

QUANTUM DEFORMATIONS OF SIMPLE LIE ALGEBRAS

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ABSTRACT. It is shown that every simple complex Lie algebra \mathfrak{g} admits a 1-parameter family \mathfrak{g}_q of deformations outside the category of Lie algebras. These deformations are derived from a tensor product decomposition for $U_q(\mathfrak{g})$ -modules; here $U_q(\mathfrak{g})$ is the quantized enveloping algebra of \mathfrak{g} . From this it follows that the multiplication on \mathfrak{g}_q is $U_q(\mathfrak{g})$ -invariant. In the special case $\mathfrak{g} = \mathfrak{sl}(2)$, the structure constants for the deformation $\mathfrak{sl}(2)_q$ are obtained from the quantum Clebsch-Gordan formula applied to $V(2)_q \otimes V(2)_q$; here $V(2)_q$ is the simple 3-dimensional $U_q(\mathfrak{sl}(2))$ -module of highest weight q^2 .

Introduction. Lyubashenko and Sudbery [LS] have suggested that the quantized enveloping algebra $U_q(\mathfrak{g})$ of a simple complex (finite dimensional) Lie algebra \mathfrak{g} ought to be regarded as the universal associative enveloping algebra of some (as yet undetermined) non-associative algebra \mathfrak{g}_q . The relation between \mathfrak{g}_q and \mathfrak{g} should be analogous to that between $U_q(\mathfrak{g})$ and $U(\mathfrak{g})$, and there should be a PBW-type theorem relating $U_q(\mathfrak{g})$ and \mathfrak{g}_q . For other work along these lines, see [DH] and [DHGZ].

The purpose of this note is to show that a natural candidate for the “quantum Lie algebra” \mathfrak{g}_q can be obtained from the decomposition of the tensor square of the $U_q(\mathfrak{g})$ -module V_q corresponding to the adjoint representation V of \mathfrak{g} . Thus in every case $\dim(\mathfrak{g}_q) = \dim(\mathfrak{g})$; deformations of \mathfrak{g} satisfying this condition appear to be new except when $\mathfrak{g} = \mathfrak{sl}(n)$. The structure constants of $\mathfrak{sl}(2)_q$ are worked out in detail using the quantum Clebsch-Gordan formula (§VII.7 of [K]).

The algebras \mathfrak{g}_q defined in this note are not Lie algebras (except for a few special values of q): this is clear since every simple complex Lie algebra \mathfrak{g} has only trivial deformations in the category of Lie algebras (Chapter XVII of [K]). However the algebras \mathfrak{g}_q are structurally

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very closely related to Lie algebras, and so Lie-theoretic techniques should be applicable to this larger class of non-associative algebras.

General references on quantum groups are [K], [J] and [CP]. We assume throughout that q is a complex number with $q \neq 0$ and q not a root of unity.

Theorem. *Let \mathfrak{g} be a simple complex Lie algebra, and let $U_q(\mathfrak{g})$ be the corresponding quantized enveloping algebra. There exists a deformation \mathfrak{g}_q of \mathfrak{g} such that*

- (1) \mathfrak{g}_q is a $U_q(\mathfrak{g})$ -module with $\dim(\mathfrak{g}_q) = \dim(\mathfrak{g})$,
- (2) the multiplication $\mathfrak{g}_q \otimes \mathfrak{g}_q \rightarrow \mathfrak{g}_q$ is a morphism of $U_q(\mathfrak{g})$ -modules,
- (3) \mathfrak{g}_q possesses a $U_q(\mathfrak{g})$ -invariant bilinear form.

Proof. Let V denote the adjoint representation of \mathfrak{g} . Then V is also a $U(\mathfrak{g})$ -module, so let V_q denote the corresponding $U_q(\mathfrak{g})$ -module. Since every finite dimensional $U_q(\mathfrak{g})$ -module is completely reducible (Theorem 10.1.14 of [CP]) and the character of V_q is given by the classical Weyl character formula (Corollary 10.1.15 of [CP]) it follows that the decomposition of any tensor product of $U_q(\mathfrak{g})$ -modules is the same as the decomposition of the corresponding tensor product of $U(\mathfrak{g})$ -modules. In particular, the Lie bracket and the Killing form on \mathfrak{g} show that $V \otimes V$ contains a copy of V and a copy of \mathbb{C} , and so $V_q \otimes V_q$ contains a copy of V_q and a copy of \mathbb{C} . The projections $V_q \otimes V_q \rightarrow V_q$ and $V_q \otimes V_q \rightarrow \mathbb{C}$ give an multiplication and a bilinear form on $\mathfrak{g}_q = V_q$ satisfying the given conditions. \square

The case $\mathfrak{g} = \mathfrak{sl}(2)$. Let U_q denote the quantized universal enveloping algebra of $\mathfrak{sl}(2)$ as defined in Chapters VI-VII of [K]; U_q has generators E, F, K, K^{-1} and relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [EF] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Let $V(n)_q$ for $n \geq 0$ denote the unique simple U_q -module with highest weight q^n ; then $\dim V(n)_q = n + 1$. If v_0 is a highest weight vector in $V(n)_q$ then the vectors $v_i = \frac{1}{[i]!} F^i v_0$ for $0 \leq i \leq n$ form a basis of $V(n)_q$. Here $[i] = (q^i - q^{-i})/(q - q^{-1})$ and $[i]! = [i][i-1] \cdots [1]$. The quantum Clebsch-Gordan formula (Theorem VII.7.1 of [K]) states that for $n \geq m \geq 0$ there is a U_q -module isomorphism

$$V(n)_q \otimes V(m)_q \cong V(n+m)_q \oplus V(n+m-2)_q \oplus \cdots \oplus V(n-m)_q.$$

In the special case $n = m = 2$ we obtain

$$(*) \quad V(2)_q \otimes V(2)_q \cong V(4)_q \oplus V(2)_q \oplus V(0)_q.$$

Let v_0 denote a highest weight vector for the copies of $V(2)_q$ on the left side of (*). Lemma VII.7.2 of [K] gives highest weight vectors x_0, y_0 and z_0 for the summands $V(4)_q, V(2)_q$ and $V(0)_q$ on the right side of (*). Let s_i denote the i -th vector in the ordered basis $\{x_0, x_1, x_2, x_3, x_4, y_0, y_1, y_2, z_0\}$ and let t_j be the j -th vector in the ordered basis

$$\{v_0 \otimes v_0, v_0 \otimes v_1, v_0 \otimes v_2, v_1 \otimes v_0, v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_0, v_2 \otimes v_1, v_2 \otimes v_2\}.$$

Then $s_i = \sum_{j=1}^9 c_{ij} t_j$ where $C = (c_{ij})$ is the matrix of quantum Clebsch-Gordan coefficients:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/q^2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/q^4 & 0 & 1/q & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/q^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (q^2 + 1)/q^3 & 0 & (q^2 - 1)/q^2 & 0 & -(q^2 + 1)/q^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1/q^2 & 0 \\ 0 & 0 & 1 & 0 & -q/(q^2 + 1) & 0 & 1/q^2 & 0 & 0 \end{pmatrix}$$

Columns 1–5, 6–8 and 9 of C^{-1} are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & q^2/(q^4 + 1) & 0 & 0 & 0 \\ 0 & 0 & q^4/(q^8 + q^6 + 2q^4 + q^2 + 1) & 0 & 0 \\ 0 & q^4/(q^4 + 1) & 0 & 0 & 0 \\ 0 & 0 & (q^7 + 2q^5 + q^3)/(q^8 + q^6 + 2q^4 + q^2 + 1) & 0 & 0 \\ 0 & 0 & 0 & q^2/(q^4 + 1) & 0 \\ 0 & 0 & q^8/(q^8 + q^6 + 2q^4 + q^2 + 1) & 0 & 0 \\ 0 & 0 & 0 & q^4/(q^4 + 1) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ q^4/(q^4 + 1) & 0 & 0 \\ 0 & q^5/(q^6 + q^4 + q^2 + 1) & 0 \\ -q^2/(q^4 + 1) & 0 & 0 \\ 0 & (q^4 - q^2)/(q^4 + 1) & 0 \\ 0 & 0 & q^4/(q^4 + 1) \\ 0 & -q^5/(q^6 + q^4 + q^2 + 1) & 0 \\ 0 & 0 & -q^2/(q^4 + 1) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ q^4/(q^4 + q^2 + 1) \\ 0 \\ -(q^3 + q)/(q^4 + q^2 + 1) \\ 0 \\ q^2/(q^4 + q^2 + 1) \\ 0 \\ 0 \end{pmatrix}$$

Now identify the copies of $V(2)_q$ on the left and right sides of (*). Let X' denote a highest weight vector in $V(2)_q$, and set $H' = FX'$ and $Y' = \frac{1}{[2]!}F^2X'$. Columns 6–8 give the structure constants of $\mathfrak{sl}(2)_q$ (written with brackets although the composition is not anticommutative in general):

$$\begin{aligned} [X'X'] &= 0 & [X'H'] &= \frac{q^4}{q^4+1}X' & [X'Y'] &= \frac{q^5}{(q^4+1)(q^2+1)}H' \\ [H'X'] &= \frac{-q^2}{q^4+1}X' & [H'H'] &= \frac{q^2(q^2-1)}{q^4+1}H' & [H'Y'] &= \frac{q^4}{q^4+1}Y' \\ [Y'X'] &= \frac{-q^5}{(q^4+1)(q^2+1)}H' & [Y'H'] &= \frac{-q^2}{q^4+1}Y' & [Y'Y'] &= 0 \end{aligned}$$

Column 9 gives the U_q -invariant bilinear form on $\mathfrak{sl}(2)_q$:

$$\begin{aligned} (X', Y') &= \frac{q^4}{(q^4 + q^2 + 1)} & (H', H') &= \frac{-q(q^2 + 1)}{(q^4 + q^2 + 1)} \\ (Y', X') &= \frac{q^2}{(q^4 + q^2 + 1)} \end{aligned}$$

and all other pairings are 0. Now define $H = aH'$, $X = bX'$, $Y = cY'$ where

$$a = b = -q^{-4}(q^6 + q^4 + q^2 + 1) \quad c = q^{-5}(q^6 + q^4 + q^2 + 1)$$

Then we obtain

$$\begin{aligned} [XX] &= 0 & [XH] &= -(1+q^2)X & [XY] &= H \\ (**) \quad [HX] &= (1+q^{-2})X & [HH] &= (q^{-2}-q^2)H & [HY] &= -(1+q^2)Y \\ [YX] &= -H & [YH] &= (1+q^{-2})Y & [YY] &= 0 \end{aligned}$$

(In the limit case $q = 1$ we obtain the $\mathfrak{sl}(2)$ relations $[HX] = 2X$, $[HY] = -2Y$, $[XY] = H$.) The bilinear form becomes

$$\begin{aligned} (X, Y) &= \frac{3(q^2+1)^2(q^4+1)^2}{4q^5(q^4+q^2+1)} & (H, H) &= \frac{3(q^2+1)^3(q^4+1)^2}{4q^7(q^4+q^2+1)} \\ (Y, X) &= \frac{3(q^2+1)^2(q^4+1)^2}{4q^3(q^4+q^2+1)} \end{aligned}$$

where we now take $-4z_0/3$ as basis for $V(0)_q$. (In the limit case $q = 1$ we obtain the Killing form $(X, Y) = 4$, $(H, H) = 8$, $(Y, X) = 4$.)

Final Remark. The algebra $\mathfrak{sl}(2)_q$ is not a quantum Lie algebra in the sense of [L] since by definition such an algebra A is anticommutative. The algebras of [L] also satisfy the quantum Jacobi identity $J_q(x, y, z) = (xy)\sigma(z) + (yz)\sigma(x) + (zx)\sigma(y) = 0$, where $A = \bigoplus_{n \in \mathbb{Z}} A_n$ is \mathbb{Z} -graded and $\sigma = (J + J^{-1})/2$, $J(a_n) = q^n a_n$ for any $a_n \in A_n$. The algebra $\mathfrak{sl}(2)_q$ fails to satisfy the quantum Jacobi identity, since any \mathbb{Z} -grading of $\mathfrak{sl}(2)_q$ must have $\deg(H) = 0$ and then $J_q(H, H, H) = 3(q^{-2} - q^2)^2 H$, which is non-zero for $q \notin \{0, \pm 1, \pm i\}$.

REFERENCES

- [CP] Chari, V. and Pressley, A., *A Guide to Quantum Groups*, Cambridge, 1994.
- [DH] Delius, G. W. and Hüffmann, A., *On quantum Lie algebras and quantum root systems*, MSRI preprints on quantum algebra and topology, q-alg/9506017.
- [DHGZ] Delius, G. W., Hüffmann, A., Gould, M. D. and Zhang, Y.-Z., *Quantum Lie algebras associated to $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{sl}_n)$* , MSRI preprints on quantum algebra and topology, q-alg/9508013.
- [J] Jantzen, J. C., *Lectures on Quantum Groups*, American Mathematical Society, 1995.
- [K] Kassel, C., *Quantum Groups*, Springer-Verlag, 1995.
- [L] Liu, K.-Q., *Characterizations of the quantum Witt algebra*, Lett. Math. Phys. **24** (1992), 257-265.
- [LS] Lyubashenko, V., and Sudbery, A., *Quantum Lie algebras of type A_n* , MSRI preprints on quantum algebra and topology, q-alg/9510004.

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