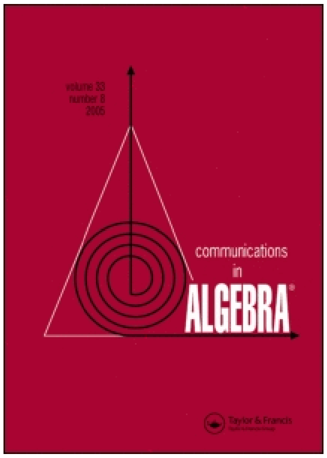

Communications in Algebra

Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

An Algebra Which Is Power Associative but not Strictly Power Associative

To cite this Article: , 'An Algebra Which Is Power Associative but not Strictly Power Associative', Communications in Algebra, 35:1, 261 - 264
xxxx:journal To link to this article: DOI: 10.1080/00927870601042068
URL: http://dx.doi.org/10.1080/00927870601042068

Communications in Algebra®, 35: 261–264, 2007

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870601042068

AN ALGEBRA WHICH IS POWER ASSOCIATIVE BUT NOT STRICTLY POWER ASSOCIATIVE

Murray R. Bremner
Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, SK, Canada

We construct a power associative algebra over the field with 3 elements which is no longer power associative when the scalars are extended to the field with 9 elements.

Key Words: Extensions fields; Finite fields; Nonassociative algebra; Power-associative algebras.

2000 MSC: Primary 17A05; Secondary 12F05.

One of the surprising properties of nonassociative algebras over finite fields is that a finite dimensional algebra can satisfy a polynomial identity even though a scalar extension of the algebra does not satisfy the identity. In this note we construct a 12-dimensional power-associative algebra over the field with 3 elements which is no longer power-associative when the scalars are extended to the field with 9 elements.

Let $\mathscr{A}$ be a nonassociative algebra over the field $\mathbb{F}$; that is, $\mathscr{A}$ is a vector space over $\mathbb{F}$ together with a bilinear multiplication $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$. We say that $\mathscr{A}$ is *power associative* if it satisfies the identities $a^k a^\ell = a^{k+\ell}$ (for all $a \in \mathscr{A}$) for $k, \ell = 1, 2, \ldots,$ where a^k is defined by $a^1 = a$, $a^k = a^{k-1}a$; equivalently, if the subalgebras generated by single elements are associative. We that $\mathscr{A}$ is *strictly power associative* if the scalar extension $\mathscr{A} \otimes_{\mathbb{F}} \mathbb{K}$ is power associative for every extension field $\mathbb{K}$ of $\mathbb{F}$. By a theorem of Albert (1948), over a field of characteristic $\neq 2, 3, 5$ every power associative algebra is strictly power associative. An example of an algebra which is power associative but not strictly power associative was given by Kokoris (1954, p. 364). However, that article presents the example without an explanation of its origin. The purpose of the present note is to provide a construction of this algebra which makes clear why it has this unusual property.

Received September 6, 2005; Revised October 18, 2005. Communicated by A. Elduque.
Address correspondence to Prof. Murray R. Bremner, Department of Mathematics and Statistics, University of Saskatchewan, McLean Hall (Room 142), 106 Wiggins Road, Saskatoon, SK S7N 5E6, Canada; E-mail: bremner@math.usask.ca

Let $\mathbb{F} = \{0, 1, 2\}$ denote the field with 3 elements. We build up a monomial basis for the free commutative nonassociative algebra $\mathscr{A}$ over $\mathbb{F}$ on two generators x, y one degree at a time:

degree 1: x, y

degree 2: x^2, xy, y^2

degree 3: $x^2x, x^2y, (xy)x, (xy)y, y^2x, y^2y$

degree 4: $(x^2x)x, (x^2y)x, ((xy)x)x, ((xy)y)x, (y^2x)x, (y^2y)x,$
$(x^2x)y, (x^2y)y, ((xy)x)y, ((xy)y)y, (y^2x)y, (y^2y)y,$
$x^2x^2, x^2(xy), x^2y^2, (xy)(xy), (xy)y^2, y^2y^2$

degree 5: $((x^2x)x)x, \ldots, ((x^2x)x)y, \ldots.$

Let $\mathscr{I}$ be the subspace of $\mathscr{A}$ spanned by all the monomials of degree ≥ 5. Then $\mathscr{I}$ is an ideal in $\mathscr{A}$ and a basis for $\mathscr{B} = \mathscr{A}/\mathscr{I}$ consists of the cosets $w + \mathscr{I}$ where w is one of the 29 monomials of degrees 1 to 4. From now on we will abuse notation (slightly) by writing w for an element of $\mathscr{B}$ when we really mean $w + \mathscr{I}$. In other words, we take the 29 monomials of degree ≤ 4 as a basis of $\mathscr{B}$ subject to the condition that any product of degree ≥ 5 is zero.

Our next step is to impose the power associative identities on $\mathscr{B}$. For any commutative nonassociative algebra, the third-power identity $a^2a = aa^2$ is clearly satisfied. For a field of characteristic 3, an algebra is power associative if and only if it satisfies the fourth-power identity $(a^2a)a = a^2a^2$ and the fifth-power identity $((a^2a)a)a = (a^2a)a^2$ (see Kokoris, 1954, p. 365). The algebra $\mathscr{B}$ automatically satisfies the latter identity since all products of five elements in $\mathscr{B}$ vanish. Therefore, the only other power associative identity of degree ≤ 4 that we need to consider is the fourth-power identity. We take $a = \lambda x + \mu y$ with $\lambda, \mu \in \mathbb{F}$ and expand, collecting terms which have the same powers of λ and μ. We obtain

$$\lambda^4 P + \lambda^3\mu Q + \lambda^2\mu^2 R + \lambda\mu^3 S + \mu^4 T = 0, \tag{0.1}$$

where (recalling that char $\mathbb{F} = 3$)

$$P = (x^2x)x + 2x^2x^2,$$
$$Q = 2((xy)x)x + (x^2y)x + (x^2x)y + 2x^2(xy),$$
$$R = (y^2x)x + 2((xy)y)x + 2((xy)x)y + ((x^2y)y + x^2y^2 + 2(xy)(xy),$$
$$S = 2((xy)y)y + (y^2x)y + (y^2y)x + 2(xy)y^2,$$
$$T = (y^2y)y + 2y^2y^2.$$

(For a detailed discussion of this process of partial linearization, see Chapter 1 of Zhevlakov et al., 1982.) Clearly, P and T are zero by the fourth-power identity, so we obtain

$$\lambda^3\mu Q + \lambda^2\mu^2 R + \lambda\mu^3 S = 0. \tag{0.2}$$

If we want to make $\mathscr{B}$ power associative, we will have to factor out an ideal $\mathscr{J}$ which contains all the elements having the form of the left side of (0.2). But now something interesting happens. Over a field containing elements λ, μ for which the scalars $\lambda^3\mu$, $\lambda^2\mu^2$, $\lambda\mu^3$ are all distinct, we can use a Vandermonde determinant to deduce that the polynomials Q, R, S must also be zero. But $\mathbb{F}$ does not have such elements! In fact every element of $\mathbb{F}$ satisfies the polynomial $\alpha^3 = \alpha$. Using this, equation (0.2) simplifies to

$$\lambda\mu Q + \lambda^2\mu^2 R + \lambda\mu S = 0, \tag{0.3}$$

and so the best we can do over $\mathbb{F}$ is to deduce that $Q + S$ and R must be zero. This is the crucial fact that permits the existence of an algebra which is power associative but not strictly power associative. (If the field is infinite this type of argument can be used to show that any power associative algebra is strictly power associative.)

So let $\mathscr{J}$ be the subspace of $\mathscr{B}$ spanned by all the monomials of degree 4 *except* $(x^2x)y$ and $(y^2x)y$. (We could equally well omit any term in Q and any term in S.) Then $\mathscr{J}$ is an ideal in $\mathscr{B}$ and the quotient algebra $\mathscr{C} = \mathscr{B}/\mathscr{J}$ is *almost* power associative. The algebra $\mathscr{C}$ has dimension 13 and basis

$$x,\ y,\ x^2,\ xy,\ y^2,\ x^2x,\ x^2y,\ (xy)x,\ (xy)y,\ y^2x,\ y^2y,\ (x^2x)y,\ (y^2x)y;$$

for monomials v, w in this list the product is vw if vw is also in the list and zero otherwise. In this algebra the polynomials P, R, T are zero, but $Q + S$ is not. To make $Q + S = 0$ we need to take one more small step. Let $\mathscr{K}$ be the subspace of $\mathscr{B}$ spanned by $(x^2x)y + (y^2x)y$ and all the monomials of degree 4 except $(x^2x)y$ and $(y^2x)y$. Then $\mathscr{K}$ is an ideal in $\mathscr{B}$; the quotient algebra $\mathscr{D} = \mathscr{B}/\mathscr{K}$ has dimension 12 and satisfies $P = R = T = Q + S = 0$, and hence is in fact power associative over $\mathbb{F}$.

Now suppose we extend the scalars of $\mathscr{D}$ from the field $\mathbb{F}$ with 3 elements to the field $\mathbb{K}$ with 9 elements (which is the splitting field over $\mathbb{F}$ of the polynomial $\alpha^9 - \alpha$). We form the scalar extension $\mathscr{D} \otimes_{\mathbb{F}} \mathbb{K}$; this algebra has the same basis and multiplication as $\mathscr{D}$ but we allow scalars from $\mathbb{K}$. The multiplicative group $\mathbb{K}^\times$ is cyclic (it's a finite subgroup of the multiplicative group of a field) and so it certainly contains an element μ of order 4 (in fact it has an element of order 8, but we don't need this). Taking $\lambda = 1$ we see that the coefficients in equation (0.2) are distinct. If the algebra $\mathscr{D} \otimes_{\mathbb{F}} \mathbb{K}$ satisfied power associativity, then using the Vandermonde determinant would show that $Q = S = 0$. But this is not true! This completes the proof that the algebra $\mathscr{D}$ is power associative but not strictly power associative.

ACKNOWLEDGMENTS

I thank Bruce Allison, Irvin Hentzel, Luiz Peresi, Michel Racine, George Seligman, and the referee for helpful remarks, and NSERC (Natural Sciences and Engineering Research Council of Canada) for financial support.

REFERENCES

Albert, A. A. (1948). On the power-associativity of rings. *Summa Brasiliensis Mathematicae* 2:21–32. See also *Collected Mathematical Papers*. Part II (Nonassociative algebras and miscellany), American Mathematical Society, Providence (1993).

Kokoris, L. A. (1954). New results on power-associative algebras. *Trans. Amer. Math. Soc.* 77:363–373.

Zhevlakov, K. A., Slinko, A. M., Shestakov, I. P., Shirshov, A. I. (1982). Rings that are nearly associative. Translated from the Russian by Harry F. Smith. Pure and Applied Math., Vol. 104, New York: Academic Press.