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Enveloping algebras of the nilpotent Malcev algebra of dimension five

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Abstract Pérez-Izquierdo and Shestakov recently extended the PBW theorem to Malcev algebras. It follows from their construction that for any Malcev algebra M over a field of characteristic $\neq 2, 3$ there is a representation of the universal nonassociative enveloping algebra $U(M)$ by linear operators on the polynomial algebra $P(M)$. For the nilpotent non-Lie Malcev algebra $\mathbb{M}$ of dimension 5, we use this representation to determine explicit structure constants for $U(\mathbb{M})$; from this it follows that $U(\mathbb{M})$ is not power-associative. We obtain a finite set of generators for the alternator ideal $I(\mathbb{M}) \subset U(\mathbb{M})$ and derive structure constants for the universal alternative enveloping algebra $A(\mathbb{M}) = U(\mathbb{M})/I(\mathbb{M})$, a new infinite dimensional alternative algebra. We verify that the map $\iota \colon \mathbb{M} \to A(\mathbb{M})$ is injective, and so $\mathbb{M}$ is special.

Keywords Malcev algebras, alternative algebras, nonassociative algebras, universal enveloping algebras, representation theory, differential operators

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1 Introduction

A nonassociative algebra A is alternative if it satisfies $(x, x, y) = 0$ and $(y, x, x) = 0$ for all $x, y \in A$ where $(x, y, z) = (xy)z - x(yz)$. Equivalent conditions are that any subalgebra generated by two elements is associative, and that the associator (x, y, z) is an alternating function of its arguments. If A is an alternative algebra, then A^- denotes the commutator algebra: the same vector space with the operation $[x, y] = xy - yx$. The algebra A^- satisfies anticommutativity $[x, x] = 0$ and the Malcev identity $[J(x, y, z), x] = J(x, y, [x, z])$ where $J(x, y, z) = [[x, y], z]+[[y, z], x]+[[z, x], y]$; these two identities define Malcev algebras. Basic references on Malcev algebras are [1], [2], [4], [7].

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Table 1.1 The nilpotent Malcev algebra $\mathbb{M}$ of dimension five

$[-,-]$	a	b	c	d	e
a	0	c	0	0	0
b	$-c$	0	0	0	0
c	0	0	0	e	0
d	0	0	$-e$	0	0
e	0	0	0	0	0

The major unsolved problem in the theory of Malcev algebras is to determine whether every Malcev algebra is isomorphic to a subalgebra of A^- for some alternative algebra A; that is, whether every Malcev algebra is special. Substantial progress was made recently by Pérez-Izquierdo and Shestakov [6]: they extended the Poincaré–Birkhoff–Witt theorem to Malcev algebras by constructing universal nonassociative enveloping algebras for Malcev algebras.

The smallest nilpotent non-Lie Malcev algebra $\mathbb{M}$ over a field of characteristic 0 has dimension 5 and is unique up to isomorphism [3]. Its structure constants, which define a nilpotent Malcev algebra over any field, are given in Table 1. In this paper we construct a representation of the universal nonassociative enveloping algebra $U(\mathbb{M})$ by differential operators on the polynomial algebra $P(\mathbb{M})$; from this we determine explicit structure constants for $U(\mathbb{M})$. We then determine a set of generators for the alternator ideal $I(\mathbb{M})$; from this we obtain explicit structure constants for the universal alternative enveloping algebra $A(\mathbb{M}) = U(\mathbb{M})/I(\mathbb{M})$, and verify the speciality of $\mathbb{M}$.

2 Theorem of Pérez-Izquierdo and Shestakov

All multilinear structures are over a field $\mathbb{F}$ of characteristic $\neq 2, 3$.

Definition 2.1 The **generalized alternative nucleus** of an algebra A is

$$N_{\text{alt}}(A) = \big\{\, a \in A \mid (a, x, y) = -(x, a, y) = (x, y, a), \text{ for all } x, y \in A \,\big\}.$$

This is a subalgebra of A^- (but not of A) and is a Malcev algebra.

Theorem 2.1 (Pérez-Izquierdo and Shestakov [6]) *For every Malcev algebra M over $\mathbb{F}$ there exists a nonassociative algebra $U(M)$ and an injective homomorphism $\iota\colon M \to U(M)^-$ such that $\iota(M) \subseteq N_{\text{alt}}(U(M))$; furthermore, $U(M)$ is a universal object with respect to such homomorphisms.*

Let $F(M)$ be the unital free nonassociative algebra over $\mathbb{F}$ on a basis of M. Let $R(M)$ be the ideal of $F(M)$ generated by the relations

$$ab - ba - [a, b], \qquad (a, x, y) + (x, a, y), \qquad (x, a, y) + (x, y, a),$$

for all $a, b \in M$ and all $x, y \in F(M)$. Define $U(M) = F(M)/R(M)$ with

$$\iota\colon M \to N_{\text{alt}}(U(M)) \subseteq U(M), \qquad a \mapsto \iota(a) = \overline{a} = a + R(M).$$

Since ι is injective, we identify M with $\iota(M) \subseteq U(M)$. Let $B = \{a_i \mid i \in \mathcal{I}\}$ be a basis of M and let $<$ be a total order on $\mathcal{I}$. Define

$$\Omega = \{\, (i_1, \dots, i_n) \mid n \geq 0;\ i_1 \leq \cdots \leq i_n;\ i_1, \dots, i_n \in \mathcal{I} \,\}.$$

For $n = 0$ the empty n-tuple gives $\overline{a}_\emptyset = 1 \in U(M)$. For $n \geq 1$ the n-tuple $I = (i_1, \ldots, i_n) \in \Omega$ defines a left-tapped monomial

$$\overline{a}_I = \overline{a}_{i_1}(\overline{a}_{i_2}(\cdots(\overline{a}_{i_{n-1}}\overline{a}_{i_n})\cdots)), \qquad |\overline{a}_I| = n.$$

The set of all $\overline{a}_I$ for $I \in \Omega$ is a basis of $U(M)$. For any $f, g \in M$, $y \in U(M)$ we write an associator using commutators:

$$(f, g, y) = \tfrac{1}{6}[[y, f], g] - \tfrac{1}{6}[[y, g], f] - \tfrac{1}{6}[[y, [f, g]].$$

The next three Lemmas, which are implicit in [6], follow from the last formula and show inductively how to multiply in $U(M)$. We first compute $[x, f]$ in $U(M)$; for $|x| = 1$ we use the product in M.

Lemma 2.1 *Let x be a basis monomial of $U(M)$ with $|x| \geq 2$; write $x = gy$ with $g \in M$. For any $f \in M$ we have*

$$[x, f] = [g, f]y + g[y, f] + \tfrac{1}{2}[[y, f], g] - \tfrac{1}{2}[[y, g], f] - \tfrac{1}{2}[y, [f, g]].$$

We next compute fx in $U(M)$; for $|x| = 1$ we have two cases: if $f \leq x$ in the ordered basis of M, then fx is an ordered monomial; otherwise, $fx = xf + [f, x]$ where $[f, x] \in M$.

Lemma 2.2 *Let x be a basis monomial of $U(M)$ with $|x| \geq 2$; write $x = gy$ with $g \in M$. For any $f \in M$ we have*

$$fx = g(fy) + [f, g]y - \tfrac{1}{3}[[y, f], g] + \tfrac{1}{3}[[y, g], f] + \tfrac{1}{3}[y, [f, g]].$$

We finally compute yz in $U(M)$; for $|y| = 1$ we use Lemma 2.2.

Lemma 2.3 *Let y and z be basis monomials of $U(M)$ with $|y| \geq 2$; write $y = fx$ with $f \in M$. We have*

$$yz = 2f(xz) - x(fz) - x[z, f] + [xz, f].$$

3 Representation by differential operators

We write $P(M)$ for the polynomial algebra on the vector space M. Theorem 2.1 gives a linear isomorphism

$$\phi\colon U(M) \to P(M), \quad \overline{a}_{i_1}(\overline{a}_{i_2}(\cdots(\overline{a}_{i_{n-1}}\overline{a}_{i_n})\cdots)) \mapsto a_{i_1}a_{i_2}\cdots a_{i_{n-1}}a_{i_n}.$$

In what follows we identify $U(M)$ with $P(M)$ by means of the linear isomorphism ϕ. This allows us to write monomials in $U(M)$ without parentheses: x represents $\phi^{-1}(x)$.

Definition 3.1 We define **right bracket** and **left multiplication** maps

$$\rho\colon U(M) \to \mathrm{End}_{\mathbb{F}}P(M), \qquad L\colon U(M) \to \mathrm{End}_{\mathbb{F}}P(M).$$

For $x \in U(M)$ we write $\rho(x)$ and $L(x)$ for the linear operators on $P(M)$ induced by $y \mapsto [y, x]$ and $y \mapsto xy$ in $U(M)$:

$$\rho(x)(f) = \phi\big([\phi^{-1}(f), x]\big) \text{ for } f \in P(M); \quad L(x)(f) = \phi\big(x\phi^{-1}(f)\big) \text{ for } f \in P(M).$$

For the rest of this paper $\mathbb{M}$ is the Malcev algebra of Table 1 over $\mathbb{F}$. We will show how to represent $\rho(x)$ and $L(x)$ as differential operators on $P(\mathbb{M})$. Throughout we assume the linear order $a < b < c < d < e$ on the basis of $\mathbb{M}$.

Definition 3.2 For $x \in \{a,b,c,d,e\}$ we write M_x for multiplication by x in $P(\mathbb{M})$ and D_x for differentiation with respect to x in $P(\mathbb{M})$. The next lemma is immediate.

Lemma 3.1 $[D_x, D_y] = 0,\ [M_x, M_y] = 0,\ [D_x, M_y] = 0\,(x \neq y),\ [D_x, M_x] = 1.$

The set $\{c,d,e\}$ spans a nilpotent Lie subalgebra $\mathbb{N} \subset \mathbb{M}$. It follows from [6] that $\mathbb{N}$ generates a subalgebra of $U(\mathbb{M})$ isomorphic to its associative universal enveloping algebra $U(\mathbb{N})$.

Proposition 3.1 *In the associative subalgebra $U(\mathbb{N})$ of $U(\mathbb{M})$ we have*

$$(c^i d^j e^k)(c^\ell d^m e^n) = \sum_{\alpha=0}^{\min(j,\ell)} (-1)^\alpha \alpha! \binom{j}{\alpha}\binom{\ell}{\alpha} c^{i+\ell-\alpha} d^{j+m-\alpha} e^{k+n+\alpha}.$$

Proof We use a differential operator to illustrate the methods we apply later to the nonassociative case. We first show that $d^j c = \big(M_c - M_e D_d\big) d^j$ by induction on j; the basis $j = 0$ is trivial. We use $[c,d] = e$, $[d,e] = 0$ to get

$$d^{j+1} c = d(cd^j - jd^{j-1}e) = cd^{j+1} - (j+1)d^j e = \big(M_c - M_e D_d\big) d^{j+1}.$$

We next show that $d^j c^\ell = \big(M_c - M_e D_d\big)^\ell d^j$ by induction on ℓ; the basis $\ell = 0$ is trivial. We use $[c,e] = 0$ to get

$$d^j c^{\ell+1} = \big((M_c - M_e D_d)^\ell d^j\big) c = \big(M_c - M_e D_d\big)^\ell (d^j c) = \big(M_c - M_e D_d\big)^{\ell+1} d^j.$$

We use $[c,e] = [d,e] = 0$ to get $(c^i d^j e^k)(c^\ell d^m e^n) = c^i (d^j c^\ell) d^m e^{k+n}$. We now apply the binomial theorem since the terms M_c and $M_e D_d$ commute. □

Corollary 3.1 *We have $d(b^q c^r d^s e^t) = b^q c^r d^{s+1} e^t - r b^q c^{r-1} d^s e^{t+1}$, and*

$$\begin{aligned}
&[b^q c^r d^s e^t, b] = 0, && [b^q c^r d^s e^t, c] = -s b^q c^r d^{s-1} e^{t+1},\\
&[b^q c^r d^s e^t, d] = r b^q c^{r-1} d^s e^{t+1}, && [b^q c^r d^s e^t, e] = 0,\\
&a(b^q c^r d^s e^t) = a b^q c^r d^s e^t, && b(b^q c^r d^s e^t) = b^{q+1} c^r d^s e^t,\\
&c(b^q c^r d^s e^t) = b^q c^{r+1} d^s e^t, && e(b^q c^r d^s e^t) = b^q c^r d^s e^{t+1}.
\end{aligned}$$

Proof Since b,c,d,e span a Lie subalgebra of $\mathbb{M}$, and b commutes with c,d,e, these all follow from Proposition 3.1. □

Lemma 3.2 *We have $[b^q c^r d^s e^t, a] = -q b^{q-1} c^{r+1} d^s e^t + \frac{qs}{2} b^{q-1} c^r d^{s-1} e^{t+1}$.*

Proof Lemma 2.1 with $f = a$ and Corollary 3.1 give

$$\begin{aligned}
[b^{q+1} c^r d^s e^t, a] &= -c(b^q c^r d^s e^t) + b[b^q c^r d^s e^t, a]\\
&\quad + \tfrac{1}{2}[[b^q c^r d^s e^t, a], b] - \tfrac{1}{2}[[b^q c^r d^s e^t, b], a] - \tfrac{1}{2}[b^q c^r d^s e^t, c]\\
&= -b^q c^{r+1} d^s e^t + b[b^q c^r d^s e^t, a] + \tfrac{1}{2}[[b^q c^r d^s e^t, a], b] + \tfrac{s}{2} b^q c^r d^{s-1} e^{t+1}\\
&= -b^q c^{r+1} d^s e^t + b\left(-q b^{q-1} c^{r+1} d^s e^t + \tfrac{qs}{2} b^{q-1} c^r d^{s-1} e^{t+1}\right) + \tfrac{s}{2} b^q c^r d^{s-1} e^{t+1}\\
&= -(q+1) b^q c^{r+1} d^s e^t + \tfrac{(q+1)s}{2} b^q c^r d^{s-1} e^{t+1},
\end{aligned}$$

which completes the induction. □

Proposition 3.2 *As operators on $P(\mathbb{M})$ we have*

(1) $\rho(a) = -M_c D_b + \frac{1}{2} M_e D_b D_d$, (2) $L(a) = M_a$,

(3) $\rho(b) = M_c D_a - \frac{1}{2} M_e D_a D_d$, (4) $L(b) = M_b - M_c D_a + \frac{1}{3} M_e D_a D_d$,

(5) $\rho(c) = -M_e D_d$, (6) $L(c) = M_c$,

(7) $\rho(d) = M_e D_c + \frac{1}{2} M_e D_a D_b$, (8) $L(d) = M_d - M_e D_c - \frac{1}{3} M_e D_a D_b$,

(9) $\rho(e) = 0$, (10) $L(e) = M_e$.

Proof We use induction on p, the exponent of a; the basis $p = 0$ is Corollary 3.1 and Lemma 3.2. For (1), Lemma 2.1 with $f = a$ gives

$$[a^{p+1} b^q c^r d^s e^t, a] = [a(a^p b^q c^r d^s e^t), a] = a[a^p b^q c^r d^s e^t, a].$$

Equation (2) is trivial. For (5), Lemma 2.1 with $f = c$ gives

$$[a^{p+1} b^q c^r d^s e^t, c] = a[a^p b^q c^r d^s e^t, c] + \tfrac{1}{2}[[a^p b^q c^r d^s e^t, c], a] - \tfrac{1}{2}[[a^p b^q c^r d^s e^t, a], c].$$

Applying (1) and induction we get

$$\begin{aligned} \rho(c)(a^{p+1} b^q c^r d^s e^t) &= -M_a (M_e D_d)(a^p b^q c^r d^s e^t) \\ &+ \tfrac{1}{2}(M_c D_b)(M_e D_d)(a^p b^q c^r d^s e^t) - \tfrac{1}{4}(M_e D_b D_d)(M_e D_d)(a^p b^q c^r d^s e^t) \\ &- \tfrac{1}{2}(M_e D_d)(M_c D_b)(a^p b^q c^r d^s e^t) + \tfrac{1}{4}(M_e D_d)(M_e D_b D_d)(a^p b^q c^r d^s e^t). \end{aligned}$$

By Lemma 3.1 the last four terms cancel and we get

$$\rho(c)(a^{p+1} b^q c^r d^s e^t) = -M_a M_e D_d (a^p b^q c^r d^s e^t) = -M_e D_d (a^{p+1} b^q c^r d^s e^t).$$

For (6), Lemma 2.2 with $f = c$ gives

$$c(a^{p+1} b^q c^r d^s e^t) = a(c(a^p b^q c^r d^s e^t)) - \tfrac{1}{3}[[a^p b^q c^r d^s e^t, c], a] + \tfrac{1}{3}[[a^p b^q c^r d^s e^t, a], c].$$

Applying (1), (2), (5) and induction gives

$$\begin{aligned} L(c)(a^{p+1} b^q c^r d^s e^t) &= (M_a M_c)(a^p b^q c^r d^s e^t) \\ &- \tfrac{1}{3}(M_c D_b)(M_e D_d)(a^p b^q c^r d^s e^t) + \tfrac{1}{6}(M_e D_b D_d)(M_e D_d)(a^p b^q c^r d^s e^t) \\ &+ \tfrac{1}{3}(M_e D_d)(M_c D_b)(a^p b^q c^r d^s e^t) - \tfrac{1}{6}(M_e D_d)(M_e D_b D_d)(a^p b^q c^r d^s e^t). \end{aligned}$$

By Lemma 3.1 the last four terms cancel and we get

$$L(c)(a^{p+1} b^q c^r d^s e^t) = M_a M_c (a^p b^q c^r d^s e^t) = M_c (a^{p+1} b^q c^r d^s e^t).$$

For (3), Lemma 2.1 with $f = b$ gives

$$\begin{aligned} [a^{p+1} b^q c^r d^s e^t, b] &= c(a^p b^q c^r d^s e^t) + a[a^p b^q c^r d^s e^t, b] \\ &+ \tfrac{1}{2}[[a^p b^q c^r d^s e^t, b], a] - \tfrac{1}{2}[[a^p b^q c^r d^s e^t, a], b] + \tfrac{1}{2}[a^p b^q c^r d^s e^t, c]. \end{aligned}$$

Applying (1), (2), (5), (6) and induction gives

$$\rho(b)(a^{p+1} b^q c^r d^s e^t) = \left(M_c + M_a M_c D_a - \tfrac{1}{2} M_a M_e D_a D_d - \tfrac{1}{2} M_e D_d\right)(a^p b^q c^r d^s e^t),$$

using Lemma 3.1. We now observe that

$$(M_c + M_c M_a D_a)(a^p b^q c^r d^s e^t) = (M_c D_a)(a^{p+1} b^q c^r d^s e^t),$$

$$(M_a M_e D_a D_d + M_e D_d)(a^p b^q c^r d^s e^t) = (M_e D_a D_d)(a^{p+1} b^q c^r d^s e^t).$$

For (4), Lemma 2.2 with $f = b$ gives

$$\begin{aligned} b(a^{p+1} b^q c^r d^s e^t) &= a(b(a^p b^q c^r d^s e^t)) - c(a^p b^q c^r d^s e^t) \\ &- \tfrac{1}{3}[[a^p b^q c^r d^s e^t, b], a] + \tfrac{1}{3}[[a^p b^q c^r d^s e^t, a], b] - \tfrac{1}{3}[a^p b^q c^r d^s e^t, c]. \end{aligned}$$

Applying (1), (2), (3), (5), (6) and induction gives

$$\begin{aligned} &L(b)(a^{p+1} b^q c^r d^s e^t) = \\ &\left(M_a M_b - M_a M_c D_a + \tfrac{1}{3} M_a M_e D_a D_d - M_c + \tfrac{1}{3} M_e D_d\right)(a^p b^q c^r d^s e^t), \end{aligned}$$

using Lemma 3.1, and now

$$M_a M_b = M_b M_a,\ M_a M_c D_a + M_c = M_c D_a M_a,\ M_a M_e D_a D_d + M_e D_d = M_e D_a D_d M_a.$$

The proofs of (7)–(10) are similar. □

Corollary 3.2 *The nonzero commutators of $L(x)$ and $\rho(x)$ are*

$$\begin{aligned} &[L(a), L(b)] = M_c - \tfrac{1}{3} M_e D_d, && [L(a), L(d)] = \tfrac{1}{3} M_e D_b, \\ &[L(b), L(d)] = -\tfrac{1}{3} M_e D_a, && [L(c), L(d)] = M_e, \\ &[\rho(a), \rho(d)] = M_e D_b, && [\rho(b), \rho(d)] = -M_e D_a, \\ &[L(a), \rho(b)] = -M_c + \tfrac{1}{2} M_e D_d, && [L(a), \rho(d)] = -\tfrac{1}{2} M_e D_b, \\ &[L(b), \rho(a)] = M_c - \tfrac{1}{2} M_e D_d, && [L(b), \rho(d)] = \tfrac{1}{2} M_e D_a, \\ &[L(c), \rho(d)] = -M_e, && [L(d), \rho(a)] = \tfrac{1}{2} M_e D_b, \\ &[L(d), \rho(b)] = -\tfrac{1}{2} M_e D_a, && [L(d), \rho(c)] = M_e. \end{aligned}$$

4 Multiplication of basis monomials

For operators D, E with $[[D, E], E] = 0$ we have $[D, E^n] = nE^{n-1}[D, E]$. The next result follows immediately from associativity for operators on $P(\mathbb{N})$, but is nontrivial for operators on $P(\mathbb{M})$.

Proposition 4.1 *We have $L(c^k d^\ell e^m) = L(c)^k L(d)^\ell L(e)^m$.*

Proof We first prove $L(e^m) = L(e)^m$ by induction on m; the basis $m = 0$ is trivial. We write $Z = a^p b^q c^r d^s e^t$. Lemma 2.3 gives

$$(e^{m+1})Z = 2e(e^m Z) - e^m(eZ) - e^m[Z, e] + [e^m Z, e].$$

We now apply Proposition 3.2. We next prove $L(d^\ell e^m) = L(d)^\ell L(e)^m$ by induction on ℓ; we have just proved the basis $\ell = 0$. Lemma 2.3 gives

$$(d^{\ell+1} e^m)Z = 2d((d^\ell e^m)Z) - (d^\ell e^m)(dZ) - (d^\ell e^m)[Z, d] + [(d^\ell e^m)Z, d].$$

Using induction we can write this as

$$L(d^{\ell+1} e^m) = 2L(d)L(d)^\ell L(e)^m - L(d)^\ell L(e)^m L(d) - [L(d)^\ell L(e)^m, \rho(d)].$$

Corollary 3.2 shows that the commutator is zero and that the first and second terms combine. We now prove $L(c^k d^\ell e^m) = L(c)^k L(d)^\ell L(e)^m$ by induction on k. Lemma 2.3 gives

$$(c^{k+1} d^\ell e^m) Z = 2c((c^k d^\ell e^m)Z) - (c^k d^\ell e^m)(cZ) - (c^k d^\ell e^m)[Z,c] + [(c^k d^\ell e^m)Z, c].$$

By induction and Proposition 3.2 we can write this as

$$\begin{aligned} L(c^{k+1} d^\ell e^m) = {} & 2L(c)L(c)^k L(d)^\ell L(e)^m - L(c)^k L(d)^\ell L(e)^m L(c) \\ & - L(c)^k L(d)^\ell L(e)^m \rho(c) + \rho(c) L(c)^k L(d)^\ell L(e)^m. \end{aligned}$$

Corollary 3.2 now gives

$$\begin{aligned} L(c^{k+1} d^\ell e^m) = {} & 2L(c)^{k+1} L(d)^\ell L(e)^m - L(c)^k \Big(L(c)L(d)^\ell - \ell L(d)^{\ell-1} L(e) \Big) L(e)^m \\ & - L(c)^k \Big(\rho(c) L(d)^\ell + \ell L(d)^{\ell-1} L(e) \Big) L(e)^m + \rho(c) L(c)^k L(d)^\ell L(e)^m, \end{aligned}$$

and cancelation completes the proof. ☐

Proposition 4.2 *We have*

$$L(b^j c^k d^\ell e^m) = \sum_{\alpha=0}^{\min(j,\ell)} \frac{\alpha!}{6^\alpha} \binom{j}{\alpha} \binom{\ell}{\alpha} D_a^\alpha L(b)^{j-\alpha} L(c)^k L(d)^{\ell-\alpha} L(e)^{m+\alpha}.$$

Proof Induction on j; the basis $j = 0$ is Proposition 4.1. We use Corollary 3.2 repeatedly. Lemma 2.3 gives

$$\begin{aligned} & (b^{j+1} c^k d^\ell e^m)(a^p b^q c^r d^s e^t) = \\ & \quad 2b((b^j c^k d^\ell e^m)(a^p b^q c^r d^s e^t)) - (b^j c^k d^\ell e^m)(b(a^p b^q c^r d^s e^t)) \\ & \quad\quad - (b^j c^k d^\ell e^m)[a^p b^q c^r d^s e^t, b] + [(b^j c^k d^\ell e^m)(a^p b^q c^r d^s e^t), b], \end{aligned}$$

which we can write as $L(b^{j+1} c^k d^\ell e^m) = A + B + C + D$ where

$$\begin{aligned} A &= 2L(b)L(b^j c^k d^\ell e^m), & B &= -L(b^j c^k d^\ell e^m)L(b), \\ C &= -L(b^j c^k d^\ell e^m)\rho(b), & D &= \rho(b)L(b^j c^k d^\ell e^m). \end{aligned}$$

Using induction and $[D_a, L(b)] = 0$ we get

$$A = 2 \sum_{\alpha=0}^{\ell} \frac{\alpha!}{6^\alpha} \binom{j}{\alpha} \binom{\ell}{\alpha} D_a^\alpha L(b)^{j+1-\alpha} L(c)^k L(d)^{\ell-\alpha} L(e)^{m+\alpha}.$$

Using induction and $[D_a, L(d)] = [D_a, L(c)] = 0$ we get $B = B' + B''$ where

$$\begin{aligned} B' &= - \sum_{\alpha=0}^{\ell} \frac{\alpha!}{6^\alpha} \binom{j}{\alpha} \binom{\ell}{\alpha} D_a^\alpha L(b)^{j+1-\alpha} L(c)^k L(d)^{\ell-\alpha} L(e)^{m+\alpha}, \\ B'' &= -\frac{1}{3} \sum_{\alpha=0}^{\ell} \frac{\alpha!}{6^\alpha} \binom{j}{\alpha} (\ell-\alpha) \binom{\ell}{\alpha} D_a^{\alpha+1} L(b)^{j-\alpha} L(c)^k L(d)^{\ell-\alpha-1} L(e)^{m+\alpha+1}. \end{aligned}$$

Using induction and $[D_a, L(c)] = 0$ we get $C = C' + C''$ where

$$C' = -\sum_{\alpha=0}^{\ell} \frac{\alpha!}{6^\alpha} \binom{j}{\alpha} \binom{\ell}{\alpha} D_a^\alpha \rho(b) L(b)^{j-\alpha} L(c)^k L(d)^{\ell-\alpha} L(e)^{m+\alpha},$$

$$C'' = \frac{1}{2} \sum_{\alpha=0}^{\ell} \frac{\alpha!}{6^\alpha} \binom{j}{\alpha} (\ell - \alpha) \binom{\ell}{\alpha} D_a^{\alpha+1} L(b)^{j-\alpha} L(c)^k L(d)^{\ell-\alpha-1} L(e)^{m+\alpha+1}.$$

Using induction and $[D_a, \rho(b)] = 0$ we get

$$D = \sum_{\alpha=0}^{\ell} \frac{\alpha!}{6^\alpha} \binom{j}{\alpha} \binom{\ell}{\alpha} D_a^\alpha \rho(b) L(b)^{j-\alpha} L(c)^k L(d)^{\ell-\alpha} L(e)^{m+\alpha}.$$

Terms A and B' combine to give

$$A + B' = \sum_{\alpha=0}^{\ell} \frac{\alpha!}{6^\alpha} \binom{j}{\alpha} \binom{\ell}{\alpha} D_a^\alpha L(b)^{j+1-\alpha} L(c)^k L(d)^{\ell-\alpha} L(e)^{m+\alpha}.$$

Terms B'' and C'' combine to give

$$B'' + C'' = \sum_{\alpha=1}^{\ell+1} \frac{\alpha!}{6^\alpha} \binom{j}{\alpha-1} \binom{\ell}{\alpha} D_a^\alpha L(b)^{j+1-\alpha} L(c)^k L(d)^{\ell-\alpha} L(e)^{m+\alpha}.$$

Terms C' and D cancel, and then $A + B'$ and $B'' + C''$ combine using Pascal's identity to give the result. □

Before we prove the formula for $L(a^i b^j c^k d^\ell e^m)$ we need a straightening Lemma for moving $L(a) + \rho(a)$ through a product of operators.

Definition 4.1 Our standard order for a product of operators will be

$$X = L(a)^s D_a^t L(b)^u D_b^v L(c)^w D_d^x L(d)^y L(e)^z.$$

Note that D_x only appears for $x = a, b, d$. Furthermore, $L(a)$ precedes D_a and $L(b)$ precedes D_b, but D_d precedes $L(d)$.

Lemma 4.1 *We have*

$$\begin{aligned} &L(a)^s D_a^t L(b)^u D_b^v L(c)^w D_d^x L(d)^y L(e)^z \big(L(a)+\rho(a)\big) = \\ &\quad \big(L(a)+\rho(a)\big) L(a)^s D_a^t L(b)^u D_b^v L(c)^w D_d^x L(d)^y L(e)^z \\ &\qquad + t L(a)^s D_a^{t-1} L(b)^u D_b^v L(c)^w D_d^x L(d)^y L(e)^z \\ &\qquad\quad - \tfrac{1}{6} u L(a)^s D_a^t L(b)^{u-1} D_b^v L(c)^w D_d^{x+1} L(d)^y L(e)^{z+1} \\ &\qquad\qquad + \tfrac{1}{6} y L(a)^s D_a^t L(b)^u D_b^{v+1} L(c)^w D_d^x L(d)^{y-1} L(e)^{z+1}. \end{aligned}$$

Therefore

$$\begin{aligned} &2L(a)X - XL(a) - X\rho(a) + \rho(a)X = \\ &\quad L(a)^{s+1} D_a^t L(b)^u D_b^v L(c)^w D_d^x L(d)^y L(e)^z \\ &\qquad - t L(a)^s D_a^{t-1} L(b)^u D_b^v L(c)^w D_d^x L(d)^y L(e)^z \\ &\qquad\quad + \tfrac{1}{6} u L(a)^s D_a^t L(b)^{u-1} D_b^v L(c)^w D_d^{x+1} L(d)^y L(e)^{z+1} \\ &\qquad\qquad - \tfrac{1}{6} y L(a)^s D_a^t L(b)^u D_b^{v+1} L(c)^w D_d^x L(d)^{y-1} L(e)^{z+1}. \end{aligned}$$

Proof We write $R(a) = L(a) + \rho(a)$: the operator of right multiplication by a in $U(\mathbb{M})$. Corollary 3.2, Lemma 3.1 and Proposition 3.2 give

$$[R(a), L(a)] = [R(a), D_b] = [R(a), L(c)] = [R(a), D_d] = [R(a), L(e)] = 0,$$
$$[R(a), L(b)] = \tfrac{1}{6} M_e D_d, \quad [R(a), L(d)] = -\tfrac{1}{6} M_e D_b, \quad [R(a), D_a] = -1.$$

These equations imply

$$\begin{aligned}&\Big[L(a)^s D_a^t L(b)^u D_b^v L(c)^w D_d^x L(d)^y L(e)^z, R(a)\Big] \\ &\quad = L(a)^s \Big[D_a^t, R(a)\Big] L(b)^u D_b^v L(c)^w D_d^x L(d)^y L(e)^z \\ &\qquad + L(a)^s D_a^t \Big[L(b)^u, R(a)\Big] D_b^v L(c)^w D_d^x L(d)^y L(e)^z \\ &\qquad\quad + L(a)^s D_a^t L(b)^u D_b^v L(c)^w D_d^x \Big[L(d)^y, R(a)\Big] L(e)^z,\end{aligned}$$

which gives the first equation. The second part follows easily. □

We use the multinomial coefficients

$$\binom{n}{i_1, \dots, i_k} = \frac{n!}{i_1! \cdots i_k! (n - i_1 - \cdots - i_k)!},$$

with the convention that

$$\binom{n}{i_1, \dots, i_k} = 0 \text{ if either } i_j < 0 \text{ for some } j \text{ or } \sum_j i_j > n.$$

Proposition 4.3 *We have*

$$\begin{aligned}&L(a^i b^j c^k d^\ell e^m) = \\ &\sum_{\alpha=0}^{\ell} \sum_{\beta=0}^{\min(i,\alpha)} \sum_{\gamma=0}^{\min(\ell-\alpha,\beta)} \sum_{\delta=0}^{\min(j-\alpha,\gamma)} (-1)^{\beta+\delta} \frac{\alpha!\beta!}{6^{\alpha+\gamma}} \binom{\alpha}{\beta-\gamma} \binom{i}{\beta} \binom{j}{\alpha, \delta} \binom{\ell}{\alpha, \gamma-\delta} \times \\ &\quad L(a)^{i-\beta} D_a^{\alpha-\beta+\gamma} L(b)^{j-\alpha-\delta} D_b^{\gamma-\delta} L(c)^k D_d^\delta L(d)^{\ell-\alpha-\gamma+\delta} L(e)^{m+\alpha+\gamma}.\end{aligned}$$

Proof Induction on i; the basis $i = 0$ is Proposition 4.2. For the inductive step we use Lemma 2.3 to get

$$\begin{aligned}&(a^{i+1} b^j c^k d^\ell e^m)(a^p b^q c^r d^s e^t) = \\ &\quad 2a((a^i b^j c^k d^\ell e^m)(a^p b^q c^r d^s e^t)) - (a^i b^j c^k d^\ell e^m)(a(a^p b^q c^r d^s e^t)) \\ &\qquad - (a^i b^j c^k d^\ell e^m)[a^p b^q c^r d^s e^t, a] + [(a^i b^j c^k d^\ell e^m)(a^p b^q c^r d^s e^t), a].\end{aligned}$$

Therefore $L(a^{i+1} b^j c^k d^\ell e^m) = A + B + C + D$ where

$$A = L(a) L(a^i b^j c^k d^\ell e^m), \qquad B = -L(a^i b^j c^k d^\ell e^m) L(a),$$
$$C = -L(a^i b^j c^k d^\ell e^m) \rho(a), \quad D = \rho(a) L(a^i b^j c^k d^\ell e^m).$$

We apply the second part of Lemma 4.1 to the monomial

$$X = L(a)^{i-\beta} D_a^{\alpha-\beta+\gamma} L(b)^{j-\alpha-\delta} D_b^{\gamma-\delta} L(c)^k D_d^\delta L(d)^{\ell-\alpha-\gamma+\delta} L(e)^{m+\alpha+\gamma}.$$

Therefore

$$A = \sum_{\alpha=0}^{\ell} \sum_{\beta=0}^{i} \sum_{\gamma=0}^{\beta} \sum_{\delta=0}^{\gamma} (-1)^{\beta+\delta} \frac{\alpha!\beta!}{6^{\alpha+\gamma}} PQRS \times$$
$$L(a)^{i-\beta+1} D_a^{\alpha-\beta+\gamma} L(b)^{j-\alpha-\delta} D_b^{\gamma-\delta} L(c)^k D_d^{\delta} L(d)^{\ell-\alpha-\gamma+\delta} L(e)^{m+\alpha+\gamma},$$
$$B = \sum_{\alpha=0}^{\ell} \sum_{\beta=0}^{i} \sum_{\gamma=0}^{\beta} \sum_{\delta=0}^{\gamma} (-1)^{\beta+\delta+1} \frac{\alpha!\beta!}{6^{\alpha+\gamma}} (\alpha-\beta+\gamma) PQRS \times$$
$$L(a)^{i-\beta} D_a^{\alpha-\beta+\gamma-1} L(b)^{j-\alpha-\delta} D_b^{\gamma-\delta} L(c)^k D_d^{\delta} L(d)^{\ell-\alpha-\gamma+\delta} L(e)^{m+\alpha+\gamma},$$
$$C = \sum_{\alpha=0}^{\ell} \sum_{\beta=0}^{i} \sum_{\gamma=0}^{\beta} \sum_{\delta=0}^{\gamma} (-1)^{\beta+\delta} \frac{\alpha!\beta!}{6^{\alpha+\gamma+1}} PQ(j-\alpha-\delta) RS \times$$
$$L(a)^{i-\beta} D_a^{\alpha-\beta+\gamma} L(b)^{j-\alpha-\delta-1} D_b^{\gamma-\delta} L(c)^k D_d^{\delta+1} L(d)^{\ell-\alpha-\gamma+\delta} L(e)^{m+\alpha+\gamma+1},$$
$$D = \sum_{\alpha=0}^{\ell} \sum_{\beta=0}^{i} \sum_{\gamma=0}^{\beta} \sum_{\delta=0}^{\gamma} (-1)^{\beta+\delta+1} \frac{\alpha!\beta!}{6^{\alpha+\gamma+1}} PQR(\ell-\alpha-\gamma+\delta) S \times$$
$$L(a)^{i-\beta} D_a^{\alpha-\beta+\gamma} L(b)^{j-\alpha-\delta} D_b^{\gamma-\delta+1} L(c)^k D_d^{\delta} L(d)^{\ell-\alpha-\gamma+\delta-1} L(e)^{m+\alpha+\gamma+1},$$

where

$$P = \binom{\alpha}{\beta-\gamma}, \qquad Q = \binom{i}{\beta}, \qquad R = \binom{j}{\alpha, \delta}, \qquad S = \binom{\ell}{\alpha, \gamma-\delta}.$$

We make the following substitutions in the summation indices: in B we replace β by $\beta-1$; in C we replace β by $\beta-1$, γ by $\gamma-1$, and δ by $\delta-1$; in D we replace β by $\beta-1$, and γ by $\gamma-1$. Using our convention on multinomial coefficients, we can write all four sums with the notation

$$\sum\nolimits^4 = \sum_{\alpha=0}^{\ell} \sum_{\beta=0}^{i+1} \sum_{\gamma=0}^{\beta} \sum_{\delta=0}^{\gamma} (-1)^{\beta+\delta} \frac{\alpha!}{6^{\alpha+\gamma}},$$
$$Y = L(a)^{i-\beta+1} D_a^{\alpha-\beta+\gamma} L(b)^{j-\alpha-\delta} D_b^{\gamma-\delta} L(c)^k D_d^{\delta} L(d)^{\ell-\alpha-\gamma+\delta} L(e)^{m+\alpha+\gamma}.$$

We obtain

$$A = \sum\nolimits^4 \big[\beta! PQRS\big] Y,$$
$$B = \sum\nolimits^4 \left[(\beta-1)!(\alpha-\beta+\gamma+1) \binom{\alpha}{\beta-\gamma-1} \binom{i}{\beta-1} RS\right] Y,$$
$$C = \sum\nolimits^4 \left[(\beta-1)! P \binom{i}{\beta-1} (j-\alpha-\delta+1) \binom{j}{\alpha, \delta-1} S\right] Y,$$
$$D = \sum\nolimits^4 \left[(\beta-1)! P \binom{i}{\beta-1} R(\ell-\alpha-\gamma+\delta+1) \binom{\ell}{\alpha, \gamma-\delta-1}\right] Y.$$

The sum of these four terms is

$$\sum\nolimits^4 \beta! \left[\frac{i+1-\beta}{i+1} + \frac{\beta-\gamma}{i+1} + \frac{\delta}{i+1} + \frac{\gamma-\delta}{i+1}\right] P \binom{i+1}{\beta} RS\, Y = \sum\nolimits^4 \left[\beta! P \binom{i+1}{\beta} RS\right] Y,$$

as required. □

Lemma 4.2 *The powers of $L(b)$ and $L(d)$ are*

$$L(b)^u = \sum_{\epsilon=0}^{u}\sum_{\zeta=0}^{u-\epsilon}(-1)^\zeta \frac{1}{3^{u-\epsilon-\zeta}}\binom{u}{\epsilon,\zeta} M_b^\epsilon M_c^\zeta M_e^{u-\epsilon-\zeta} D_a^{u-\epsilon} D_d^{u-\epsilon-\zeta},$$

$$L(d)^y = \sum_{\eta=0}^{y}\sum_{\theta=0}^{y-\eta}(-1)^{y-\eta} \frac{1}{3^{y-\eta-\theta}}\binom{y}{\eta,\theta} M_d^\eta M_e^{y-\eta} D_a^{y-\eta-\theta} D_b^{y-\eta-\theta} D_c^\theta.$$

Proof We use the trinomial theorem, since the terms in $L(b)$ and $L(d)$ commute. □

Lemma 4.3 *The expansion of the monomial X of Definition 4.1 is*

$$L(a)^s D_a^t L(b)^u D_b^v L(c)^w D_d^x L(d)^y L(e)^z =$$

$$\sum_{\epsilon=0}^{u}\sum_{\zeta=0}^{u-\epsilon}\sum_{\eta=0}^{y}\sum_{\theta=0}^{y-\eta}\sum_{\lambda=0}^{\min(u-\epsilon-\zeta+x,\eta)} \frac{(-1)^{\zeta+y-\eta}\lambda!}{3^{u-\epsilon-\zeta+y-\eta-\theta}}\binom{u}{\epsilon,\zeta}\binom{y}{\eta,\theta}\binom{u-\epsilon-\zeta+x}{\lambda}\binom{\eta}{\lambda}$$

$$M_a^s M_b^\epsilon M_c^{\zeta+w} M_d^{\eta-\lambda} M_e^{u-\epsilon-\zeta+y-\eta+z} D_a^{t+u-\epsilon+y-\eta-\theta} D_b^{v+y-\eta-\theta} D_c^\theta D_d^{u+x-\epsilon-\zeta-\lambda}.$$

Proof Proposition 3.2 and Lemma 4.2 give

$$L(a)^s D_a^t L(b)^u D_b^v L(c)^w D_d^x L(d)^y L(e)^z =$$

$$\sum_{\epsilon=0}^{u}\sum_{\zeta=0}^{u-\epsilon}\sum_{\eta=0}^{y}\sum_{\theta=0}^{y-\eta} \frac{(-1)^{\zeta+y-\eta}}{3^{u-\epsilon-\zeta+y-\eta-\theta}}\binom{u}{\epsilon,\zeta}\binom{y}{\eta,\theta}\times$$

$$M_a^s D_a^t M_b^\epsilon M_c^\zeta M_e^{u-\epsilon-\zeta} D_a^{u-\epsilon} D_d^{u-\epsilon-\zeta} D_b^v M_c^w D_d^x M_d^\eta M_e^{y-\eta} D_a^{y-\eta-\theta} D_b^{y-\eta-\theta} D_c^\theta M_e^z.$$

Using Lemma 3.1 we can move M_x to the left and D_x to the right, and collect the remaining noncommuting factors on the right:

$$M_a^s M_b^\epsilon M_c^{\zeta+w} M_e^{u-\epsilon-\zeta+y-\eta+z} D_a^{t+u-\epsilon+y-\eta-\theta} D_b^{v+y-\eta-\theta} D_c^\theta \big(D_d^{u-\epsilon-\zeta+x} M_d^\eta\big)$$

To complete the proof we use the commutation formula

$$D_d^m M_d^n = \sum_{i=0}^{\min(m,n)} i!\binom{m}{i}\binom{n}{i} M_d^{n-i} D_d^{m-i}.$$

(Compare Proposition 3.1.) □

Proposition 4.4 *We have*

$$L(a^i b^j c^k d^\ell e^m) =$$

$$\sum_{\alpha=0}^{\ell}\sum_{\beta=0}^{i}\sum_{\gamma=0}^{\beta}\sum_{\delta=0}^{\gamma}\sum_{\epsilon=0}^{j-\alpha-\delta}\sum_{\zeta=0}^{j-\alpha-\delta-\epsilon}\sum_{\eta=0}^{\ell-\alpha-(\gamma-\delta)}\sum_{\theta=0}^{\ell-\alpha-(\gamma-\delta)-\eta}\sum_{\lambda=0}^{\eta}$$

$$(-1)^{\beta+\zeta+\ell-\alpha-\gamma-\eta}\frac{\alpha!\beta!\lambda!}{2^{\alpha+\gamma}3^{j-\epsilon-\zeta+\ell-\alpha-\eta-\theta}}\times$$

$$\binom{\alpha}{\beta-\gamma}\binom{i}{\beta}\binom{j}{\alpha,\delta,\epsilon,\zeta}\binom{j-\alpha-\epsilon-\zeta}{\lambda}\binom{\ell}{\alpha,\gamma-\delta,\eta,\theta}\binom{\eta}{\lambda}\times$$

$$M_a^{i-\beta} M_b^\epsilon M_c^{\zeta+k} M_d^{\eta-\lambda} M_e^{j-\alpha-\epsilon-\zeta+\ell-\eta+m}\times$$

$$D_a^{j-\beta-\epsilon+\ell-\alpha-\eta-\theta} D_b^{\ell-\alpha-\eta-\theta} D_c^\theta D_d^{j-\alpha-\epsilon-\zeta-\lambda}.$$

Proof In Proposition 4.3 take $s = i-\beta$, $t = \alpha-\beta+\gamma$, $u = j-\alpha-\delta$, $v = \gamma-\delta$, $w = k$, $x = \delta$, $y = \ell-\alpha-\gamma+\delta$, $z = m+\alpha+\gamma$, and apply Lemma 4.3. □

In the next result we use the notation

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 1, \; \begin{bmatrix} n \\ k \end{bmatrix} = n(n-1)\cdots(n-k+1), \text{ so that } D_x^k(x^n) = \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k}.$$

Theorem 4.1 (Universal structure constants) *In $U(\mathbb{M})$ we have*

$$\begin{aligned}
&(a^i b^j c^k d^\ell e^m)(a^p b^q c^r d^s e^t) = \\
&\sum_{\alpha=0}^{\ell} \sum_{\beta=0}^{i} \sum_{\gamma=0}^{\beta} \sum_{\delta=0}^{\gamma} \sum_{\epsilon=0}^{j-\alpha-\delta} \sum_{\zeta=0}^{j-\alpha-\delta-\epsilon} \sum_{\eta=0}^{\ell-\alpha-(\gamma-\delta)} \sum_{\theta=0}^{\ell-\alpha-(\gamma-\delta)-\eta} \sum_{\lambda=0}^{\eta} \\
&(-1)^{\beta+\zeta+\ell-\alpha-\gamma-\eta} \frac{\alpha!\beta!\lambda!}{2^{\alpha+\gamma}3^{j-\epsilon-\zeta+\ell-\alpha-\eta-\theta}} \times \\
&\binom{\alpha}{\beta-\gamma} \binom{i}{\beta} \binom{j}{\alpha,\delta,\epsilon,\zeta} \binom{j-\alpha-\epsilon-\zeta}{\lambda} \binom{\ell}{\alpha,\gamma-\delta,\eta,\theta} \binom{\eta}{\lambda} \times \\
&\begin{bmatrix} p \\ j-\beta-\epsilon+\ell-\alpha-\eta-\theta \end{bmatrix} \begin{bmatrix} q \\ \ell-\alpha-\eta-\theta \end{bmatrix} \begin{bmatrix} r \\ \theta \end{bmatrix} \begin{bmatrix} s \\ j-\alpha-\epsilon-\zeta-\lambda \end{bmatrix} \times \\
&a^{i+p-j+\epsilon-\ell+\alpha+\eta+\theta} b^{\epsilon+q-\ell+\alpha+\eta+\theta} c^{\zeta+k+r-\theta} d^{\eta+s+\alpha+\epsilon+\zeta-j} e^{j-\alpha-\epsilon-\zeta+\ell-\eta+m+t}.
\end{aligned}$$

5 The universal alternative enveloping algebra

The algebra $U(\mathbb{M})$ is not power-associative, since

$$(abd, abd, abd) = \tfrac{1}{6}abcd^2e - \tfrac{1}{6}abde^2 - \tfrac{1}{6}c^2d^2e + \tfrac{11}{36}cde^2 - \tfrac{1}{12}e^3.$$

In this section we construct the maximal alternative quotient of $U(\mathbb{M})$.

Definition 5.1 The **alternator ideal** of a nonassociative algebra A is

$$I(A) = \langle\, (x, x, y),\, (y, x, x) \mid x, y \in A \,\rangle$$

If M is a Malcev algebra then we write $I(M) = I(U(M))$. The **universal alternative enveloping algebra** of M is $A(M) = U(M)/I(M)$.

Remark 5.1 The speciality problem for Malcev algebra is equivalent to the question of the injectivity of the natural mapping $M \to A(M)$.

In $U(\mathbb{M})$ we have the alternators

$$(ab, ab, d) = -\tfrac{1}{6}ce, \qquad (bd, bd, a^2) = \tfrac{1}{18}e^2.$$

Definition 5.2 We write $I = I(\mathbb{M})$ and $J = \text{ideal}\langle ce, e^2\rangle$; it is clear that $J \subseteq I$.

Lemma 5.1 *A basis of the ideal J consists of the set of monomials*

$$\{a^i b^j c^k d^\ell e^m \mid m \geq 2\} \cup \{a^i b^j c^k d^\ell e \mid k \geq 1\}.$$

Proof Linear independence is clear. Every monomial $a^i b^j c^k d^\ell e^m$ ($m \ge 2$) belongs to J. Since $cd^\ell e = \ell d^{\ell-1}e^2 + d^\ell ce$, every monomial $a^i b^j c^k d^\ell e$ ($k \ge 1$) belongs to J. Since the generators of J belong to this set, it suffices to show that the span of these monomials is an ideal in $U(\mathbb{M})$. This follows from Theorem 4.1: the product of $a^i b^j c^k d^\ell e^m$ and $a^p b^q c^r d^s e^t$ is a linear combination of monomials in which the exponents of e satisfy

$$j - \alpha - \epsilon - \zeta + \ell - \eta + m + t \ge \delta + \ell - \eta + m + t \ge \alpha + \gamma + m + t \ge m + t.$$

This exponent is 0 if and only if $m = t = 0$. For $(m,t) \ne (0,0)$, this exponent is 1 if and only if $(m,t) \in \{(1,0),(0,1)\}$, $\delta = j - \alpha - \epsilon - \zeta$, and $\delta + \ell - \eta = \alpha + \gamma$. This forces $\theta = 0$, and then the exponent of c is $k + r + \zeta \ge k + r$. It follows that the product of an arbitrary monomial with a monomial in the set is a linear combination of monomials in the set. □

We now determine the structure constants for the quotient algebra $U(\mathbb{M})/J$. We will show that $U(\mathbb{M})/J$ is an alternative algebra; it will then follow that $I = J$ and that $A(\mathbb{M}) = U(\mathbb{M})/J$. A spanning set for $U(\mathbb{M})/J$ (in fact a basis, by Lemma 5.1) consists of the cosets of the monomials $m = a^i b^j d^\ell e$ (type 1: the exponent of e is 1, and so the exponent of c is 0) and $m = a^i b^j c^k d^\ell$ (type 2: the exponent of e is 0). For type 1, since $\{c,d,e\}$ spans a Lie subalgebra of $\mathbb{M}$, we have $[c,d^\ell] = \ell d^{\ell-1}e$ and so $cd^\ell e = \ell d^{\ell-1}e^2 + d^\ell ce$; thus c cannot occur. In the next result we write m for the coset $m + J$.

Theorem 5.1 (Alternative structure constants) *In $U(\mathbb{M})/J$ we have*

$$(a^i b^j d^\ell e)(a^p b^q d^s e) = 0, \tag{5.1}$$

$$(a^i b^j c^k d^\ell)(a^p b^q d^s e) = \delta_{k0} a^{i+p} b^{j+q} d^{\ell+s} e, \tag{5.2}$$

$$(a^i b^j d^\ell e)(a^p b^q c^r d^s) = \delta_{r0} a^{i+p} b^{j+q} d^{\ell+s} e, \tag{5.3}$$

$$(a^i b^j c^k d^\ell)(a^p b^q c^r d^s) = \sum_{\mu=0}^{j} (-1)^\mu \mu! \binom{j}{\mu}\binom{p}{\mu} a^{i+p-\mu} b^{j+q-\mu} c^{k+r+\mu} d^{\ell+s} \tag{5.4}$$

$$+ \delta_{k0}\delta_{r0}\left(\tfrac{1}{6}ijs - \tfrac{1}{6}i\ell q + \tfrac{1}{2}j\ell p + \tfrac{1}{3}jps - \tfrac{1}{3}\ell pq\right) a^{i+p-1} b^{j+q-1} d^{\ell+s-1} e$$

$$- \delta_{k0}\delta_{r1} \ell a^{i+p} b^{j+q} d^{\ell+s-1} e.$$

Proof We need only the terms on the right side of Theorem 4.1 which are nonzero modulo J:

$$\begin{array}{ll} \text{either} & j-\alpha-\epsilon-\zeta+\ell-\eta+m+t = 0, \\ \text{or} & j-\alpha-\epsilon-\zeta+\ell-\eta+m+t = 1, \quad \zeta+k+r-\theta = 0. \end{array}$$

We write the exponent of e as the sum of three nonnegative terms:

$$(j-\alpha-\epsilon-\zeta-\delta) + (\ell-\eta+\delta-\alpha-\gamma) + (\alpha+\gamma+m+t). \tag{5.5}$$

Equation (5.1): We have $m = t = 1$: on the right side of Theorem 4.1, the exponent of e in every term is ≥ 2, so every term becomes zero in $U(\mathbb{M})/J$.
Equation (5.2): We have $m = 0$, $r = 0$ and $t = 1$: the exponent of e must be 1 and hence the exponent of c must be 0. Therefore

$$j-\alpha-\epsilon-\zeta-\delta = 0, \quad \ell-\eta+\delta-\alpha-\gamma = 0, \quad \alpha = 0, \quad \gamma = 0, \quad \zeta+k-\theta = 0.$$

The factor $\binom{\alpha}{\beta-\gamma}$ will be zero unless $\beta = 0$. From the sum on δ we get $\delta = 0$. We now get $j-\epsilon-\zeta = 0$ and $\ell-\eta = 0$, so $\zeta = j-\epsilon$ and $\eta = \ell$. From the sum on θ we get $\theta = 0$. Hence $\zeta+k = 0$ and so $\zeta = k = 0$; then $\epsilon = j$. The factor $\binom{j-\alpha-\epsilon-\zeta}{\lambda}$ will be zero unless $\lambda = 0$. Only one term remains.
Equation (5.3): Similar to Equation (5.2).
Equation (5.4): In Theorem 4.1 we have $m = 0$ and $t = 0$. We write the result as $T_0 + T_1$, collecting terms with the same exponent of e. For T_0, all three terms in (5.5) must be 0. Since $\alpha+\gamma = 0$ we get $\alpha = \gamma = 0$, and hence $\beta = \delta = 0$. Then $j-\epsilon-\zeta = 0$ and $\ell-\eta = 0$, which imply $\zeta = j-\epsilon$ and $\eta = \ell$. Then $\theta = 0$, and the factor $\binom{j-\alpha-\epsilon-\zeta}{\lambda}$ will be zero unless $\lambda = 0$. The remaining terms correspond to Proposition 3.1 with the new factor $d^{\ell+s}$:

$$\begin{aligned} T_0 &= \sum_{\epsilon=0}^{j} (-1)^{j-\epsilon} \binom{j}{\epsilon} \begin{bmatrix} p \\ j-\epsilon \end{bmatrix} a^{i-(j-\epsilon)+p} b^{\epsilon+q} c^{j-\epsilon+k+r} d^{\ell+s} \\ &= \sum_{\mu=0}^{j} (-1)^{\mu} \mu! \binom{j}{\mu} \binom{p}{\mu} a^{i+p-\mu} b^{j+q-\mu} c^{k+r+\mu} d^{\ell+s}. \end{aligned}$$

For T_1, we have

$$(j-\alpha-\epsilon-\zeta-\delta) + (\ell-\eta+\delta-\alpha-\gamma) + (\alpha+\gamma) = 1, \quad \zeta+k+r-\theta = 0.$$

The four cases

$$\begin{array}{lllll} (1) & j-\alpha-\epsilon-\zeta-\delta = 1, & \ell-\eta+\delta-\alpha-\gamma = 0, & \alpha = 0, & \gamma = 0 \\ (2) & j-\alpha-\epsilon-\zeta-\delta = 0, & \ell-\eta+\delta-\alpha-\gamma = 1, & \alpha = 0, & \gamma = 0 \\ (3) & j-\alpha-\epsilon-\zeta-\delta = 0, & \ell-\eta+\delta-\alpha-\gamma = 0, & \alpha = 1, & \gamma = 0 \\ (4) & j-\alpha-\epsilon-\zeta-\delta = 0, & \ell-\eta+\delta-\alpha-\gamma = 0, & \alpha = 0, & \gamma = 1 \end{array}$$

produce respectively 1, 2, 1, 2 terms, giving

$$\begin{aligned} T_1 = {} & \delta_{k0}\delta_{r0} \left(\tfrac{1}{6} ijs - \tfrac{1}{6} i\ell q + \tfrac{1}{2} j\ell p + \tfrac{1}{3} jps - \tfrac{1}{3} \ell pq \right) a^{i+p-1} b^{j+q-1} d^{\ell+s-1} e \\ & - \delta_{k0}\delta_{r1} \ell a^{i+p} b^{j+q} d^{\ell+s-1} e. \end{aligned}$$

This completes the proof. □

Corollary 5.1 *We have $(m, m', m'') = (m', m, m'') = (m', m'', m) = 0$ in $U(\mathbb{M})/J$, where m has type 1 and m', m'' are arbitrary.*

Proof This follows easily from Theorem 5.1. □

Corollary 5.2 *The associator of type 2 monomials is*

$$\begin{aligned} & (a^i b^j c^k d^\ell, a^p b^q c^r d^s, a^v b^w c^x d^y) = \\ & \delta_{k0}\delta_{r0}\delta_{x0} \tfrac{1}{6} \big(iqy - isw - jpy + jsv + \ell pw - \ell qv \big) a^{i+p+v-1} b^{j+q+w-1} d^{\ell+s+y-1} e. \end{aligned}$$

Proof We write $m_1 = a^i b^j c^k d^\ell$, $m_2 = a^p b^q c^r d^s$, $m_3 = a^v b^w c^x d^y$. We use the notation $N(a^i b^j c^k, a^p b^q c^r)$ for the associative multiplication in the enveloping algebra of the nilpotent Lie subalgebra of $\mathbb{M}$ with basis $\{a, b, c\}$ (compare Proposition 3.1); we extend N in the obvious way to linear combinations of monomials. Theorem 5.1 shows that

$$(a^i b^j c^k d^\ell)(a^p b^q c^r d^s) = N(a^i b^j c^k, a^p b^q c^r) d^{\ell+s} + T_1(i, j, k, \ell, p, q, r, s),$$

where T_1 denotes the terms involving e. We define

$$C(i, j, \ell, p, q, s) = \tfrac{1}{6} ijs - \tfrac{1}{6} i\ell q + \tfrac{1}{2} j\ell p + \tfrac{1}{3} jps - \tfrac{1}{3} \ell pq.$$

We obtain

$$\begin{aligned}
(m_1 m_2) m_3 &= \big((a^i b^j c^k d^\ell)(a^p b^q c^r d^s)\big)(a^v b^w c^x d^y) \\
&= \big(N(a^i b^j c^k, a^p b^q c^r) d^{\ell+s}\big)(a^v b^w c^x d^y) + T_1(i, j, k, \ell, p, q, r, s)(a^v b^w c^x d^y) \\
&= N(N(a^i b^j c^k, a^p b^q c^r), a^v b^w c^x) d^{\ell+s+y} \\
&\quad + \delta_{k0}\delta_{r0}\delta_{x0} \big[C(i{+}p, j{+}q, \ell{+}s, v, w, y) + C(i, j, \ell, p, q, s)\big] \times \\
&\qquad a^{i+p+v-1} b^{j+q+w-1} d^{\ell+s+y-1} e \\
&\quad - \delta_{k0}\delta_{r0}\delta_{x1} (\ell{+}s) a^{i+p+v} b^{j+q+w} d^{\ell+s+y-1} e \\
&\quad - \delta_{k0}\delta_{r1}\delta_{x0} \ell a^{i+p+v} b^{j+q+w} d^{\ell+s+y-1} e, \\
m_1(m_2 m_3) &= (a^i b^j c^k d^\ell)\big((a^p b^q c^r d^s)(a^v b^w c^x d^y)\big) \\
&= (a^i b^j c^k d^\ell)\big(N(a^p b^q c^r, a^v b^w c^x) d^{s+y}\big) + (a^i b^j c^k d^\ell) T_1(p, q, r, s, v, w, x, y) \\
&= N(a^i b^j c^k, N(a^p b^q c^r, a^v b^w c^x)) d^{\ell+s+y} \\
&\quad + \delta_{k0}\delta_{r0}\delta_{x0} \big[C(i, j, \ell, p{+}v, q{+}w, s{+}y) + \ell qv + C(p, q, s, v, w, y)\big] \times \\
&\qquad a^{i+p+v-1} b^{j+q+w-1} d^{\ell+s+y-1} e \\
&\quad - \delta_{k0}\delta_{r1}\delta_{x0} \ell a^{i+p+v} b^{j+q+w} d^{\ell+s+y-1} e \\
&\quad - \delta_{k0}\delta_{r0}\delta_{x1} (\ell{+}s) a^{i+p+v} b^{j+q+w} d^{\ell+s+y-1} e.
\end{aligned}$$

The associator (m_1, m_2, m_3) is therefore

$$\begin{aligned}
\delta_{k0}\delta_{r0}\delta_{x0} \big[&C(i{+}p, j{+}q, \ell{+}s, v, w, y) + C(i, j, \ell, p, q, s) \\
&- C(i, j, \ell, p{+}v, q{+}w, s{+}y) - \ell qv - C(p, q, s, v, w, y)\big] \times \\
&a^{i+p+v-1} b^{j+q+w-1} d^{\ell+s+y-1} e.
\end{aligned}$$

The expression in square brackets simplifies as required. □

Corollary 5.3 *The algebra* $U(\mathbb{M})/J$ *is alternative.*

Proof Corollaries 5.1 and 5.2 show that the associator alternates. □

Corollary 5.4 *The alternator ideal* $I(\mathbb{M})$ *is generated by* ce *and* e^2.

Corollary 5.5 *The universal alternative enveloping algebra* $A(\mathbb{M})$ *is isomorphic to the algebra with basis* $\{\, a^i b^j c^k d^\ell,\ a^i b^j d^\ell e \mid i, j, k, \ell \geq 0 \,\}$ *and structure constants of Proposition 5.1.*

Proof This follows from Lemma 5.1.

Corollary 5.6 *The nilpotent non-Lie Malcev algebra* $\mathbb{M}$ *is special: it is isomorphic to a subalgebra of* A^- *for some alternative algebra* A.

Proof The alternator ideal contains no elements of degree 1, and so the canonical map from $\mathbb{M}$ to $A(\mathbb{M})$ is injective. Speciality of $\mathbb{M}$ also follows from Pchelintsev [5]. □

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