LIE INVARIANTS OF DEGREE TEN

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Abstract. Let $L = \oplus_{n=1}^{\infty} L_n$ denote the free Lie algebra over $\mathbb{C}$ on 2 generators of degree 1; so $\dim(L_1) = 2$, and L_n is the subspace of homogeneous Lie polynomials of degree n. The Lie group $SL_2(\mathbb{C})$ acts naturally on L by identifying L_1 with the 2-dimensional natural module and extending the group action to all of L. The fixed points of $SL_2(\mathbb{C})$ form a free Lie subalgebra K. This paper uses the generalized Witt formula for the dimensions of the homogeneous subspaces of a free Lie algebra, together with known results on the invariants of the symmetric group in a free Lie algebra, to determine, for $n \leq 40$, (i) the dimension of the subspace $K_n \subseteq L_n$ of homogeneous $SL_2(\mathbb{C})$-invariants of degree n, and (ii) the dimension of the subspace $J_n \subseteq K_n$ of primitive invariants (i.e. free generators of K). The paper concludes with an explicit description of all primitive invariants of degree 10. (The primitive invariants of degree < 10 have previously been computed.)

Introduction. Let L denote the free Lie algebra on generators a, b over the complex numbers $\mathbb{C}$. We regard $L = \oplus_{n=1}^{\infty} L_n$ as a graded Lie algebra by setting $\deg a = \deg b = 1$. Let $G = SL_2(\mathbb{C})$ act on the complex vector space with basis $\{a, b\}$ in the natural way. This action extends to L, and the fixed points $K = L^G$ form a Lie subalgebra of L, which is also free by the Shirshov-Witt Theorem [R] (§2.2, p. 44). The free Lie algebra K is known as the algebra of Lie invariants (for $SL_2(\mathbb{C})$ in the natural representation). See [M], [We] and [Bu], and especially [R] (§8.6.2, p. 207).

Let K_n denote the subspace of K consisting of the homogeneous invariants of degree n. It is known that

$$K_n = \{0\} \text{ for } n \text{ odd}, \quad \dim K_2 = 1, \quad \dim K_4 = 0, \quad \dim K_6 = 1.$$

Invariants of degrees 2 and 6 are

$$(1) \qquad I_2 = [ba] \quad \text{and} \quad I_6 = [[[ba]b][[ba]a]].$$

Let M_n denote the module over the symmetric group S_n spanned by the multilinear Lie polynomials of degree n (so M_n is a subspace of the free Lie algebra on n generators). The dimension of K_{2m} equals the multiplicity, in the S_{2m}-module M_{2m}, of the simple S_{2m}-module corresponding to the partition m^2. The multiplicity in M_n of the simple S_n-module corresponding to the partition λ is given by the formula [R] (§8.6.1, p. 206)

$$\alpha_\lambda = \frac{1}{n} \sum_{f|n} \mu(f) \chi^{\lambda}_{f^{n/f}}.$$

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Here μ is the Möbius function, and $\chi^{\lambda}_{\lambda'}$ denotes the (λ, λ') entry of the character table for S_n. So we obtain

$$\dim K_{2m} = \frac{1}{2m} \sum_{f|2m} \mu(f) \chi^{m^2}_{f^{2m/f}}. \tag{2}$$

We are primarily interested in primitive invariants, i.e. a set of free generators for K. Let J_n denote a subspace of primitive invariants in K_n. (To be precise, we take J_n to be the orthogonal complement in K_n of the span of the non-primitive invariants in K_n, with respect to the scalar product defined below.) In order to compute the dimension of J_n we need a generalization of the Witt dimension formula.

Generalized Witt formula. In this section, let K denote the free Lie algebra over $\mathbb{C}$ with d_i generators of degree i. Here i runs over the positive integers, d_i is a non-negative integer for each i, and the total number $\sum_i d_i$ of generators may be finite or infinite. We need a formula expressing $\dim K_n$ in terms of the d_i. In the case $d_1 = d$, $d_i = 0$ for $i \geq 2$, the result has been obtained by Witt:

$$\dim K_n = \frac{1}{n} \sum_{f|n} \mu(f) d^{n/f}. \tag{3}$$

See [Wi] or [Ba] (§3.1.3, pp. 73-75). The following result appears in [K] (§2).

Lemma. *The dimension of the homogeneous subspace of degree n, in the free Lie algebra K on d_i generators of degree i, is*

$$\dim K_n = \frac{1}{n} \sum_{f|n} \mu(f) W(n/f), \quad \text{where} \quad W(p) = p \sum_{j=1}^{p} \frac{1}{j} \sum_{i_1 + \cdots + i_j = p} d_{i_1} \cdots d_{i_j}. \tag{4}$$

The inner sum in $W(p)$ is over all j-compositions of p, i.e. all ordered partitions of p into j parts; there are $\binom{p-1}{j-1}$ of them [S] *(p. 14).* □

Table of dimensions. We now apply the generalized Witt formula to the computation of $\dim J_n$. Suppose that we know $\dim J_i$ for $i < n$, and let K'_n denote the free Lie algebra on $\dim J_i$ generators of degree i for $i < n$. We can use the generalized Witt formula to compute $\dim K'_n$; then clearly

$$\dim J_n = \dim K_n - \dim K'_n. \tag{5}$$

This calculation was programmed in Maple V.4 and executed on a Sun Ultra 1 Sparcstation. Table 1 at the end of this paper gives, for $1 \leq n \leq 40$, the numbers $\dim L_n$ (computed using (3) with $d = 2$), $\dim K_n$ (computed using (2)), and $\dim J_n$ (computed using (4) and (5)). In particular, we see that the only primitive invariants of degree < 10 are those given in (1), and that there is a 4-dimensional space of primitive invariants in degree 10.

A basis for L_{10}. We use the method of [R] (Prop. 4.1, p. 85) to construct a Hall basis of L_{10}. Let $A = \{a, b\}$, and define a total order $<_A$ on A by setting $b <_A a$. Let M denote the free magma on A; we can think of M as the set of all nonassociative words in $\{a, b\}$. Any $X \in M$ can be written uniquely as $X = (X')(X'')$ with $X', X'' \in M$; we omit the parentheses if the enclosed factor is in A. For any $X \in M$ we let $|X|$ denote the length of X (i.e. the number of letters in X). We define a total order $<_M$ (read "X comes after Y") on M by making $<_M$ agree with $<_A$ on A; then for $X, Y \in M - A$ we define $X <_M Y$ if and only if

(*i*) $|X| > |Y|$, or
(*ii*) $|X| = |Y|$ but $X' <_M Y'$, or
(*iii*) $|X| = |Y|$ and $X' = Y'$ but $X'' <_M Y''$.

(This total order is different from that used in [R] (Example 4.1); in particular, it is compatible with the length function on M.) We now define the Hall set $H \subset M$ recursively, by assuming that $A \subset H$, and then putting $X \in M - A$ into H if and only if

(*i*) $X', X'' \in H$, and
(*ii*) $X' <_M X''$, and
(*iii*) either $X' \in A$ or $(X')'' \geq_M X''$.

We can now determine the elements $X \in H$ with $|X| \leq 10$. To illustrate the recursive computation of H, we list the Lie monomials $X \in H$ for $1 \leq |X| \leq 5$:

1: $a, \quad b,$
2: $ba,$
3: $(ba)a, \quad (ba)b,$
4: $((ba)a)a, \quad ((ba)a)b, \quad ((ba)b)b,$
5: $((ba)a)(ba), \quad ((ba)b)(ba), \quad (((ba)a)a)a, \quad (((ba)a)a)b, \quad (((ba)a)b)b, \quad (((ba)b)b)b.$

We omit the monomials $X \in H$ for $6 \leq |X| \leq 9$, since these are intermediate results which are not needed in the remainder of this paper. The monomials X_i $(1 \leq i \leq 99)$ which form a basis of L_{10} are listed in Table 2 at the end of this paper.

Theorem. *Any nonzero linear combination of the following 4 homogeneous Lie polynomials is a primitive invariant of degree 10 for the natural action of $SL_2(\mathbb{C})$ on the free Lie algebra generated by a, b:*

$$
\begin{aligned}
I^1_{10} &= [[[[ba]b][ba]][[[ba]a][ba]]] + [[[[[ba]a][ba]][ba]][[ba]b]] - [[[[[ba]b][ba]][ba]][[ba]a]],\\
I^2_{10} &= [[[[[ba]a]a]b][[[ba]b][ba]]] - 2[[[[[ba]a]b]b][[[ba]a][ba]]] - 3[[[[[ba]a]b]b][[[[ba]a]a]b]]\\
&\quad + [[[[[ba]b]b]b][[[[ba]a]a]a]],\\
I^3_{10} &= [[[[ba]b][[ba]a]][[[ba]a]b]] + [[[[[ba]a]a][[ba]b]][[ba]b]] - 2[[[[[ba]a]b][[ba]a]][[ba]b]]\\
&\quad + [[[[[ba]b]b][[ba]a]][[ba]a]],\\
I^4_{10} &= [[[[[ba]a]a][ba]][[[ba]b]b]] - 2[[[[[ba]a]b][ba]][[[ba]a]b]] + [[[[[ba]b]b][ba]][[[ba]a]a]].
\end{aligned}
$$

Proof. Let sl_2 denote the Lie algebra of SL_2; then sl_2 has basis $\{E, F, H\}$ and commutation relations

$$[HE] = 2E, \qquad [HF] = -2F, \qquad [EF] = H.$$

We regard L_1 as a copy of the simple highest weight sl_2-module with highest weight 1. The action of sl_2 is

$$E.a = 0, \quad E.b = a, \qquad F.a = b, \quad F.b = 0, \qquad H.a = a, \quad H.b = -b. \tag{6}$$

This action extends to all of L by linearity and the Leibniz rule for derivations.

We need to determine which linear combinations of the Hall monomials X_i ($1 \le i \le 99$) are invariant under the action of SL_2. That is, we want to find elements $Z \in L_{10}$ which satisfy $D.Z = 0$ for all $D \in sl_2$. Since L_{10} is finite dimensional, it decomposes as a direct sum of simple highest weight modules, and so it suffices to impose the conditions $H.Z = 0$ and $E.Z = 0$. We define subspaces of L_{10} by

$$L_{10}^k = \{ Z \in L_{10} \mid H.Z = kZ \} = \text{span}\{ X_i \mid \#a(X_i) - \#b(X_i) = k \}.$$

The invariants $Z \in L_{10}$ are the kernel of the mapping $E \colon L_{10}^0 \to L_{10}^2$. The weight space L_{10}^k has a basis consisting of the monomials X_i with $i \in B_k$ as follows:

$$
\begin{aligned}
&B_8 = \{91\}, \quad B_6 = \{28,64,84,92\}, \quad B_4 = \{2,6,19,29,31,43,56,65,66,79,85,93\},\\
&B_2 = \{3,4,9,16,20,22,30,32,34,44,46,52,57,58,67,68,76,80,86,94\},\\
&B_0 = \{1,5,7,10,13,17,21,23,25,33,35,37,45,47,49,53,54,59,60,69,70,77,81,87,95\},\\
&B_{-2} = \{8,11,14,18,24,26,36,38,40,48,50,55,61,62,71,72,78,82,88,96\},\\
&B_{-4} = \{12,15,27,39,41,51,63,73,74,83,89,97\}, \quad B_{-6} = \{42,75,90,98\}, \quad B_{-8} = \{99\}.
\end{aligned}
$$

The dimensions of the weight spaces are

k	8	6	4	2	0	-2	-4	-6	-8
$\dim L_{10}^k$	1	4	12	20	25	20	12	4	1

From this we obtain the decomposition of L_{10} as an sl_2-module:

$$L_{10} \approx V(8) \oplus V(6)^3 \oplus V(4)^8 \oplus V(2)^8 \oplus V(0)^5,$$

where $V(k)$ denote the simple sl_2-module of highest weight k, and the superscripts indicate multiplicities.

We next compute the action of E on each of the basis vectors X_i with $i \in B_0$. We obtain

$$
\begin{aligned}
&E.X_1 = 0, \qquad E.X_5 = X_3 + X_4, \qquad E.X_7 = 2X_4, \qquad E.X_{10} = -X_4 + X_9,\\
&E.X_{13} = -X_3 + 3X_9, \qquad E.X_{17} = X_{16}, \qquad E.X_{21} = 2X_{20}, \qquad E.X_{23} = X_{20} + X_{22},\\
&E.X_{25} = 2X_{22}, \qquad E.X_{33} = X_{30} + 2X_{32}, \qquad E.X_{35} = X_{20} + 2X_{32} + X_{34},\\
&E.X_{37} = -2X_{16} + 3X_{22} + 3X_{34}, \qquad E.X_{45} = -X_{16} + 2X_{44}, \qquad E.X_{47} = X_{44} + X_{46},\\
&E.X_{49} = 2X_{46}, \qquad E.X_{53} = X_{52}, \qquad E.X_{54} = X_{52}, \qquad E.X_{59} = X_{57} + X_{58},\\
&E.X_{60} = X_{52} + 2X_{58}, \qquad E.X_{69} = X_{57} + 2X_{67} + X_{68},
\end{aligned}
$$

$$E.X_{70} = -2X_{16} + 2X_{44} + 3X_{58} + 3X_{68}, \qquad E.X_{77} = X_{76}, \qquad E.X_{81} = X_{76} + 2X_{80},$$
$$E.X_{87} = X_3 + X_{57} + 2X_{80} + 3X_{86}, \qquad E.X_{95} = 3X_{30} + 8X_{67} + 6X_{68} + 4X_{94}.$$

These results are straighforward consequences of the Jacobi identity. For example:

$$\begin{aligned}
E.X_{70} &= E.[[[[[[ba]a]a]b]b]b][[ba]a]] \\
&= [[[[[[ba]a]a]a]b]b][[ba]a]] + [[[[[[ba]a]a]b]a]b][[ba]a]] + [[[[[[ba]a]a]b]b]a][[ba]a]] \\
&\qquad \text{using the derivation rule and (6)} \\
&= X_{68} + \{\ X_{68} + [[[[[[ba]a]a][ba]]b][[ba]a]]\ \} \\
&\qquad + \{\ X_{68} + [[[[[[ba]a]a][ba]]b][[ba]a]] + [[[[[ba]a]a]b][ba]][[ba]a]]\ \} \\
&\qquad \text{using the derivation rule for } [-,a] \\
&= 3X_{68} + 2[[[[[ba]a]a][ba]]b][[ba]a]] + X_{58} \\
&= X_{58} + 3X_{68} + 2\{\ [[[[[ba]a]a]b][ba]][[ba]a]] + [[[[ba]a]a][[ba]b]][[ba]a]]\ \} \\
&\qquad \text{using the derivation rule for } [-,b] \\
&= 3X_{58} + 3X_{68} + 2\{\ X_{44} - X_{16}\ \}.
\end{aligned}$$

We form the 20×25 matrix representing $E\colon L^0_{10} \to L^2_{10}$, compute its reduced row-echelon form, and obtain the following basis for the kernel:

$$\tilde{I}^1_{10} = X_1, \qquad I^2_{10} = X_5 - 2X_7 - 3X_{10} + X_{13}, \qquad I^3_{10} = X_{17} + X_{45} - 2X_{47} + X_{49},$$
$$I^4_{10} = X_{21} - 2X_{23} + X_{25}, \qquad I^5_{10} = X_{53} - X_{54}.$$

These elements of L_{10} span the subspace K_{10} of SL_2-invariants of degree 10.

In degree < 10, the only primitive invariants are given in (1); the invariants I_2 and I_6 generate a non-primitive invariant in K_{10}, namely

$$[[I_6 I_2] I_2] = 2X_1 - X_{53} + X_{54}.$$

We define an SL_2-invariant scalar product on L^0_{10} by declaring the Hall basis words to be an orthonormal basis. (Any scalar product on a sum of trivial SL_2-modules is invariant.) We define J_{10} to be the orthogonal complement of $[[I_6 I_2] I_2]$ in K_{10}. We now have the following basis of the space J_{10} of primitive invariants of degree 10:

$$I^1_{10} = X_1 + X_{53} - X_{54}, \quad \text{together with} \quad I^2_{10}, I^3_{10}, I^4_{10}.$$

This completes the proof. □

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Table 1: Dimensions of Spaces of Invariants

n	$\dim L_n$	$\dim K_n$	$\dim J_n$
1	2	.	.
2	1	1	1
3	2	.	.
4	3	.	.
5	6	.	.
6	9	1	1
7	18	.	.
8	30	1	.
9	56	.	.
10	99	5	4
11	186	.	.
12	335	9	4
13	630	.	.
14	1,161	33	23
15	2,182	.	.
16	4,080	85	48
17	7,710	.	.
18	14,532	276	182
19	27,594	.	.
20	52,377	827	513
21	99,858	.	.
22	190,557	2,693	1,755
23	364,722	.	.
24	698,870	8,626	5,539
25	1,342,176	.	.
26	2,580,795	28,639	18,764
27	4,971,008	.	.
28	9,586,395	95,393	62,455
29	18,512,790	.	.
30	35,790,267	323,367	213,677
31	69,273,666	.	.
32	134,215,680	1,104,525	731,998
33	260,300,986	.	.
34	505,286,415	3,813,797	2,541,406
35	981,706,806	.	.
36	1,908,866,960	13,266,366	8,870,448
37	3,714,566,310	.	.
38	7,233,615,333	46,509,357	31,219,865
39	14,096,302,710	.	.
40	27,487,764,474	164,098,390	110,496,441

Table 2: Lie Monomials of Degree Ten

$(((ba)b)(ba))(((ba)a)(ba))_{1}$	$((((ba)a)a)a)(((ba)a)(ba))_{2}$	$((((ba)a)a)a)(((ba)b)(ba))_{3}$
$((((ba)a)a)b)(((ba)a)(ba))_{4}$	$((((ba)a)a)b)(((ba)b)(ba))_{5}$	$((((ba)a)a)b)((((ba)a)a)a)_{6}$
$((((ba)a)b)b)(((ba)a)(ba))_{7}$	$((((ba)a)b)b)(((ba)b)(ba))_{8}$	$((((ba)a)b)b)((((ba)a)a)a)_{9}$
$((((ba)a)b)b)((((ba)a)a)b)_{10}$	$((((ba)b)b)b)(((ba)a)(ba))_{11}$	$((((ba)b)b)b)(((ba)b)(ba))_{12}$
$((((ba)b)b)b)((((ba)a)a)a)_{13}$	$((((ba)b)b)b)((((ba)a)a)b)_{14}$	$((((ba)b)b)b)((((ba)a)b)b)_{15}$
$(((ba)b)((ba)a))(((ba)a)a)_{16}$	$(((ba)b)((ba)a))(((ba)a)b)_{17}$	$(((ba)b)((ba)a))(((ba)b)b)_{18}$
$((((ba)a)a)(ba))(((ba)a)a)_{19}$	$((((ba)a)a)(ba))(((ba)a)b)_{20}$	$((((ba)a)a)(ba))(((ba)b)b)_{21}$
$((((ba)a)b)(ba))(((ba)a)a)_{22}$	$((((ba)a)b)(ba))(((ba)a)b)_{23}$	$((((ba)a)b)(ba))(((ba)b)b)_{24}$
$((((ba)b)b)(ba))(((ba)a)a)_{25}$	$((((ba)b)b)(ba))(((ba)a)b)_{26}$	$((((ba)b)b)(ba))(((ba)b)b)_{27}$
$(((((ba)a)a)a)a)(((ba)a)a)_{28}$	$(((((ba)a)a)a)a)(((ba)a)b)_{29}$	$(((((ba)a)a)a)a)(((ba)b)b)_{30}$
$(((((ba)a)a)a)b)(((ba)a)a)_{31}$	$(((((ba)a)a)a)b)(((ba)a)b)_{32}$	$(((((ba)a)a)a)b)(((ba)b)b)_{33}$
$(((((ba)a)a)b)b)(((ba)a)a)_{34}$	$(((((ba)a)a)b)b)(((ba)a)b)_{35}$	$(((((ba)a)a)b)b)(((ba)b)b)_{36}$
$(((((ba)a)b)b)b)(((ba)a)a)_{37}$	$(((((ba)a)b)b)b)(((ba)a)b)_{38}$	$(((((ba)a)b)b)b)(((ba)b)b)_{39}$
$(((((ba)b)b)b)b)(((ba)a)a)_{40}$	$(((((ba)b)b)b)b)(((ba)a)b)_{41}$	$(((((ba)b)b)b)b)(((ba)b)b)_{42}$
$((((ba)a)a)((ba)a))((ba)a)_{43}$	$((((ba)a)a)((ba)a))((ba)b)_{44}$	$((((ba)a)a)((ba)b))((ba)b)_{45}$
$((((ba)a)b)((ba)a))((ba)a)_{46}$	$((((ba)a)b)((ba)a))((ba)b)_{47}$	$((((ba)a)b)((ba)b))((ba)b)_{48}$
$((((ba)b)b)((ba)a))((ba)a)_{49}$	$((((ba)b)b)((ba)a))((ba)b)_{50}$	$((((ba)b)b)((ba)b))((ba)b)_{51}$
$((((ba)a)(ba))(ba))((ba)a)_{52}$	$((((ba)a)(ba))(ba))((ba)b)_{53}$	$((((ba)b)(ba))(ba))((ba)a)_{54}$
$((((ba)b)(ba))(ba))((ba)b)_{55}$	$(((((ba)a)a)a)(ba))((ba)a)_{56}$	$(((((ba)a)a)a)(ba))((ba)b)_{57}$
$(((((ba)a)a)b)(ba))((ba)a)_{58}$	$(((((ba)a)a)b)(ba))((ba)b)_{59}$	$(((((ba)a)b)b)(ba))((ba)a)_{60}$
$(((((ba)a)b)b)(ba))((ba)b)_{61}$	$(((((ba)b)b)b)(ba))((ba)a)_{62}$	$(((((ba)b)b)b)(ba))((ba)b)_{63}$
$((((((ba)a)a)a)a)a)((ba)a)_{64}$	$((((((ba)a)a)a)a)a)((ba)b)_{65}$	$((((((ba)a)a)a)a)b)((ba)a)_{66}$
$((((((ba)a)a)a)a)b)((ba)b)_{67}$	$((((((ba)a)a)a)b)b)((ba)a)_{68}$	$((((((ba)a)a)a)b)b)((ba)b)_{69}$
$((((((ba)a)a)b)b)b)((ba)a)_{70}$	$((((((ba)a)a)b)b)b)((ba)b)_{71}$	$((((((ba)a)b)b)b)b)((ba)a)_{72}$
$((((((ba)a)b)b)b)b)((ba)b)_{73}$	$((((((ba)b)b)b)b)b)((ba)a)_{74}$	$((((((ba)b)b)b)b)b)((ba)b)_{75}$
$(((((ba)a)a)(ba))(ba))(ba)_{76}$	$(((((ba)a)b)(ba))(ba))(ba)_{77}$	$(((((ba)b)b)(ba))(ba))(ba)_{78}$
$((((((ba)a)a)a)a)(ba))(ba)_{79}$	$((((((ba)a)a)a)b)(ba))(ba)_{80}$	$((((((ba)a)a)b)b)(ba))(ba)_{81}$
$((((((ba)a)b)b)b)(ba))(ba)_{82}$	$((((((ba)b)b)b)b)(ba))(ba)_{83}$	$(((((((ba)a)a)a)a)a)a)(ba)_{84}$
$(((((((ba)a)a)a)a)a)b)(ba)_{85}$	$(((((((ba)a)a)a)a)b)b)(ba)_{86}$	$(((((((ba)a)a)a)b)b)b)(ba)_{87}$
$(((((((ba)a)a)b)b)b)b)(ba)_{88}$	$(((((((ba)a)b)b)b)b)b)(ba)_{89}$	$(((((((ba)b)b)b)b)b)b)(ba)_{90}$
$((((((((ba)a)a)a)a)a)a)a)a_{91}$	$((((((((ba)a)a)a)a)a)a)a)b_{92}$	$((((((((ba)a)a)a)a)a)a)b)b_{93}$
$((((((((ba)a)a)a)a)a)b)b)b_{94}$	$((((((((ba)a)a)a)a)b)b)b)b_{95}$	$((((((((ba)a)a)a)b)b)b)b)b_{96}$
$((((((((ba)a)a)b)b)b)b)b)b_{97}$	$((((((((ba)a)b)b)b)b)b)b)b_{98}$	$((((((((ba)b)b)b)b)b)b)b)b_{99}$

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