CLASSIFYING VARIETIES OF ANTI-COMMUTATIVE ALGEBRAS

Murray Bremner
Department of Mathematics and Statistics
University of Saskatchewan
Room 142, McLean Hall, 106 Wiggins Road
Saskatoon, SK, S7N 5E6
Canada

Abstract. This paper classifies varieties of anti-commutative algebras defined by identities of degree $n \leq 7$ by computing the decomposition of the S_n-module of multi-linear anti-commutative polynomials of degree n into simple submodules. The results are valid over any field of characteristic 0. In the case $n = 4$ these varieties were classified by Kass and Witthoft, using Osborn's method of irreducible identities, which does not explicitly involve any representation theory.

Introduction

This paper is concerned with varieties of anti-commutative algebras, and the identities which define them. If A is an anti-commutative algebra, the product on A will be denoted $[a, b]$, or $[ab]$ for short. All the algebras we consider will be vector spaces over a field F of characteristic 0. From this it follows [O, Cor. 3.7, p. 181] that we may restrict attention to multi-linear identities[1].

Let f be a multi-linear anti-commutative polynomial, and let A be any anti-commutative algebra. When we say that f is an identity of A we mean that $f \equiv 0$ on A. When we say that f is an identity, without specifying the algebra A, we mean merely that f is a multi-linear anti-commutative polynomial.

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In the case of degree 1, there is only one identity, namely a, which defines the variety consisting of the zero algebra alone. In degree 2, there is again only one identity, namely $[ab]$, which defines the variety of abelian algebras. The symmetric group S_2 acts on the 1-dimensional space with basis $[ab]$ by permuting the letters a, b; this gives the signature character of S_2. In degree 3, the space of possible identities has basis $[[ab]c], [[ac]b], [[bc]a]$. The symmetric group S_3 acts on this space by permuting the letters a, b, c. One easily checks that this 3-dimensional S_3-module decomposes as the direct sum of two simple submodules: (i) one spanned by the Jacobi identity $[[ab]c] - [[ac]b] + [[bc]a]$, which defines the variety of Lie algebras (this submodule gives the signature character of S_3), and (ii) the other spanned by the two unequal but equivalent identities $[[ac]b] + [[bc]a]$ and $[[ab]c] + [[ac]b]$ (this submodule gives the 2-dimensional simple S_3-module).

In general, we can consider the vector space P_n with basis β consisting of all multi-linear anti-commutative monomials of degree n. This is a subspace of the free anti-commutative algebra on n generators. The symmetric group S_n acts on P_n by permuting the n letters. Since each $\pi \in S_n$ sends one monomial to another, with a change of sign in some cases, the matrix $[\pi]_\beta$ representing $\pi \in S_n$ is a signed permutation matrix. We define a scalar product on P_n by decreeing that the monomials form an orthonormal basis; then the columns of $[\pi]_\beta$ form another orthonormal basis. Hence S_n preserves the scalar product on P_n, so we may assume that the submodules in the decomposition of P_n as an S_n-module are orthogonal subspaces.

We can easily determine the dimension a_n of P_n. When the brackets in each anti-commutative monomial are expanded in the free non-associative algebra, each of the $n-1$ pairs of brackets doubles the number of terms, so we obtain an expression with 2^{n-1} terms. For example,

$$[[ab]c] \quad \longmapsto \quad (ab)c - (ba)c - c(ab) + c(ba).$$

When we expand the a_n anti-commutative monomials this way, we obtain each non-associative monomial once and only once. The number of arrangements of letters in a non-associative monomial is clearly $n!$, and the number of possible bracket arrangements is the Catalan number $c_n = \frac{1}{n}\binom{2n-2}{n-1}$. We therefore obtain the equation $2^{n-1}a_n = n!c_n$ which gives

$$a_n = (2n-3)!! = (2n-3)(2n-5)\cdots 3 \cdot 1.$$

So we have the table

n	2	3	4	5	6	7	8	9
a_n	1	3	15	105	945	10395	135135	2027025

If I is a set of identities in P_n, then by $S(I)$ we mean the submodule of P_n generated by I. If an anti-commutative algebra satisfies the identities in I, then it also satisfies all linear combinations of permuted forms of I, but this is exactly $S(I)$. So varieties of anti-commutative algebras defined by identities of degree n are in bijective correspondence with S_n-submodules of P_n.

In this paper we compute the decomposition of P_n as an S_n-module for $n \leq 7$. Most of these calculations were done using Maple V.3 on a Sun Sparcstation LX.

The problem of finding the S_n-module decomposition of the multi-linear subspace of degree n of the free algebra on n generators can be posed for any variety of algebras. See [B, Ch. 3] and [R, Ch. 8] for complete results in the case of Lie algebras.

Degree Four

The space P_4 has a basis consisting of 15 monomials:

$$\begin{array}{llll}
b_1 = [[[ab]c]d] & b_2 = [[[ab]d]c] & b_3 = [[[ac]b]d] & b_4 = [[[ac]d]b] \\
b_5 = [[[ad]b]c] & b_6 = [[[ad]c]b] & b_7 = [[[bc]a]d] & b_8 = [[[bc]d]a] \\
b_9 = [[[bd]a]c] & b_{10} = [[[bd]c]a] & b_{11} = [[[cd]a]b] & b_{12} = [[[cd]b]a] \\
b_{13} = [[ab][cd]] & b_{14} = [[ac][bd]] & b_{15} = [[ad][bc]]. &
\end{array}$$

The monomials $b_1, \ldots, b_{12}$ have the form $[[[**]*]*]$ and span a subspace P_4'; the monomials b_{13}, b_{14}, b_{15} have the form $[[**][**]]$ and span a subspace P_4''. Within each type of bracket arrangement, the monomials have been lexicographically ordered. Since permuting the letters in a monomial does not change the bracket arrangement, the subspaces P_4' and P_4'' are S_4-submodules of P_4, and $P_4 = P_4' \oplus P_4''$ (orthogonal direct sum).

We enumerate the elements $\pi = \pi(1)\pi(2)\pi(3)\pi(4)$ of S_4 in lexicographical order:

$$\pi_1 = 1234 \qquad \pi_2 = 1243 \qquad \pi_3 = 1324 \qquad \ldots \qquad \pi_{23} = 4312 \qquad \pi_{24} = 4321.$$

We identify $1, 2, 3, 4$ with a, b, c, d and regard the elements of S_4 as permuting the letters, not the positions. For example,

$$\pi_{23}.b_1 = 4312.[[[ab]c]d] = [[[dc]a]b] = -[[[cd]a]b] = -b_{11}.$$

The character table of S_4 [JK, p. 349] is

	$[1^4]$	$[21^2]$	$[2^2]$	$[31]$	$[4]$
$[4]$	1	1	1	1	1
$[31]$	3	1	-1	0	-1
$[2^2]$	2	0	2	-1	0
$[21^2]$	3	-1	-1	0	1
$[1^4]$	1	-1	1	1	-1

The column labels are the cycle-type partitions identifying the conjugacy classes, and the row labels are the partitions identifying the simple modules. As representatives of the conjugacy classes we take $()$, (34), $(12)(34)$, (234), (1234). On the space C of class functions on S_4 we have the scalar product

$$(f, g) = \frac{1}{24}\{f_1 g_1 + 6 f_2 g_2 + 3 f_3 g_3 + 8 f_4 g_4 + 6 f_5 g_5\},$$

where $f_1 = f([1^4])$, $f_2 = f([21^2])$, $\ldots$, $f_5 = f([4])$. The irreducible characters form an orthonormal basis of C with respect to (f, g).

Theorem. $P_4 \cong [1^4] \oplus [2^2] \oplus [31] \oplus [21^2] \oplus [21^2] \oplus [21^2]$.

Proof. We can express the action of each class representative on P_4' by a signed permutation of $b_1, \ldots, b_{12}$. One easily computes

$$\begin{array}{rcrrrrrrrrrrrr}
(12) & = & -1 & -2 & 7 & 8 & 9 & 10 & 3 & 4 & 5 & 6 & 12 & 11 \\
(12)(34) & = & -2 & -1 & 9 & 10 & 7 & 8 & 5 & 6 & 3 & 4 & 12 & 11 \\
(123) & = & 7 & 8 & -1 & -2 & 10 & 9 & -3 & -4 & 12 & 11 & 5 & 6 \\
(1234) & = & 8 & 7 & 10 & 9 & -1 & -2 & 12 & 11 & -3 & -4 & -5 & -6
\end{array}$$

From this we find that the character of S_4 on P_4' is $(12, -2, 0, 0, 0)$. Writing this character as a linear combination of the irreducible characters we obtain

$$P_4' \cong [1^4] \oplus [2^2] \oplus [31] \oplus [21^2] \oplus [21^2].$$

Similarly, for P_4'' one computes these signed permutations of $b_{13}, \ldots, b_{15}$:

$$\begin{array}{rcrrr}
(12) & = & -13 & -15 & -14 \\
(12)(34) & = & 13 & -14 & -15
\end{array}
\qquad\qquad
\begin{array}{rcrrr}
(123) & = & -15 & -13 & 14 \\
(1234) & = & 15 & 14 & -13
\end{array}$$

We find that the character of S_4 on P_4'' is $(3, -1, -1, 0, 1)$ and obtain $P_4'' \cong [21^2]$.
$\square$

Since the module of type $[1^4]$ occurs exactly once, there is a unique subspace of P_4 affording this representation; and similarly for the modules of types $[2^2]$ and $[31]$. However, the module of type $[21^2]$ occurs three times: there is a unique subspace of P_4 affording the representation $[21^2] \oplus [21^2] \oplus [21^2]$, but this subspace can be decomposed into three subspaces each affording the representation $[21^2]$ in uncountably many different ways.

Equivalent results in this case were derived by Kass and Witthoft using Osborn's method of irreducible identities [O]. In [KW, Th. 3] we find the following:

Theorem. *Every anti-commutative algebra over a field F of characteristic different from 2 that satisfies a homogeneous identity of degree no higher than 4 not already implied by the anti-commutative law satisfies one of the following:*

(1) $[[[xy]x]x]$,

(2) $[x, y, [xy]]$,

(3) $a[[xy][xz]] + b\{[[[xy]x]z] - [[[xz]x]y]\} + c\{[[[xy]z]x] - [[[xz]y]x]\} + (b+c)$ $[[[yz]x]x]$, *for some* $a, b, c \in F$,

(4) $[[x, y, z], w] - [[w, x, y], z] + [[z, w, x], y] - [[y, z, w], x]$. □

Here $[x, y, z]$ denotes $[[xy]z] + [[yz]x] + [[zx]y]$.

If we linearize identity (1) we obtain a generator of the submodule of P_4 of type [31]. If we linearize identity (2) we obtain a generator of the submodule of P_4 of type $[2^2]$. If we take $(a, b, c) = (1, 0, 0)$ in (3) we obtain $[[xy][xz]]$. The linearized form of this identity generates the submodule P'' isomorphic to $[21^2]$. If we take (a, b, c) to be $(0, 1, 0)$ or $(0, 0, 1)$ in (3) and linearize, we obtain two identities that generate distinct submodules of type $[21^2]$ inside P'. The identity (4) is a spanning vector of the submodule of type $[1^4]$. Note that the number of parameters in the identity (3) equals the multiplicity of $[21^2]$ as a submodule of P_4.

Over a field of characteristic 0, Osborn's method of finding irreducible identities is equivalent to finding generators for the simple S_n-submodules of the space of multi-linear anti-commutative polynomials. Osborn's method implicitly uses the representation theory of the symmetric group in its consideration of identities which are preserved by interchanging certain pairs of variables and negated by interchanging certain other pairs of variables. This method has the advantage that it deals with non-multi-linear identities which generally have fewer terms than the corresponding multi-linear identities. However it does not give the explicit S_n-module decomposition of P_n, and it is not easy to program on a computer (so it would be difficult to extend the calculations of [KW] to $n \geq 5$).

The remainder of this paper extends the results of this section to the cases $5 \leq n \leq 7$.

Degrees Five, Six and Seven

A problem that immediately arises when we consider higher degrees is the enumeration of the possible anti-commutative bracket arrangements in each degree. We saw above that in degree 4 there are only two, namely [[[∗∗]∗]∗] and [[∗∗][∗∗]]. It is easy to see that in degree n, the number b_n of anti-commutative bracket arrangements is the same as the number of commutative (non-associative) parenthesizations, and this equals the number of homeomorphism classes of rooted binary trees with n leaf nodes. There is a simple recursion formula for these numbers [W]; we have the following table:

n	2	3	4	5	6	7	8	9
b_n	1	1	2	3	6	11	23	46

No closed-form expression for these numbers is known.

We can now decompose the S_n-module P_n into the direct sum of b_n submodules corresponding to the different bracket arrangements (as we decomposed P_4 into $P_4' \oplus P_4''$ in the previous section). We then compute the decomposition of each of these submodules into simple S_n-modules using the methods described above; then the multiplicity of an simple summand in the decomposition of P_n is just the sum of its multiplicities in the submodules.

We now present the results for $5 \leq n \leq 7$. To make the notation clear, we first re-express the results of the previous section in the tabular form we will use in this section:

		4	31	2^2	21^2	1^4
		1	3	2	3	1
[[[∗∗]∗]∗]	12	0	1	1	2	1
[[∗∗][∗∗]]	3	0	0	0	1	0
P_4	15	0	1	1	3	1

Here the first two rows give the partitions of n identifying the simple S_n-modules, and the dimensions of these modules. The next b_n rows give the decompositions of the submodules of P_n corresponding to the different bracket arrangements. The last row gives the decomposition of P_n. In the last $b_n + 1$ rows the first entry is the submodule label and the second entry is the dimension of the submodule; the remaining entries are the multiplicities of the simple S_n-modules.

Here are the results for degree 5:

		5	41	32	31^2	2^21	21^3	1^5
		1	4	5	6	5	4	1
[[[[**]*]*]*]	60	0	1	2	3	3	3	1
[[[**][**]]*]	15	0	0	0	1	1	1	0
[[[**]*][**]]	30	0	0	1	1	2	2	1
P_5	105	0	1	3	5	6	6	2

Here are the results for degree 6:

		6	51	42	41^2	3^2	321	31^3	2^3	2^21^2	21^4	1^6
		1	5	9	10	5	16	10	5	9	5	1
[[[[[**]*]*]*]*]	360	0	1	3	4	2	8	6	3	6	4	1
[[[[**][**]]*]*]	90	0	0	0	1	0	2	2	1	2	1	0
[[[[**]*][**]]*]	180	0	0	1	1	1	4	3	2	4	3	1
[[[[**]*]*][**]]	180	0	0	1	1	1	4	3	2	4	3	1
[[[**][**]][**]]	45	0	0	0	0	0	1	1	1	1	1	0
[[[**]*][[**]*]]	90	0	0	0	1	1	2	1	0	3	1	1
P_6	945	0	1	5	8	5	21	16	9	20	13	4

The results for degree 7 are on the following page.

The Maple procedures used to compute these tables may be obtained from the author.

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E-mail address: bremner@math.usask.ca