vdash b) \dashv (a \vdash a)$$

$$
\begin{aligned}
&\quad + (a \dashv a) \vdash (b \dashv a) + (a \dashv a) \vdash (a \vdash b) + (a \vdash a) \vdash (b \dashv a) + (a \vdash a) \vdash (a \vdash b) \\
&= 2\widehat{b}aaa + 2a\widehat{b}aa + 2aa\widehat{b}a + 2aaa\widehat{b} \\
&= (b \dashv (a \dashv a)) \dashv a + (b \dashv (a \vdash a)) \dashv a + ((a \dashv a) \vdash b) \dashv a + ((a \vdash a) \vdash b) \dashv a \\
&\quad + a \vdash (b \dashv (a \dashv a)) + a \vdash (b \dashv (a \vdash a)) + a \vdash ((a \dashv a) \vdash b) + a \vdash ((a \vdash a) \vdash b) \\
&= (b \dashv (a \dashv a + a \vdash a) + (a \dashv a + a \vdash a) \vdash b) \dashv a \\
&\quad + a \vdash (b \dashv (a \dashv a + a \vdash a) + (a \dashv a + a \vdash a) \vdash b) \\
&= (ba^2)a.
\end{aligned}
$$

Remark 5. The (right) quasi-Jordan product does not satisfy $a^2(ab) = a(a^2b)$:

$$
\begin{aligned}
a^2(ab) &= (a \dashv a + a \vdash a) \dashv (a \dashv b + b \vdash a) + (a \dashv b + b \vdash a) \vdash (a \dashv a + a \vdash a) \\
&= (a \dashv a) \dashv (a \dashv b) + (a \dashv a) \dashv (b \vdash a) + (a \vdash a) \dashv (a \dashv b) + (a \vdash a) \dashv (b \vdash a) \\
&\quad + (a \dashv b) \vdash (a \dashv a) + (a \dashv b) \vdash (a \vdash a) + (b \vdash a) \vdash (a \dashv a) + (b \vdash a) \vdash (a \vdash a) \\
&= \widehat{a}aab + \widehat{a}aba + a\widehat{a}ab + a\widehat{a}ba + ab\widehat{a}a + aba\widehat{a} + ba\widehat{a}a + baa\widehat{a}, \\
a(a^2b) &= a \dashv ((a \dashv a + a \vdash a) \dashv b + b \vdash (a \dashv a + a \vdash a)) \\
&\quad + ((a \dashv a + a \vdash a) \dashv b + b \vdash (a \dashv a + a \vdash a)) \vdash a \\
&= a \dashv ((a \dashv a) \dashv b) + a \dashv ((a \vdash a) \dashv b) + a \dashv (b \vdash (a \dashv a)) + a \dashv (b \vdash (a \vdash a)) \\
&\quad + ((a \dashv a) \dashv b) \vdash a + ((a \vdash a) \dashv b) \vdash a + (b \vdash (a \dashv a)) \vdash a + (b \vdash (a \vdash a)) \vdash a \\
&= 2\widehat{a}aab + 2\widehat{a}baa + 2aab\widehat{a} + 2baa\widehat{a}.
\end{aligned}
$$

In this paper we determine a set of generators for the identities of degree ≤ 4 satisfied by the quasi-Jordan product over a field of characteristic 0. We may assume that the identities are multilinear by Zhevlakov et al. [10], Chapter 1. We use the computer algebra system Maple [6] for calculations with large matrices.

Theorem 6. *Every polynomial identity of degree ≤ 3 satisfied by the quasi-Jordan product is a consequence of right commutativity.*

Proof. The two association types $(**)*$, $*(**)$ and six permutations of a, b, c give 12 nonassociative monomials, ordered as follows:

$$(ab)c, (ac)b, (ba)c, (bc)a, (ca)b, (cb)a, a(bc), a(cb), b(ac), b(ca), c(ab), c(ba).$$

There are 18 dialgebra monomials, ordered as follows:

$$\widehat{a}bc, \widehat{a}cb, \widehat{b}ac, \widehat{b}ca, \widehat{c}ab, \widehat{c}ba, a\widehat{b}c, a\widehat{c}b, b\widehat{a}c, b\widehat{c}a, c\widehat{a}b, c\widehat{b}a, ab\widehat{c}, ac\widehat{b}, ba\widehat{c}, bc\widehat{a}, ca\widehat{b}, cb\widehat{a}.$$

We expand each nonassociative monomial with the quasi-Jordan product:

$$
\begin{aligned}
(ab)c &= (a \dashv b + b \vdash a) \dashv c + c \vdash (a \dashv b + b \vdash a) = \widehat{a}bc + b\widehat{a}c + c\widehat{a}b + cb\widehat{a}, \\
a(bc) &= a \dashv (b \dashv c + c \vdash b) + (b \dashv c + c \vdash b) \vdash a = \widehat{a}bc + \widehat{a}cb + bc\widehat{a} + cb\widehat{a}.
\end{aligned}
$$

By permutation of a, b, c we obtain the remaining expansions:

$$
\begin{aligned}
(ac)b &= \widehat{a}cb + c\widehat{a}b + b\widehat{a}c + bc\widehat{a}, & a(cb) &= \widehat{a}cb + \widehat{a}bc + cb\widehat{a} + bc\widehat{a}, \\
(ba)c &= \widehat{b}ac + a\widehat{b}c + c\widehat{b}a + ca\widehat{b}, & b(ac) &= \widehat{b}ac + \widehat{b}ca + ac\widehat{b} + ca\widehat{b}, \\
(bc)a &= \widehat{b}ca + c\widehat{b}a + a\widehat{b}c + ac\widehat{b}, & b(ca) &= \widehat{b}ca + \widehat{b}ac + ca\widehat{b} + ac\widehat{b}, \\
(ca)b &= \widehat{c}ab + a\widehat{c}b + b\widehat{c}a + ba\widehat{c}, & c(ab) &= \widehat{c}ab + \widehat{c}ba + ab\widehat{c} + ba\widehat{c}, \\
(cb)a &= \widehat{c}ba + b\widehat{c}a + a\widehat{c}b + ab\widehat{c}, & c(ba) &= \widehat{c}ba + \widehat{c}ab + ba\widehat{c} + ab\widehat{c}.
\end{aligned}
$$

$$
\left[\begin{array}{cccccccccccc}
1 & . & . & . & . & . & 1 & 1 & . & . & . & . \\
. & 1 & . & . & . & . & 1 & 1 & . & . & . & . \\
. & . & 1 & . & . & . & . & . & 1 & 1 & . & . \\
. & . & . & 1 & . & . & . & . & 1 & 1 & . & . \\
. & . & . & . & 1 & . & . & . & . & . & 1 & 1 \\
. & . & . & . & . & 1 & . & . & . & . & 1 & 1 \\
. & . & 1 & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & 1 & . & . & . & . & . & . \\
1 & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & 1 & . & . & . & . & . & . \\
1 & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & 1 & 1 \\
. & . & . & 1 & . & . & . & . & 1 & 1 & . & . \\
. & . & . & . & 1 & . & . & . & . & . & 1 & 1 \\
. & 1 & . & . & . & . & 1 & 1 & . & . & . & . \\
. & . & 1 & . & . & . & . & . & 1 & 1 & . & . \\
1 & . & . & . & . & . & 1 & 1 & . & . & . & .
\end{array}\right]
\qquad
\left[\begin{array}{cccccccccccc}
1 & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & 1 \\
\hline
. & . & . & . & . & . & -1 & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & -1 & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & -1 & 1
\end{array}\right]
$$

TABLE 1. The matrix E_3, its row canonical form and nullspace basis

Let E_3 be the 18×12 matrix whose (i,j) entry is the coefficient of the i-th dialgebra monomial in the expansion of the j-th nonassociative monomial. The nullspace of E_3 consists of the identities in degree 3 satisfied by the quasi-Jordan product. The matrix E_3 is on the left side of Table 1 (. for 0); on the right side is the row canonical form (upper block) and the canonical basis of the nullspace (lower block). The nullspace basis vectors are the coefficient vectors of the permutations of the right commutative identity: $-a(bc)+a(cb)$, $-b(ac)+b(ca)$, $-c(ab)+c(ba)$. $\square$

Theorem 7. *Every polynomial identity of degree ≤ 4 satisfied by the quasi-Jordan product is a consequence of right commutativity and these identities in degree 4:*

$$
\begin{aligned}
J &= (a(bc))d + (a(bd))c + (a(cd))b - (ab)(cd) - (ac)(bd) - (ad)(bc), \\
K &= ((ab)d)c + ((ac)d)b - (a(bc))d - (a(bd))c - (a(cd))b + a((bc)d).
\end{aligned}
$$

The identities J and K are independent: neither implies the other.

Proof. The 5 association types are $((**)*)*$, $(*(**))*$, $(**)(**)$, $*((**)*)$, $*(*(**))$. Right commutativity eliminates type 5, $a(b(cd)) = a((cd)b)$, and implies symmetries in types 1–4: $((ab)c)d$ has no symmetry, but $(a(bc))d = (a(cb))d$, $(ab)(cd) = (ab)(dc)$, and $a((bc)d) = a((cb)d)$ since $a((bc)d) = a(d(bc)) = a(d(cb)) = a((cb)d)$. Each symmetry divides the number of monomials by 2, so types 1, 2, 3, 4 have 24, 12, 12, 12 monomials. We order these 60 monomials by association type and by lex order of the permutation (Table 2). There are 96 dialgebra monomials: in each permutation each factor can be the center. We order these monomials by the position of the center and by lex order of the permutation (Table 3). We expand each nonassociative monomial and replace each dialgebra term by its normal form:

$$
\begin{aligned}
&((ab)c)d \\
&= ((a \dashv b) \dashv c) \dashv d + d \vdash ((a \dashv b) \dashv c) + (c \vdash (a \dashv b)) \dashv d + d \vdash (c \vdash (a \dashv b)) \\
&\quad + ((b \vdash a) \dashv c) \dashv d + d \vdash ((b \vdash a) \dashv c) + (c \vdash (b \vdash a)) \dashv d + d \vdash (c \vdash (b \vdash a)) \\
&= \widehat{a}bcd + d\widehat{a}bc + c\widehat{a}bd + dc\widehat{a}b + b\widehat{a}cd + db\widehat{a}c + cb\widehat{a}d + dcb\widehat{a}, \\
&(a(bc))d \\
&= (a \dashv (b \dashv c)) \dashv d + d \vdash (a \dashv (b \dashv c)) + ((b \dashv c) \vdash a) \dashv d + d \vdash ((b \dashv c) \vdash a) \\
&\quad + (a \dashv (c \vdash b)) \dashv d + d \vdash (a \dashv (c \vdash b)) + ((c \vdash b) \vdash a) \dashv d + d \vdash ((c \vdash b) \vdash a)
\end{aligned}
$$

$$= \widehat{a}bcd + d\widehat{a}bc + bc\widehat{a}d + dbc\widehat{a} + \widehat{a}cbd + d\widehat{a}cb + cb\widehat{a}d + dcb\widehat{a},$$

$(ab)(cd)$

$$= (a \dashv b) \dashv (c \dashv d) + (c \dashv d) \vdash (a \dashv b) + (a \dashv b) \dashv (d \vdash c) + (d \vdash c) \vdash (a \dashv b)$$
$$+ (b \vdash a) \dashv (c \dashv d) + (c \dashv d) \vdash (b \vdash a) + (b \vdash a) \dashv (d \vdash c) + (d \vdash c) \vdash (b \vdash a)$$
$$= \widehat{a}bcd + cd\widehat{a}b + \widehat{a}bdc + dc\widehat{a}b + b\widehat{a}cd + cdb\widehat{a} + b\widehat{a}dc + dcb\widehat{a},$$

$a((bc)d)$

$$= a \dashv ((b \dashv c) \dashv d) + ((b \dashv c) \dashv d) \vdash a + a \dashv (d \vdash (b \dashv c)) + (d \vdash (b \dashv c)) \vdash a$$
$$+ a \dashv ((c \vdash b) \dashv d) + ((c \vdash b) \dashv d) \vdash a + a \dashv (d \vdash (c \vdash b)) + (d \vdash (c \vdash b)) \vdash a$$
$$= \widehat{a}bcd + bcd\widehat{a} + \widehat{a}dbc + dbc\widehat{a} + \widehat{a}cbd + cbd\widehat{a} + \widehat{a}dcb + dcb\widehat{a}.$$

Because of its large size, we omit the 96×60 matrix E_4 whose (i,j) entry is the coefficient of the i-th dialgebra monomial in the expansion of the j-th nonassociative monomial. Table 4 gives the row canonical form of E_4 in the upper block ($+$ for 1, $-$ for -1), and the canonical basis of the nullspace of E_4 in the lower block. The rank is 44 and the nullity is 16. Nullspace basis vectors 1–4 represent the first 4 identities in Table 5 which span an S_4-module generated by the identity J. In J, if we (simultaneously) replace a by b, and b, c, d by a, then we obtain $3(ba^2)a - 3(ba)a^2$; this is equivalent to the right quasi-Jordan identity (since $\operatorname{char} \mathbb{F} \neq 3$). If we linearize the right quasi-Jordan identity and apply right commutativity then we obtain

$$(a(bc))d + (a(bd))c + (a(cb))d + (a(cd))b + (a(db))c + (a(dc))b$$
$$- (ab)(cd) - (ab)(dc) - (ac)(bd) - (ac)(db) - (ad)(bc) - (ad)(cb)$$
$$= 2(a(bc))d + 2(a(bd))c + 2(a(cd))b - 2(ab)(cd) - 2(ac)(bd) - 2(ad)(bc);$$

this is equivalent to identity J (since $\operatorname{char} \mathbb{F} \neq 2$). Nullspace basis vectors 5–16 represent the last 12 identities in Table 5 which span an S_4-module generated by the identity K. The S_4-submodules generated by J and K have zero intersection, and so J and K are independent. □

Remark 8. If we write $D_{a,d}(x) = (a, x, d) = (ax)d - a(xd)$ then K represents a form of the derivation rule: $D_{a,d}(bc) = D_{a,d}(b)c + D_{a,d}(c)b$. In K, if we (simultaneously) replace a by b, d by c, and b, c by a, then we obtain

$$((ba)c)a + ((ba)c)a - (ba^2)c - (b(ac))a - (b(ac))a + b(a^2c)$$
$$= 2((ba)c)a - 2(b(ac))a - (ba^2)c + b(a^2c) = 2(b, a, c)a - (b, a^2, c),$$

If we linearize $2(a, b, d)b - (a, b^2, d)$ and apply right commutativity then we obtain an identity equivalent to K (since $\operatorname{char} \mathbb{F} \neq 2$):

$$2(a, b, d)c + 2(a, c, d)b - (a, bc, d) - (a, cb, d)$$
$$= 2((ab)d)c - 2(a(bd))c + 2((ac)d)b - 2(a(cd))b - (a(bc))d + a((bc)d)$$
$$- (a(cb))d + a((cb)d)$$
$$= 2((ab)d)c + 2((ac)d)b - 2(a(bc))d - 2(a(bd))c - 2(a(cd))b + 2a((bc)d).$$

Remark 9. We have the following direct proof that $(b, a^2, c) = 2(b, a, c)a$:

$$(b, a^2, c) = (ba^2)c - b(a^2c)$$
$$= (b \dashv (a \dashv a + a \vdash a) + (a \dashv a + a \vdash a) \vdash b) \dashv c$$

$((ab)c)d$	$((ab)d)c$	$((ac)b)d$	$((ac)d)b$	$((ad)b)c$	$((ad)c)b$	$((ba)c)d$	$((ba)d)c$
$((bc)a)d$	$((bc)d)a$	$((bd)a)c$	$((bd)c)a$	$((ca)b)d$	$((ca)d)b$	$((cb)a)d$	$((cb)d)a$
$((cd)a)b$	$((cd)b)a$	$((da)b)c$	$((da)c)b$	$((db)a)c$	$((db)c)a$	$((dc)a)b$	$((dc)b)a$
$(a(bc))d$	$(a(bd))c$	$(a(cd))b$	$(b(ac))d$	$(b(ad))c$	$(b(cd))a$	$(c(ab))d$	$(c(ad))b$
$(c(bd))a$	$(d(ab))c$	$(d(ac))b$	$(d(bc))a$	$(ab)(cd)$	$(ac)(bd)$	$(ad)(bc)$	$(ba)(cd)$
$(bc)(ad)$	$(bd)(ac)$	$(ca)(bd)$	$(cb)(ad)$	$(cd)(ab)$	$(da)(bc)$	$(db)(ac)$	$(dc)(ab)$
$a((bc)d)$	$a((bd)c)$	$a((cd)b)$	$b((ac)d)$	$b((ad)c)$	$b((cd)a)$	$c((ab)d)$	$c((ad)b)$
$c((bd)a)$	$d((ab)c)$	$d((ac)b)$	$d((bc)a)$.				

TABLE 2. Nonassociative monomials in degree 4

$\widehat{a}bcd$	$\widehat{a}bdc$	$\widehat{a}cbd$	$\widehat{a}cdb$	$\widehat{a}dbc$	$\widehat{a}dcb$	$\widehat{b}acd$	$\widehat{b}adc$	$\widehat{b}cad$	$\widehat{b}cda$	$\widehat{b}dac$	$\widehat{b}dca$
$\widehat{c}abd$	$\widehat{c}adb$	$\widehat{c}bad$	$\widehat{c}bda$	$\widehat{c}dab$	$\widehat{c}dba$	$\widehat{d}abc$	$\widehat{d}acb$	$\widehat{d}bac$	$\widehat{d}bca$	$\widehat{d}cab$	$\widehat{d}cba$
$a\widehat{b}cd$	$a\widehat{b}dc$	$a\widehat{c}bd$	$a\widehat{c}db$	$a\widehat{d}bc$	$a\widehat{d}cb$	$b\widehat{a}cd$	$b\widehat{a}dc$	$b\widehat{c}ad$	$b\widehat{c}da$	$b\widehat{d}ac$	$b\widehat{d}ca$
$c\widehat{a}bd$	$c\widehat{a}db$	$c\widehat{b}ad$	$c\widehat{b}da$	$c\widehat{d}ab$	$c\widehat{d}ba$	$d\widehat{a}bc$	$d\widehat{a}cb$	$d\widehat{b}ac$	$d\widehat{b}ca$	$d\widehat{c}ab$	$d\widehat{c}ba$
$ab\widehat{c}d$	$ab\widehat{d}c$	$ac\widehat{b}d$	$ac\widehat{d}b$	$ad\widehat{b}c$	$ad\widehat{c}b$	$ba\widehat{c}d$	$ba\widehat{d}c$	$bc\widehat{a}d$	$bc\widehat{d}a$	$bd\widehat{a}c$	$bd\widehat{c}a$
$ca\widehat{b}d$	$ca\widehat{d}b$	$cb\widehat{a}d$	$cb\widehat{d}a$	$cd\widehat{a}b$	$cd\widehat{b}a$	$da\widehat{b}c$	$da\widehat{c}b$	$db\widehat{a}c$	$db\widehat{c}a$	$dc\widehat{a}b$	$dc\widehat{b}a$
$abc\widehat{d}$	$abd\widehat{c}$	$acb\widehat{d}$	$acd\widehat{b}$	$adb\widehat{c}$	$adc\widehat{b}$	$bac\widehat{d}$	$bad\widehat{c}$	$bca\widehat{d}$	$bcd\widehat{a}$	$bda\widehat{c}$	$bdc\widehat{a}$
$cab\widehat{d}$	$cad\widehat{b}$	$cba\widehat{d}$	$cbd\widehat{a}$	$cda\widehat{b}$	$cdb\widehat{a}$	$dab\widehat{c}$	$dac\widehat{b}$	$dba\widehat{c}$	$dbc\widehat{a}$	$dca\widehat{b}$	$dcb\widehat{a}$.

TABLE 3. Dialgebra monomials in degree 4

$$
\begin{aligned}
&\quad + c \vdash (b \dashv (a \dashv a + a \vdash a) + (a \dashv a + a \vdash a) \vdash b) \\
&\quad - b \dashv ((a \dashv a + a \vdash a) \dashv c + c \vdash (a \dashv a + a \vdash a)) \\
&\quad - ((a \dashv a + a \vdash a) \dashv c + c \vdash (a \dashv a + a \vdash a)) \vdash b \\
&= (b \dashv (a \dashv a)) \dashv c + (b \dashv (a \vdash a)) \dashv c + ((a \dashv a) \vdash b) \dashv c + ((a \vdash a) \vdash b) \dashv c \\
&\quad + c \vdash (b \dashv (a \dashv a)) + c \vdash (b \dashv (a \vdash a)) + c \vdash ((a \dashv a) \vdash b) + c \vdash ((a \vdash a) \vdash b) \\
&\quad - b \dashv ((a \dashv a) \dashv c) - b \dashv ((a \vdash a) \dashv c) - b \dashv (c \vdash (a \dashv a)) - b \dashv (c \vdash (a \vdash a)) \\
&\quad - ((a \dashv a) \dashv c) \vdash b - ((a \vdash a) \dashv c) \vdash b - (c \vdash (a \dashv a)) \vdash b - (c \vdash (a \vdash a)) \vdash b \\
&= 2\widehat{b}aac + 2aa\widehat{b}c + 2c\widehat{b}aa + 2caa\widehat{b} - 2\widehat{b}aac - 2\widehat{b}caa - 2aac\widehat{b} - 2caa\widehat{b} \\
&= -2\widehat{b}caa + 2c\widehat{b}aa + 2aa\widehat{b}c - 2aac\widehat{b}, \\
&(b,a,c)a = ((ba)c)a - (b(ac))a \\
&= ((b \dashv a + a \vdash b) \dashv c + c \vdash (b \dashv a + a \vdash b)) \dashv a \\
&+ a \vdash ((b \dashv a + a \vdash b) \dashv c + c \vdash (b \dashv a + a \vdash b)) \\
&\quad - (b \dashv (a \dashv c + c \vdash a) + (a \dashv c + c \vdash a) \vdash b) \dashv a \\
&- a \vdash (b \dashv (a \dashv c + c \vdash a) + (a \dashv c + c \vdash a) \vdash b) \\
&= ((b \dashv a) \dashv c) \dashv a + ((a \vdash b) \dashv c) \dashv a + (c \vdash (b \dashv a)) \dashv a + (c \vdash (a \vdash b)) \dashv a \\
&\quad + a \vdash ((b \dashv a) \dashv c) + a \vdash ((a \vdash b) \dashv c) + a \vdash (c \vdash (b \dashv a)) + a \vdash (c \vdash (a \vdash b)) \\
&\quad - (b \dashv (a \dashv c)) \dashv a - (b \dashv (c \vdash a)) \dashv a - ((a \dashv c) \vdash b) \dashv a - ((c \vdash a) \vdash b) \dashv a \\
&\quad - a \vdash (b \dashv (a \dashv c)) - a \vdash (b \dashv (c \vdash a)) - a \vdash ((a \dashv c) \vdash b) - a \vdash ((c \vdash a) \vdash b) \\
&= \widehat{b}aca + a\widehat{b}ca + c\widehat{b}aa + ca\widehat{b}a + a\widehat{b}ac + aa\widehat{b}c + ac\widehat{b}a + aca\widehat{b}
\end{aligned}
$$

+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.	.	.	.	.	.	.
.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.	.	.	.	.	.	.	.
.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.	.	.	.	.	.
.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.	.	.	.	.	.
.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.	.	.	.
.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	+	+	+	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	+	+	+	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	+	+	+	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	+	+	+	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	+	+	+	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	+	+	+	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	+	+	+	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	+	+	+	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	+	+	+	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	+	+	+
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	+	+	+
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	+	+	+
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	−	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	−	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	+	+	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	+	+	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	+	+	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	+	+	+	.	.	.	.	.	.	.	.	.	.	.	.
.	+	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.
+	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.
.	.	+	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	+	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.	.
.	.	.	.	.	.	+	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	+	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	+	.	.	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	+	.	.	.	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	+	.	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	.	.	.	+	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+	.	+	.	.	.	.	.	.	.	.	.	.	−	−	−	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	+

TABLE 4. The row canonical form and the nullspace basis of E_4

$$
\begin{array}{r}
-(a(bc))d-(a(bd))c-(a(cd))b+(ab)(cd)+(ac)(bd)+(ad)(bc)\\
-(b(ac))d-(b(ad))c-(b(cd))a+(ba)(cd)+(bc)(ad)+(bd)(ac)\\
-(c(ab))d-(c(ad))b-(c(bd))a+(ca)(bd)+(cb)(ad)+(cd)(ab)\\
-(d(ab))c-(d(ac))b-(d(bc))a+(da)(bc)+(db)(ac)+(dc)(ab)\\
((ab)d)c+((ac)d)b-(a(bc))d-(a(bd))c-(a(cd))b+a((bc)d)\\
((ab)c)d+((ad)c)b-(a(bc))d-(a(bd))c-(a(cd))b+a((bd)c)\\
((ac)b)d+((ad)b)c-(a(bc))d-(a(bd))c-(a(cd))b+a((cd)b)\\
((ba)d)c+((bc)d)a-(b(ac))d-(b(ad))c-(b(cd))a+b((ac)d)\\
((ba)c)d+((bd)c)a-(b(ac))d-(b(ad))c-(b(cd))a+b((ad)c)\\
((bc)a)d+((bd)a)c-(b(ac))d-(b(ad))c-(b(cd))a+b((cd)a)\\
((ca)d)b+((cb)d)a-(c(ab))d-(c(ad))b-(c(bd))a+c((ab)d)\\
((ca)b)d+((cd)b)a-(c(ab))d-(c(ad))b-(c(bd))a+c((ad)b)\\
((cb)a)d+((cd)a)b-(c(ab))d-(c(ad))b-(c(bd))a+c((bd)a)\\
((da)c)b+((db)c)a-(d(ab))c-(d(ac))b-(d(bc))a+d((ab)c)\\
((da)b)c+((dc)b)a-(d(ab))c-(d(ac))b-(d(bc))a+d((ac)b)\\
((db)a)c+((dc)a)b-(d(ab))c-(d(ac))b-(d(bc))a+d((bc)a)
\end{array}
$$

TABLE 5. Nullspace basis identities 1–16

$$
\begin{aligned}
&-\widehat{b}aca-\widehat{b}caa-ac\widehat{b}a-ca\widehat{b}a-a\widehat{b}ac-a\widehat{b}ca-aac\widehat{b}-aca\widehat{b}\\
&=-\widehat{b}caa+c\widehat{b}aa+aa\widehat{b}c-aac\widehat{b}.
\end{aligned}
$$

Definition 10. Velásquez and Felipe [8] define a **(right) quasi-Jordan algebra** to be a nonassociative algebra satisfying the right-commutative identity $a(bc) = a(cb)$ and the right quasi-Jordan identity $(ba^2)a = (ba)a^2$. We propose the following definition which includes the (nonlinear version of the) identity K: a **semispecial** (right) quasi-Jordan algebra over a field of characteristic $\neq 2, 3$ is a (right) quasi-Jordan algebra satisfying the **associator-derivation identity**:

$$(b, a^2, c) = 2(b, a, c)a.$$

Remark 11. If D is an (associative) dialgebra, then the **plus algebra** of D is the algebra D^+ with the same underlying vector space but with the operation $ab = a \dashv b + b \vdash a$. By analogy with the theory of Jordan algebras, the natural definition of a **special** (right) quasi-Jordan algebra is one which is isomorphic to a subalgebra of D^+ for some (associative) dialgebra D. By the results of this paper, it is clear that every special (right) quasi-Jordan algebra is a semispecial (right) quasi-Jordan algebra. However, the converse is false: it has been shown recently by Bremner and Peresi [1] that there exist polynomial identities in degree 8 satisfied by the quasi-Jordan product in every (associative) dialgebra which do not follow from the three identities of Definition 10. We therefore have three classes of algebras, related by strict containments, where RQJ means (right) quasi-Jordan:

$$\text{special RQJ algebras} \subsetneq \text{semispecial RQJ algebras} \subsetneq \text{RQJ algebras}$$

This motivates our choice of the term semispecial in Definition 10.

Remark 12. Any associative algebra is a dialgebra in which the operations $\dashv$ and $\vdash$ coincide. It follows from Remark 9 that every special Jordan algebra is a special quasi-Jordan algebra and hence satisfies the associator-derivation identity. The referee raised the question of whether every Jordan algebra satisfies the associator-derivation identity. The linearization of the Jordan identity $(a^2b)a - a^2(ba)$, assuming commutativity and characteristic $\neq 2$, is

$$J(a,b,c,d) = ((ac)b)d + ((ad)b)c + ((cd)b)a - (ab)(cd) - (ac)(bd) - (ad)(bc).$$

The linearization of the associator-derivation identity $(b,a^2,c) = 2(b,a,c)a$, assuming commutativity and characteristic $\neq 2$, is

$$K(a,b,c,d) = ((ab)c)d - ((ac)b)d - ((ad)b)c + ((ad)c)b + ((bd)c)a - ((cd)b)a.$$

Then we have $K(a,b,c,d) = J(a,c,b,d) - J(a,b,c,d)$, which implies that every Jordan algebra satisfies the associator-derivation identity. Hentzel and Peresi [2] consider the identity $(b,a^2,a) = 2(b,a,a)a$ which is a weak form of the associator-derivation identity obtained by setting $c = a$. They prove that over a field of characteristic $\neq 2, 3$ any commutative algebra which satisfies this weaker identity, and is also semiprime, must be a Jordan algebra.

Remark 13. After this work was completed, the author became aware of two closely related papers. In Kolesnikov [3] the polynomial identities (26–27) are equivalent (collectively) to the opposite versions of the right quasi-Jordan identity and the associator-derivation identity, $a(a^2b) = a^2(ab)$ and $(b,a^2,c) = 2a(b,a,c)$, assuming left commutativity $(ab)c = (ba)c$. Pozhidaev [7] gives a simple algorithm to obtain from any variety of algebras a corresponding variety of dialgebras; in the case of Jordan algebras we obtain the variety of semispecial quasi-Jordan algebras.

Acknowledgements. I thank Raúl Felipe, Luiz Peresi, Alexandre Pozhidaev and the referee for very helpful comments. This research was partially supported by NSERC, the Natural Sciences and Engineering Research Council of Canada.

References

[1] M. R. Bremner and L. A. Peresi: Special identities for quasi-Jordan algebras. Preprint, June 2009. http://math.usask.ca/~bremner/research/publications
[2] I. R. Hentzel and L. A. Peresi: Almost Jordan rings. *Proc. Amer. Math. Soc.* 104 (1988), no. 2, 343–348. MR0962796
[3] P. S. Kolesnikov: Varieties of dialgebras and conformal algebras. *Sibirsk. Mat. Zh.* 49 (2008), no. 2, 322–339; translation in *Sib. Math. J.* 49 (2008), no. 2, 257–272. MR2419658
[4] J.-L. Loday: Algèbres ayant deux opérations associatives (digèbres). *C. R. Acad. Sci. Paris Sér. I Math.* 321 (1995), no. 2, 141–146. MR1345436
[5] J.-L. Loday: Dialgebras. Pages 7–66 of *Dialgebras and Related Operads*, Lecture Notes in Math., 1763, Springer, Berlin, 2001. MR1860994
[6] Maplesoft: A Division of Waterloo Maple Inc. http://www.maplesoft.com
[7] A. P. Pozhidaev: 0-dialgebras with bar unity, Rota-Baxter and 3-Leibniz algebras. Pages 245–256 of *Groups, Rings and Group Rings*, edited by A. Giambruno, C. Polcino Milies and S. K. Sehgal, Contemporary Mathematics 499, American Mathematical Society, 2009.
[8] R. Velásquez and R. Felipe: Quasi-Jordan algebras. *Comm. Algebra* 36 (2008), no. 4, 1580–1602. MR2410352
[9] R. Velásquez and R. Felipe: Split dialgebras, split quasi-Jordan algebras, and regular elements. *J. Algebra Appl.* 8 (2009), no. 2, 191–218. MR2514855
[10] K. A. Zhevlakov, A. M. Slinko, I. P. Shestakov and A. I. Shirshov: *Rings that are Nearly Associative.* Translated from the Russian by Harry F. Smith. Pure and Applied Mathematics, 104. Academic Press, Inc., New York, 1982. MR0668355

Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Saskatchewan, Canada
E-mail address: bremner@math.usask.ca