NONHOMOGENEOUS SUBALGEBRAS OF LIE AND SPECIAL JORDAN SUPERALGEBRAS

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Abstract. We consider polynomial identities satisfied by nonhomogeneous subalgebras of Lie and special Jordan superalgebras: we ignore the grading and regard the superalgebra as an ordinary algebra. The Lie case has been studied by Volichenko and Baranov: they found identities in degrees 3, 4 and 5 which imply all the identities in degrees ≤ 6. We simplify their identities in degree 5, and show that there are no new identities in degree 7. The Jordan case has not previously been studied: we find identities in degrees 3, 4, 5 and 6 which imply all the identities in degrees ≤ 6, and demonstrate the existence of further new identities in degree 7. Our proofs depend on computer algebra; we use the representation theory of the symmetric group, the Hermite normal form of an integer matrix, the LLL algorithm for lattice basis reduction, and the Chinese remainder theorem.

1. Introduction

In this paper, an **algebra** is a vector space A over $\mathbb{Q}$ with a bilinear product $A \times A \to A$ denoted $(a, b) \mapsto ab$. A **superalgebra** is a graded vector space $A = A_0 \oplus A_1$ with graded bilinear products $A_\alpha \times A_\beta \mapsto A_{\alpha+\beta}$ where addition of subscripts is modulo 2; these extend in the obvious way to a product on A. A **subsuperalgebra** is a graded subspace $B \subseteq A$ (that is, $B = B_0 \oplus B_1$ where $B_0 = B \cap A_0$ and $B_1 = B \cap A_1$) which is closed under the graded products. A **nonhomogeneous subalgebra** is an arbitrary subspace $B \subseteq A$ (that is, B is not necessarily graded) which is closed under the product; here we regard A as an ordinary algebra. A superalgebra is **associative** if it is associative as an ordinary algebra.

Volichenko (unpublished) investigated polynomial identities satisfied by all nonhomogeneous subalgebras of Lie superalgebras (Volichenko algebras), and believed that his identities implied all the identities in degrees ≤ 5. However, Baranov [1] found an identity in degree 5 that does not follow from Volichenko's identities, and stated that Volichenko's identities together with his new identity imply all the identities in degrees ≤ 6. (For a classification of simple finite dimensional Volichenko algebras, see Leites and Serganova [4, 5, 7].)

In §§5–9 we use the representation theory of the symmetric group to verify that the identities of Volichenko and Baranov in degrees ≤ 5 imply all the identities in degrees ≤ 6 for Volichenko algebras; we further show that there are no new identities in degree 7. We find a new identity that is equivalent to the two previously known identities in degree 5, and show that every identity of degree ≤ 7 follows from 11 irreducible identities in degrees ≤ 5.

In §§10–14, we investigate, for the first time, the polynomial identities satisfied by all nonhomogeneous subalgebras of special Jordan superalgebras. We find identities in degrees 3, 4, 5 and 6, and demonstrate the existence of many further identities in degree 7. We discover new identities using the Hermite normal form of an integer matrix, the LLL algorithm for lattice basis reduction, and the Chinese remainder theorem. We show that every identity of degree ≤ 7 follows from 101 irreducible identities in degrees ≤ 7.

2. FREE NONHOMOGENEOUS SUBALGEBRAS

Definition 2.1. Let $X = \{a_i, b_i : i = 1, \dots, n\}$ be a set of $2n$ symbols. Let A be the free associative algebra generated by X over the field $\mathbb{Q}$. A basis for A as a vector space consists of the monomials $x = x_1 \cdots x_d$ ($d \geq 0$) where $x_j \in X$ for $j = 1, \dots, d$. For the monomial $x = x_1 \cdots x_d$ we define the **parity vector** $\gamma(x) = (\gamma_1, \gamma_2, \dots, \gamma_d)$ by $\gamma_j = 0$ (respectively $\gamma_j = 1$) if $x_j = a_i$ (respectively $x_j = b_i$) for some i; the components of the parity vector are attached to the positions, not to the symbols. We define the **parity** $|x|$ to be $\gamma_1 + \cdots + \gamma_d$ (mod 2). For $\alpha \in \{0,1\}$ we define A_α to be the subspace with basis $\{x : |x| = \alpha\}$; then $A = A_0 \oplus A_1$ (vector space direct sum) and $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$. We call A the **free associative superalgebra** on **even** generators $a_1, \dots, a_n$ and **odd** generators $b_1, \dots, b_n$.

Definition 2.2. For $x \in A_\alpha$ and $y \in A_\beta$ we define two **nonassociative operations**, the **Lie superbracket** $\{x, y\}$ and the **Jordan superproduct** $x \bullet y$, as follows:

$$\{x, y\} = xy - (-1)^{\alpha\beta} yx \,, \qquad x \bullet y = xy + (-1)^{\alpha\beta} yx \,.$$

Both operations respect the grading and extend bilinearly to A. We write A^- (respectively A^+) for the algebra with the same underlying vector space as A but with the operation $\{x, y\}$ (respectively $x \bullet y$). The subalgebra of A^- (respectively A^+) generated by X is the **free Lie superalgebra** (respectively the **free special Jordan superalgebra**) on n odd and n even generators.

Definition 2.3. Let $c_i = a_i + b_i$ for $1 \leq i \leq n$. We write V_n for the subalgebra of A^- generated by c_1, ..., c_n. This can be regarded as the **free nonhomogeneous subalgebra of a Lie superalgebra**; it is also called the **free Volichenko algebra**. We write W_n for the subalgebra of A^+ generated by c_1, ..., c_n. This can be regarded as the **free nonhomogeneous subalgebra of a special Jordan superalgebra**.

Definition 2.4. In a nonassociative algebra with product ab we define

$$\begin{aligned} &[a, b] = ab - ba, \qquad a \circ b = ab + ba, \qquad (a, b, c) = (ab)c - a(bc), \\ &j(a, b, c) = [a, b \circ c] + [b, c \circ a] + [c, a \circ b]. \end{aligned}$$

Theorem 2.5. Baranov [1]. *The free Volichenko algebras V_n satisfy these independent multilinear identities, where we write xy for $\{x, y\}$:*

$$[[a, b], c] + [[b, c], a] + [[c, a], b] \equiv 0 \,, \tag{1}$$

$$(a \circ b) \circ c \equiv 0 \,, \tag{2}$$

$$[a \circ b, c \circ d] - [a \circ b, c] \circ d - [a \circ b, d] \circ c \equiv 0 \,, \tag{3}$$

$$j(a, b, c) \circ d \equiv 0 \,, \tag{4}$$

$$[j(a, b, c), d \circ e] - [j(a, b, c), d] \circ e - [j(a, b, c), e] \circ d \equiv 0 \,. \tag{5}$$

Theorem 2.6. Baranov [1]. *The algebras V_n satisfy an identity in degree 5 which does not follow from the identities of Theorem 2.5.*

3. NONASSOCIATIVE POLYNOMIALS

Definition 3.1. By a (multilinear) **associative monomial** of degree n we mean a permutation of $x_1 x_2 \cdots x_n$; the ordering is the lexicographical ordering of the permutations. By an **association type** in degree n we mean a valid placement of $n-1$ pairs of parentheses into

n factors (we usually omit the outermost pair); the ordering is defined inductively as follows: x_1x_2 precedes $x'_1x'_2$ if either (i) $\deg(x_1) > \deg(x'_1)$, or (ii) $\deg(x_1) = \deg(x'_1)$ and x_1 precedes x'_1, or (iii) $x_1 = x'_1$ and x_2 precedes x'_2. By a **basic nonassociative monomial** of degree n we mean $x_1 \cdots x_n$ with an association type. By a (multilinear) **nonassociative monomial** of degree n we mean an associative monomial with an association type; the ordering is first by the association type and then by the associative monomial.

Lemma 3.2. *The number of association types (and the number of basic nonassociative monomials) in degree n is the (shifted) Catalan number C_n, and the number of nonassociative monomials in degree n is $n!C_n$:*

$$C_n = \frac{1}{n}\binom{2n-2}{n-1}, \qquad n!C_n = \frac{(2n-2)!}{(n-1)!}$$

Example 3.3. In degree 3, there are 2 association types and 12 nonassociative monomials:

$$(x_1x_2)x_3,\ (x_1x_3)x_2,\ (x_2x_1)x_3,\ (x_2x_3)x_1,\ (x_3x_1)x_2,\ (x_3x_2)x_1,$$
$$x_1(x_2x_3),\ x_1(x_3x_2),\ x_2(x_1x_3),\ x_2(x_3x_1),\ x_3(x_1x_2),\ x_3(x_2x_1).$$

In degree 4, there are 5 association types and 120 nonassociative monomials:

$$((x_1x_2)x_3)x_4, \dots, (x_1(x_2x_3))x_4, \dots, (x_1x_2)(x_3x_4), \dots, x_1((x_2x_3)x_4), \dots, x_1(x_2(x_3x_4)), \dots.$$

In degree 5, there are 14 association types and 1680 nonassociative monomials:

$$(((x_1x_2)x_3)x_4)x_5, \dots, ((x_1(x_2x_3))x_4)x_5, \dots, ((x_1x_2)(x_3x_4))x_5, \dots, (x_1((x_2x_3)x_4))x_5, \dots,$$
$$(x_1(x_2(x_3x_4)))x_5, \dots, ((x_1x_2)x_3)(x_4x_5), \dots, (x_1(x_2x_3))(x_4x_5), \dots, (x_1x_2)((x_3x_4)x_5), \dots,$$
$$(x_1x_2)(x_3(x_4x_5)), \dots, x_1(((x_2x_3)x_4)x_5), \dots, x_1((x_2(x_3x_4))x_5), \dots, x_1((x_2x_3)(x_4x_5)), \dots,$$
$$x_1(x_2((x_3x_4)x_5)), \dots, x_1(x_2(x_3(x_4x_5))), \dots.$$

Definition 3.4. Let P_n be the (multilinear) **nonassociative polynomials** of degree n: the vector space over $\mathbb{Q}$ whose basis is the nonassociative monomials of degree n. We define an **action of the symmetric group** S_n on the nonassociative monomials by permuting x_1, x_2, …, x_n and leaving the association type unchanged:

$$\sigma \,.\, x_1x_2 \cdots x_n = x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)} \text{ for } \sigma \in S_n \text{ (parentheses omitted).}$$

This action extends linearly to P_n, making P_n into an S_n-module isomorphic to the direct sum of C_n copies of the group algebra $\mathbb{Q}S_n$. We say that $I \in P_n$ is **irreducible** if it generates a simple S_n-submodule of P_n.

Lemma 3.5. *If A is a nonassociative algebra, and $Q \subseteq P_n$ consists of the polynomial identities of degree n satisfied by A, then Q is an S_n-submodule of P_n.*

Definition 3.6. Let $I(x_1, x_2, \dots, x_n)$ be a nonassociative polynomial of degree n. We obtain from I a set of $n+2$ distinct **lifted identities** in degree $n+1$; we introduce the new symbol x_{n+1} and then perform n substitutions and 2 multiplications:

$$I(x_1x_{n+1}, x_2, \dots, x_n), \qquad I(x_1, x_2x_{n+1}, \dots, x_n), \qquad \dots, \qquad I(x_1, x_2, \dots, x_nx_{n+1}),$$
$$x_{n+1}I(x_1, x_2, \dots, x_n), \qquad I(x_1, x_2, \dots, x_n)x_{n+1}.$$

Lemma 3.7. *If $I \in P_n$ is a polynomial identity of degree n then the subspace of P_{n+1} consisting of the identities which follow from I is generated as an S_n-module by the lifted identities obtained from I.*

Definition 3.8. Fix a nonassociative operation ($\{x,y\}$ or $x \bullet y$), and denote it by xy. Let x be the basic nonassociative monomial for association type t in degree n. For $i = 1, \dots, n$ we set $x_i = a_i + b_i$ in the free associative superalgebra A, and expand x using the nonassociative operation. The result is an element $e_t \in A$ which is homogeneous of degree n; we call e_t the **expansion of association type** t. Each term of e_t consists of a sign $\epsilon = \pm 1$, a permutation $\pi \in S_n$, and a parity vector $\alpha \in \mathbb{Z}_2^n$:

$$(\epsilon, \pi, \gamma) \;\longleftrightarrow\; \epsilon\, y_{\pi(1)}^{(\gamma_1)} \cdots y_{\pi(n)}^{(\gamma_n)} \text{ where } y_i^{(0)} = a_i, \; y_i^{(1)} = b_i \,.$$

Remark 3.9. We order the parity vectors as binary numerals: $\gamma = (\gamma_1, \dots, \gamma_n)$ has index $(\gamma_1 2^{n-1} + \gamma_2 2^{n-2} + \cdots + \gamma_n) + 1$. We collect the terms in e_t with the same parity vector,

$$e_t = \sum_{\gamma=(0,\dots,0)}^{(1,\dots,1)} f_t^{(\gamma)}.$$

(Each $f_t^{(\gamma)}$ contains 2^{n-1} terms; altogether e_t contains 2^{2n-1} terms.) Within each $f_t^{(\gamma)}$, the terms differ only by a permutation of the subscripts, since the parity vector determines whether each position has a_i or b_i. Thus $f_t^{(\gamma)}$ can be regarded as an element of $\mathbb{Q}S_n$, and e_t is an element of $(\mathbb{Q}S_n)^{2^n}$.

4. COMPUTATIONAL METHODS

For a partition λ of n, let $R_\lambda \colon \mathbb{Q}S_n \to M_d(\mathbb{Q})$ where $d = d_\lambda$ be the corresponding irreducible representation of S_n by $d \times d$ matrices; we regard R_λ as projection onto a simple ideal in $\mathbb{Q}S_n$.

Definition 4.1. We write $\lambda = (n_1, \dots, n_k)$ with $n = n_1 + \cdots + n_k$ and $n \geq n_1 \geq \cdots \geq n_k \geq 1$. The **frame** for λ consists of n empty boxes arranged in k left-justified rows with n_i boxes in row i. A **tableau** consists of the boxes filled with the integers $1, \dots, n$; a tableau is **standard** if the integers increase from left to right and from top to bottom. We order the standard tableaux lexicographically as $T_1, \dots, T_d$ where $d = d_\lambda$. We define the **semi-representation matrix** A_π^λ of size $d \times d$ by giving an algorithm for computing its (i,j) entry following Clifton [2]. Let T_i be the standard tableau corresponding to row i, and let πT_j be the (possibly non-standard) tableau obtained by applying π to the integers in the standard tableau T_j corresponding to column j. If there exist two integers which appear both in a column of T_i and in a row of πT_j, then we set $(A_\pi^\lambda)_{ij} = 0$. Otherwise, there is a permutation $\sigma \in S_n$ which leaves the columns of T_i fixed as sets and moves the integers in T_i into the rows they occupy in πT_j; in this case we set $(A_\pi^\lambda)_{ij} = \operatorname{sign}(\sigma)$.

Lemma 4.2. Clifton [2]. *The semi-representation matrix A_{id}^λ for the identity permutation may not be the identity matrix; however it is invertible.*

Theorem 4.3. Clifton [2]. *For every partition λ of n and permutation $\pi \in S_n$, the integral representation matrix for π in the irreducible representation corresponding to λ is*

$$R_\lambda(\pi) = (A_{\mathrm{id}}^\lambda)^{-1} A_\pi^\lambda \,.$$

Remark 4.4. Together, Definition 4.1 and Theorem 4.3 give an algorithm to compute the isomorphism ϕ from the group algebra to the direct sum of matrix algebras:

$$\phi \colon \mathbb{Q}S_n \longrightarrow \bigoplus_{k=1}^{\nu} M_{d_{\lambda_k}}(\mathbb{Q}) \,, \qquad \phi(\pi) = \big(R_{\lambda_1}(\pi), \dots, R_{\lambda_\nu}(\pi) \big) \,,$$

where ν is the number of partitions of n. To compute ϕ^{-1}, we need to find the group algebra element corresponding to the matrix unit E^{λ}_{ij} in the summand for partition λ. Let R (resp. C) be the subgroup of S_n which leaves the rows (resp. columns) of T_i fixed as sets. Let s_{ij} be the permutation for which $s_{ij}T_i = T_j$. We define elements $D_{ij} \in \mathbb{Q}S_n$ as follows:

$$D_{ii} = \frac{d}{n!} \sum_{\sigma \in R} \sum_{\tau \in C} \operatorname{sign}(\tau)\, \sigma\tau, \qquad D_{ij} = D_{ii} s_{ij}^{-1}.$$

Let (a_{ij}) be the matrix $(A^{\lambda}_{\mathrm{id}})^{-1}$. We then have

$$\phi^{-1}(E^{\lambda}_{ij}) = \sum_{k=1}^{d_\lambda} a_{jk} D_{ik}.$$

Definition 4.5. Given a partition λ of n, we represent the expansions $e_i(x)$ of the association types in degree n by a matrix E_λ, called the **expansion representation matrix** for λ. The matrix E_λ consists of $d \times d$ blocks ($d = d_\lambda$), with C_n blocks vertically and $2^n + C_n$ blocks horizontally; it contains a left part with 2^n columns of blocks (corresponding to the parity vectors), and a right part with C_n columns of blocks (corresponding to the association types). For $i = 1, \dots, C_n$ we set the (i,i) block of the right part to $-I_d$ representing $-m_i$ where m_i is the basic nonassociative monomial in association type i; for $i \neq j$ we set the (i,j) block of the right part to zero. For $i = 1, \dots, C_n$ and $j = 1, \dots, 2^n$ we set the (i,j) block of the left part to $R_\lambda\big(f_i^{(\gamma)}\big)$ representing the group algebra element $f_i^{(\gamma)}$, where γ is the parity vector with index j. Thus the i-th row of $d \times d$ blocks in E_λ represents $e_i(x) - m_i(x)$. Figure 1 shows the block structure of E_λ for $n = 3$.

parity vectors					association types	
$\alpha = 000$	$\alpha = 001$	$\cdots$	$\alpha = 110$	$\alpha = 111$	$t_1 = (x_1x_2)x_3$	$t_2 = x_1(x_2x_3)$
$R_\lambda\big(f_1^{000}\big)$	$R_\lambda\big(f_1^{001}\big)$	$\cdots$	$R_\lambda\big(f_1^{110}\big)$	$R_\lambda\big(f_1^{111}\big)$	$-I_d$	O_d
$R_\lambda\big(f_2^{000}\big)$	$R_\lambda\big(f_2^{001}\big)$	$\cdots$	$R_\lambda\big(f_2^{110}\big)$	$R_\lambda\big(f_2^{111}\big)$	O_d	$-I_d$

Figure 1. The expansion representation matrix E_λ in degree 3

Definition 4.6. To each partition λ of n there corresponds a simple ideal $M_\lambda \subseteq \mathbb{Q}S_n$ isomorphic to the algebra of $d \times d$ matrices over $\mathbb{Q}$ where $d = d_\lambda$. Since P_n is isomorphic to the direct sum of C_n copies of $\mathbb{Q}S_n$, we write $\bigoplus_t M_\lambda^{(t)}$ for the direct sum over all association types t of the C_n copies of M_λ; clearly this is an S_n-submodule of P_n.

Definition 4.7. Given a nonassociative operation $*$, we write $\mathrm{Id}_n(*)$ for the S_n-submodule of P_n consisting of the polynomial identities of degree n satisfied by $*$.

Lemma 4.8. *Let $*$ be either the Lie superbracket or the Jordan superproduct. Let* $\mathrm{RCF}(E_\lambda)$ *be the row canonical form of E_λ. If there are any rows of* $\mathrm{RCF}(E_\lambda)$ *which have leading 1s in the right part of the matrix, then these rows represent irreducible polynomial identities in degree n satisfied by $*$. Furthermore, these rows are a minimal set of generators for* $\bigoplus M_\lambda^{(t)} \cap \mathrm{Id}_n(*)$ *as an S_n-submodule of P_n.*

Definition 4.9. The number of rows of $\mathrm{RCF}(E_\lambda)$ which have leading 1s in the right part of the matrix will be called the **expansion rank** for partition λ of the operation $*$, and denoted e-rank(λ).

Definition 4.10. Suppose we have a minimal set of irreducible generators for $\bigoplus M_\lambda^{(t)} \cap \mathrm{Id}_n(*)$, the polynomial identities satisfied by $*$ in degree n. We need to determine which generators are consequences of identities satisfied by $*$ in degree $n-1$. Suppose that we have a list of generators of these latter identities, say $I_k(x_1,\dots,x_{n-1})$ for $k=1,\dots,\ell$. From I_1, …, I_ℓ we obtain the set J_1, …, $J_{\ell(n+1)}$ of lifted identities in degree n. Given a partition λ of n, we represent the lifted identities in a matrix L_λ, called the **lifted representation matrix** for λ. The matrix L_λ consists of $d \times d$ blocks with $\ell(n+1)$ blocks vertically and C_n blocks horizontally. For $i=1,\dots,\ell(n+1)$ and $j=1,\dots,C_n$ we set the (i,j) block to $R_\lambda\big(J_{i,j}\big)$ where $J_{i,j}$ consists of the terms of J_i in association type j. The i-th row of $d \times d$ blocks in L_λ represents the projection of J_i onto the sum of the simple ideals corresponding to λ.

Lemma 4.11. *Let $*$ be either the Lie superbracket or the Jordan superproduct. Let $\mathrm{RCF}(L_\lambda)$ be the row canonical form of L_λ. The nonzero rows of $\mathrm{RCF}(L_\lambda)$ represent irreducible polynomial identities in degree n satisfied by $*$ which are consequences of the identities of degree $n-1$. These rows are a minimal set of generators for the S_n-submodule of the intersection $\bigoplus M_\lambda^{(t)} \cap \mathrm{Id}_n(*)$ consisting of the consequences of the identities of degree $n-1$.*

Definition 4.12. The number of nonzero rows of $\mathrm{RCF}(L_\lambda)$ will be called the **lifted rank** for partition λ of the operation $*$, and denoted ℓ-rank(λ).

Lemma 4.13. *Let $*$ be either the Lie superbracket or the Jordan superproduct. We have ℓ-rank$(\lambda) \le e$-rank(λ) for all n and for all partitions λ. If the ranks are equal, then there are no new polynomial identities for λ. If the inequality is strict, then there are new identities, and the difference e-rank$(\lambda) - \ell$-rank(λ) is the number of new irreducible identities for λ.*

Remark 4.14. Even for a sparse matrix with small integer entries, rational entries with very large numerators and denominators can appear during the computation of the RCF. This becomes a serious problem for larger values of the degree n of the polynomial identities. We can reduce the time and space required by the computations using modular arithmetic. If we choose a prime $p > n$, then the following lemma guarantees that the ranks will be the same over $\mathbb{F}_p$ and over $\mathbb{Q}$.

Lemma 4.15. *Let $T_0 \subseteq (\mathbb{Q}S_n)^\ell$ be the left ideal generated by the elements $I_1,\dots,I_k \in (\mathbb{Z}S_n)^\ell$. Let p be a prime number with $p > n$, and let $\mathbb{F}_p$ be the field with p elements. Let $I_1',\dots,I_k' \in (\mathbb{F}_pS_n)^\ell$ be the elements obtained by reducing the coefficients of $I_1,\dots,I_k$ modulo p, and let $T_p \subseteq (\mathbb{F}_pS_n)^\ell$ be the left ideal generated by $I_1',\dots,I_k'$. Then $\dim_{\mathbb{Q}} T_0 = \dim_{\mathbb{F}_p} T_p$.*

Proof. Let λ be a partition of n and let $d = d_\lambda$ be the dimension of the corresponding irreducible representation of S_n. For $i=1,\dots,k$ and $j=1,\dots,\ell$ let $I_{i,j}$ consists of the terms of I_i in component j. Let $G_0^{(\lambda)}$ be the $kd \times \ell d$ matrix over $\mathbb{Z}$ consisting of $d \times d$ blocks in which block (i,j) is the integral representation matrix $R_\lambda(I_{i,j})$ computed according to the algorithm of Definition 4.1. Since $\mathbb{Q}S_n$ is semisimple, we have

$$\dim_{\mathbb{Q}} T_0 = \sum_\lambda \mathrm{rank}_{\mathbb{Q}} G_0^{(\lambda)}.$$

Let $G_p^{(\lambda)}$ be the reduction of $G_0^{(\lambda)}$ modulo p; then since $\mathbb{F}_p S_n$ is semisimple, we have

$$\dim_{\mathbb{F}_p} T_p = \sum_{\lambda} \mathrm{rank}_{\mathbb{F}_p} G_p^{(\lambda)}.$$

The classical isomorphism of $\mathbb{Q}S_n$ with a direct product of full matrix algebras expresses the matrix units as linear combinations of permutations in which the denominators of the coefficients are divisors of $n!$. It follows that the same group algebra elements are defined over $\mathbb{F}_p$, and that they satisfy the same matrix unit relations,

$$E_{ij} E_{k\ell} = \delta_{jk} E_{i\ell}.$$

Since computation of the row canonical form can be expressed as a sequence of left multiplications by elementary matrices, it follows that

$$\mathrm{rank}_{\mathbb{Q}} G_0^{(\lambda)} = \mathrm{rank}_{\mathbb{F}_p} G_p^{(\lambda)},$$

and this completes the proof. □

Definition 4.16. We now consider monomials which are not necessarily multilinear. We want polynomial identities in degree n for the operation $*$ in which each term consists of a scalar coefficient, an association type, and some permutation of $y_1^{n_1} \cdots y_k^{n_k}$ where $\mu = (n_1, \dots, n_k)$ is a partition of n; that is, y_i occurs n_i times for $i = 1, \dots, k$. The number of these **nonlinear nonassociative monomials** is

$$t = C_n \binom{n}{n_1, \dots, n_k}.$$

We expand each nonassociative monomial by setting $y_i = a_i + b_i$ and using $*$. Each term in the expansion consists of a permutation of $y_1^{n_1} \cdots y_k^{n_k}$ with a parity vector. The number of these **nonlinear associative monomials** is

$$s = 2^n \binom{n}{n_1, \dots, n_k}.$$

The **combinatorial expansion matrix** X_μ has size $s \times t$; entry $(X_\mu)_{ij}$ is the coefficient of the i-th associative monomial in the expansion of the j-th nonassociative monomial. The polynomial identities are the non-trivial linear dependence relations on the columns of X_μ: that is, the nonzero elements of the nullspace. Since the entries of X_μ are in $\mathbb{Z}$, we consider the **nullspace lattice** $N = \{ B \in \mathbb{Z}^n \mid X_\mu B = O \}$.

Remark 4.17. Let r be the rank of X_μ; then $t-r$ is the rank of N. We want a lattice basis consisting of short vectors: $I_1, \dots I_{t-r} \in N$ such that $N = \mathbb{Z}I_1 \oplus \cdots \oplus \mathbb{Z}I_{t-r}$ and the Euclidean norms $\|I_i\|$ are small. We can find such a basis using the Hermite normal form [3] for an integer matrix and the LLL algorithm [6] for lattice basis reduction. The first step is to compute matrices H ($t \times s$, unique) and U ($t \times t$, not unique): H is the HNF of the transpose X_μ^t, and U is an integer matrix with $\det(U) = \pm 1$ and $UX_\mu^t = H$. Since H has rank r, the last $t-r$ rows of H are zero, and so the last $t-r$ rows of U are in the left nullspace of X_μ^t (the right nullspace of X_μ). The last $t-r$ rows of U are a lattice basis for N. The second step is to use the LLL algorithm to reduce this lattice basis. We use the Maple package `LinearAlgebra` and the command

```
U := HermiteForm( Tranpose(E), output='U', method='integer[reduced]' ):
```

We take the output and reduce the basis again, using our own Maple program to implement the LLL algorithm with the limiting value 1 of the reduction parameter. The third step is to sort these vectors by increasing norm, linearize the corresponding polynomial identities, and then use the representation theory of the symmetric group to process these identities for a given partition λ. We find the simplest identity which increases the rank beyond the rank of the identities already known.

5. LIE SUPERBRACKET: DEGREE 3

In §§5–9 we simplify and extend the results of Volichenko (unpublished) and Baranov [1] on polynomial identities for nonhomogeneous subalgebras of Lie superalgebras.

Lemma 5.1. *For the Lie superbracket in degree 3 the expansions of the association types* $t_1 = \{\{c_1, c_2\}, c_3\}$ *and* $t_2 = \{c_1, \{c_2, c_3\}\}$ *have the form* $e_i(x) = \sum_{\gamma=000}^{111} f_i^\gamma$ $(i = 1, 2)$ *where*

$$\begin{aligned}
f_1^{000} &= a_1a_2a_3 - a_2a_1a_3 - a_3a_1a_2 + a_3a_2a_1, & f_1^{001} &= a_1a_2b_3 - a_2a_1b_3 - a_3a_1b_2 + a_3a_2b_1,\\
f_1^{010} &= a_1b_2a_3 - a_2b_1a_3 - a_3b_1a_2 + a_3b_2a_1, & f_1^{011} &= a_1b_2b_3 - a_2b_1b_3 - a_3b_1b_2 - a_3b_2b_1,\\
f_1^{100} &= b_1a_2a_3 - b_2a_1a_3 - b_3a_1a_2 + b_3a_2a_1, & f_1^{101} &= b_1a_2b_3 - b_2a_1b_3 + b_3a_1b_2 - b_3a_2b_1,\\
f_1^{110} &= b_1b_2a_3 + b_2b_1a_3 + b_3b_1a_2 - b_3b_2a_1, & f_1^{111} &= b_1b_2b_3 + b_2b_1b_3 - b_3b_1b_2 - b_3b_2b_1,\\
f_2^{000} &= a_1a_2a_3 - a_1a_3a_2 - a_2a_3a_1 + a_3a_2a_1, & f_2^{001} &= a_1a_2b_3 - a_1a_3b_2 - a_2a_3b_1 + a_3a_2b_1,\\
f_2^{010} &= a_1b_2a_3 - a_1b_3a_2 - a_2b_3a_1 + a_3b_2a_1, & f_2^{011} &= a_1b_2b_3 + a_1b_3b_2 + a_2b_3b_1 - a_3b_2b_1,\\
f_2^{100} &= b_1a_2a_3 - b_1a_3a_2 - b_2a_3a_1 + b_3a_2a_1, & f_2^{101} &= b_1a_2b_3 - b_1a_3b_2 + b_2a_3b_1 - b_3a_2b_1,\\
f_2^{110} &= b_1b_2a_3 - b_1b_3a_2 - b_2b_3a_1 - b_3b_2a_1, & f_2^{111} &= b_1b_2b_3 + b_1b_3b_2 - b_2b_3b_1 - b_3b_2b_1.
\end{aligned}$$

Proof. We give details only for t_1:

$$\begin{aligned}
e_1(x) &= \{\{a_1 + b_1, a_2 + b_2\}, a_3 + b_3\}\\
&= \{\{a_1, a_2\} + \{a_1, b_2\} + \{b_1, a_2\} + \{b_1, b_2\}, a_3 + b_3\}\\
&= \{(a_1a_2 - a_2a_1) + (a_1b_2 - b_2a_1) + (b_1a_2 - a_2b_1) + (b_1b_2 + b_2b_1), a_3 + b_3\}\\
&= \{a_1a_2, a_3\} - \{a_2a_1, a_3\} + \{a_1b_2, a_3\} - \{b_2a_1, a_3\} + \{b_1a_2, a_3\} - \{a_2b_1, a_3\}\\
&\quad + \{b_1b_2, a_3\} + \{b_2b_1, a_3\} + \{a_1a_2, b_3\} - \{a_2a_1, b_3\} + \{a_1b_2, b_3\} - \{b_2a_1, b_3\}\\
&\quad + \{b_1a_2, b_3\} - \{a_2b_1, b_3\} + \{b_1b_2, b_3\} + \{b_2b_1, b_3\}\\
&= (a_1a_2a_3 - a_3a_1a_2) - (a_2a_1a_3 - a_3a_2a_1) + (a_1b_2a_3 - a_3a_1b_2) - (b_2a_1a_3 - a_3b_2a_1)\\
&\quad + (b_1a_2a_3 - a_3b_1a_2) - (a_2b_1a_3 - a_3a_2b_1) + (b_1b_2a_3 - a_3b_1b_2) + (b_2b_1a_3 - a_3b_2b_1)\\
&\quad + (a_1a_2b_3 - b_3a_1a_2) - (a_2a_1b_3 - b_3a_2a_1) + (a_1b_2b_3 + b_3a_1b_2) - (b_2a_1b_3 + b_3b_2a_1)\\
&\quad + (b_1a_2b_3 + b_3b_1a_2) - (a_2b_1b_3 + b_3a_2b_1) + (b_1b_2b_3 - b_3b_1b_2) + (b_2b_1b_3 - b_3b_2b_1)\,.
\end{aligned}$$

We collect these 32 terms into 8 sums of 4 terms corresponding to the parity vectors. □

Theorem 5.2. *For nonhomogeneous subalgebras of Lie superalgebras, there are three irreducible identities in degree 3, one for each partition* λ.

Proof. We use the expansions from Lemma 5.1. For $\lambda = 3$ we get the representation $R_\lambda(\pi) = 1$ for all $\pi \in S_3$. The matrix E_λ and its RCF are

$$\left[\begin{array}{cccccccc|cc} . & . & . & -2 & . & . & 2 & . & -1 & . \\ . & . & . & 2 & . & . & -2 & . & . & -1 \end{array}\right] \qquad \left[\begin{array}{cccccccc|cc} . & . & . & 1 & . & . & -1 & . & . & -1/2 \\ . & . & . & . & . & . & . & . & 1 & 1 \end{array}\right]$$

(In E_λ we usually replace the entry 0 by a dot.) For $\lambda = 21$ we get this representation:

$$
\begin{array}{ccccccc}
\pi \in S_3 & 123 & 132 & 213 & 231 & 312 & 321 \\
R_\lambda(\pi) & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}
\end{array}
$$

The matrix E_λ and its RCF are

$$
\left[\begin{array}{cccccccccccccccc|cccc}
-1 & 2 & -1 & 2 & -1 & 2 & 1 & 2 & -1 & 2 & 1 & . & 3 & -2 & 3 & . & -1 & . & . & . \\
-2 & 4 & -2 & 4 & -2 & 4 & . & 2 & -2 & 4 & 2 & . & 2 & -2 & . & . & . & -1 & . & . \\
1 & -2 & 1 & -2 & 1 & -2 & 1 & 2 & 1 & -2 & 1 & . & 3 & -2 & 3 & . & . & . & -1 & . \\
-1 & 2 & -1 & 2 & -1 & 2 & 1 & . & -1 & 2 & -1 & . & 1 & . & 3 & . & . & . & . & -1
\end{array}\right]
$$

$$
\left[\begin{array}{cccccccccccccccc|cccc}
1 & -2 & 1 & -2 & 1 & -2 & . & . & 1 & -2 & . & . & . & . & . & . & . & 1/4 & -1/4 & 1/4 \\
. & . & . & . & . & . & 1 & . & . & . & -1 & . & 1 & . & 3 & . & . & 1/4 & -1/4 & -3/4 \\
. & . & . & . & . & . & . & 1 & . & . & 1 & . & 1 & -1 & . & . & . & -1/4 & -1/4 & 1/4 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & -1/2 & -1/2 & -1/2
\end{array}\right]
$$

For $\lambda = 1^3$ we get $R_\lambda(\pi) = \mathrm{sign}(\pi)$ for all $\pi \in S_3$. The matrix E_λ and its RCF are

$$
\left[\begin{array}{cccccccc|cc}
. & . & . & 2 & . & 4 & 2 & . & -1 & . \\
. & . & . & 2 & . & 4 & 2 & . & . & -1
\end{array}\right]
\qquad
\left[\begin{array}{cccccccc|cc}
. & . & . & 1 & . & 2 & 1 & . & . & -1/2 \\
. & . & . & . & . & . & . & . & 1 & -1
\end{array}\right]
$$

For each λ, only the last row of $\mathrm{RCF}(E_\lambda)$ has its leading 1 in the right part. ☐

Corollary 5.3. *The three irreducible identities of Theorem 5.2 are equivalent to identities (1) and (2) of Theorem 2.5.*

Proof. Expanding identities (1) and (2) gives

$$
(ab)c - (ac)b - (ba)c + (bc)a + (ca)b - (cb)a - a(bc) + a(cb) + b(ac) - b(ca) - c(ab) + c(ba)\,,
$$

$$
(ab)c + (ba)c + c(ab) + c(ba)\,.
$$

For each expansion we compute the representation matrices for each partition λ:

$$
(1) \qquad \lambda = 3\colon \begin{array}{c|c} t_1 & t_2 \\ \hline 0 & 0 \end{array}
\qquad \lambda = 21\colon \begin{array}{cc|cc} \multicolumn{2}{c}{t_1} & \multicolumn{2}{c}{t_2} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}
\qquad \lambda = 1^3\colon \begin{array}{c|c} t_1 & t_2 \\ \hline 6 & -6 \end{array}
$$

$$
(2) \qquad \lambda = 3\colon \begin{array}{c|c} t_1 & t_2 \\ \hline 2 & 2 \end{array}
\qquad \lambda = 21\colon \begin{array}{cc|cc} \multicolumn{2}{c}{t_1} & \multicolumn{2}{c}{t_2} \\ \hline 2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}
\qquad \lambda = 1^3\colon \begin{array}{c|c} t_1 & t_2 \\ \hline 0 & 0 \end{array}
$$

We compute the RCFs of these matrices, identify the nonzero rows, and compare these nonzero rows with the rows with leading 1s in the right part of the matrices from Lemma 5.2. Identity (1), respectively (2), is equivalent to the identity for $\lambda = 1^3$, respectively the identities for $\lambda = 3, 21$. ☐

6. Lie superbracket: degree 4

The partitions of 4 are $\lambda = 4, 31, 2^2, 21^2, 1^4$ and the dimensions of the irreducible representations of S_4 are $d = 1, 3, 2, 3, 1$ respectively.

Lemma 6.1. *For nonhomogeneous subalgebras of Lie superalgebras, there are* 4*,* 9*,* 6*,* 9*,* 3 *irreducible identities in degree 4 corresponding to partitions* 4*,* 31*,* 2^2*,* 21^2*,* 1^4 *respectively.*

$$\lambda = 4\colon \quad \begin{bmatrix} 1 & . & . & . & -1 \\ . & 1 & . & . & 1 \\ . & . & 1 & . & . \\ . & . & . & 1 & 1 \end{bmatrix}$$

$$\lambda = 31\colon \quad \begin{bmatrix} 1 & . & . & . & . & -1 & . & . & 1 & . & . & -1 & . & -1 & -1 \\ . & 1 & . & . & . & -1 & . & . & . & . & . & -1 & . & -1 & . \\ . & . & 1 & . & . & -2 & . & . & 3 & . & . & -2 & -1 & -3 & -4 \\ . & . & . & 1 & . & . & . & . & -1 & . & . & . & . & . & 1 \\ . & . & . & . & 1 & 1 & . & . & -3 & . & . & . & 1 & 1 & 4 \\ . & . & . & . & . & . & 1 & . & -1 & . & . & . & . & . & 2 \\ . & . & . & . & . & . & . & 1 & -2 & . & . & . & . & . & 2 \\ . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & 1 \\ . & . & . & . & . & . & . & . & . & . & 1 & 1 & 1 & 1 & . \end{bmatrix}$$

$$\lambda = 2^2\colon \quad \begin{bmatrix} 1 & . & . & . & . & . & . & . & -1 & . \\ . & 1 & . & . & . & . & . & . & . & -1 \\ . & . & 1 & . & . & . & . & 1 & -2 & 1 \\ . & . & . & 1 & . & . & . & -1 & . & . \\ . & . & . & . & 1 & -1/2 & . & . & . & . \\ . & . & . & . & . & . & 1 & 1 & -2 & 1 \end{bmatrix}$$

$$\lambda = 21^2\colon \quad \begin{bmatrix} 1 & . & . & . & . & . & . & . & 1 & . & . & . & . & . & -1 \\ . & 1 & . & . & . & . & . & -1 & 1 & . & . & . & 1 & . & . \\ . & . & 1 & . & . & . & . & . & . & . & . & . & 1 & -1 & 1 \\ . & . & . & 1 & . & . & . & 1 & . & . & . & 1/2 & -1/2 & . & -1 \\ . & . & . & . & 1 & . & . & . & 1 & . & . & -1/2 & 1/2 & -1 & . \\ . & . & . & . & . & 1 & . & . & . & . & . & . & 1 & -1 & 1 \\ . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & 1 & . & 1/2 & -1/2 & . & -1 \\ . & . & . & . & . & . & . & . & . & . & 1 & -1/2 & 1/2 & -1 & . \end{bmatrix}$$

$$\lambda = 1^4\colon \quad \begin{bmatrix} 1 & . & . & . & -1 \\ . & 1 & . & . & -1 \\ . & . & . & 1 & -1 \end{bmatrix}$$

Table 1. Matrices for the proof of Lemma 6.1

$$\lambda = 4\colon \quad \begin{bmatrix} 1 & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & 1 \end{bmatrix}$$

$$\lambda = 31\colon \quad \begin{bmatrix} 1 & . & . & . & . & -1 & . & . & 1 & . & . & -1 & . & -1 & -1 \\ . & 1 & . & . & . & -1 & . & . & . & . & . & -1 & . & -1 & . \\ . & . & 1 & . & 1 & -1 & . & . & . & . & . & -2 & . & -2 & . \\ . & . & . & 1 & . & . & . & . & -1 & . & . & . & . & . & 1 \\ . & . & . & . & . & . & 1 & -1 & 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & 1 \\ . & . & . & . & . & . & . & . & . & . & 1 & 1 & 1 & 1 & . \end{bmatrix}$$

$$\lambda = 2^2\colon \quad \begin{bmatrix} 1 & . & . & -1 & . & . & . & 1 & -1 & . \\ . & 1 & . & . & . & . & . & . & . & -1 \\ . & . & 1 & -1 & . & . & . & 2 & -2 & 1 \\ . & . & . & . & 1 & -1/2 & . & . & . & . \\ . & . & . & . & . & . & 1 & 1 & -2 & 1 \end{bmatrix}$$

Table 2. Matrices for the proof of Lemma 6.2

Proof. For each partition λ, we compute E_λ and its RCF. The rows with leading 1s in the right part of the matrix are displayed in Table 1; the number of rows is the number of irreducible identities. $\square$

Lemma 6.2. *For nonhomogeneous subalgebras of Lie superalgebras, there are* 3, 7, 5, 9, 3 *irreducible identities in degree 4 which follow from the identities in degree 3 and correspond to partitions* 4, 31, 2^2, 21^2, 1^4 *respectively.*

Proof. Let $I(a,b,c)$ and $J(a,b,c)$ be identities (1) and (2) of Theorem 2.5. Since I alternates in a, b, c, it gives only three lifted identities: $I(ab,c,d)$, $I(a,b,c)d$, $aI(b,c,d)$. Since J is symmetric in a, b it gives only four lifted identities: $J(ab,c,d)$, $J(a,b,cd)$, $J(a,b,c)d$, $aJ(b,c,d)$. We compute $\mathrm{RCF}(L_\lambda)$; the nonzero rows are displayed in Table 2. (For $\lambda = 21^2$ and $\lambda = 1^4$ we get the same matrices as in Lemma 6.1.) ☐

Theorem 6.3. *For nonhomogeneous subalgebras of Lie superalgebras, there are* 1, 2, 1, 0, 0 *new irreducible identities in degree 4 corresponding to partitions* 4, 31, 2^2, $21^2, 1^4$ *respectively.*

Proof. We compare the matrices in the proofs of Lemmas 6.1 and 6.2. The new identities correspond to the matrix rows in Lemma 6.1 which have leading 1s in columns which do not have leading 1s in Lemma 6.2. For $\lambda = 4$, we get row 2; for $\lambda = 31$, we get rows 5 and 7; for $\lambda = 2^2$, we get row 4; and for $\lambda = 2^2$ and $\lambda = 1^4$, there are no such rows. ☐

Corollary 6.4. *The four irreducible identities of Theorem 6.3 are equivalent to identities (3) and (4) of Theorem 2.5.*

Proof. Expanding identities (3) and (4) gives

$$
\begin{aligned}
&(ab)(cd) + (ab)(dc) + (ba)(cd) + (ba)(dc) - (cd)(ab) - (cd)(ba) - (dc)(ab) - (dc)(ba) \\
&- ((ab)c)d - ((ba)c)d + (c(ab))d + (c(ba))d - d((ab)c) + d(c(ab)) - d((ba)c) + d(c(ba)) \\
&- ((ab)d)c - ((ba)d)c + (d(ab))c + (d(ba))c - c((ab)d) + c(d(ab)) - c((ba)d) + c(d(ba)), \\
&(a(bc))d + (a(cb))d - ((bc)a)d - ((cb)a)d + (b(ca))d + (b(ac))d - ((ca)b)d - ((ac)b)d \\
&+ (c(ab))d + (c(ba))d - ((ab)c)d - ((ba)c)d + d(a(bc)) + d(a(cb)) - d((bc)a) - d((cb)a) \\
&+ d(b(ca)) + d(b(ac)) - d((ca)b) - d((ac)b) + d(c(ab)) + d(c(ba)) - d((ab)c) - d((ba)c).
\end{aligned}
$$

For each λ we compute the RCF of the representation matrix for these identities. For $\lambda = 4, 31, 2^2$, we obtain:

$$
\begin{bmatrix} 1 & -1 & . & 1 & -1 \end{bmatrix}
$$

$$
\begin{bmatrix}
1 & . & -1/2 & -1 & 1/2 & 1/2 & -1/2 & -1/2 & 1/2 & . & -1/2 & -1/2 & 1/2 & 1/2 & . \\
. & 1 & -1/2 & . & 1/2 & 1/2 & -3/2 & -3/2 & 3/2 & 1 & -1/2 & -1/2 & 1/2 & 1/2 & -1
\end{bmatrix}
$$

$$
\begin{bmatrix} 1 & -1/2 & 1/2 & 1/2 & . & . & -1/2 & -1/2 & -1 & 1/2 \end{bmatrix}
$$

For $\lambda = 21^2, 1^4$ we obtain the zero matrix. For each λ, we stack these rows together with the rows from Lemma 6.2, compute the RCF, and obtain the rows from Lemma 6.1. ☐

7. LIE SUPERBRACKET: DEGREE 5

The partitions of 5 and the dimensions of the irreducible representations of S_5 are:

partition λ	5	41	32	31^2	$2^2 1$	21^3	1^5
dimension d	1	4	5	6	5	4	1

In degree 4 there were 7 lifted identities and 2 new identities, for a total of 9, giving 54 lifted identities in degree 5.

Lemma 7.1. *For nonhomogeneous subalgebras of Lie superalgebras, the number of irreducible identities in degree 5 corresponding to each partition is as follows:*

partition λ	5	41	32	31^2	2^21	21^3	1^5
e-rank(λ)	12	45	54	66	54	43	11

Proof. Similar to the proof of Lemma 6.1. ☐

Lemma 7.2. *For nonhomogeneous subalgebras of Lie superalgebras, the number of irreducible identities in degree 5 which follow from identities in lower degrees and correspond to each partition is as follows:*

partition λ	5	41	32	31^2	2^21	21^3	1^5
ℓ-rank(λ)	11	44	53	66	53	43	11

Proof. Similar to the proof of Lemma 6.2. ☐

Lemma 7.3. *For nonhomogeneous subalgebras of Lie superalgebras, the number of new irreducible identities in degree 5 corresponding to each partition is as follows:*

partition λ	5	41	32	31^2	2^21	21^3	1^5
new identities	1	1	1	0	1	0	0

Proof. An immediate consequence of Lemmas 7.1 and 7.2. ☐

Theorem 7.4. *Identity (5) of Theorem 2.5 implies the new irreducible identities for partitions* 5, 41, 32. *It does not imply the new irreducible identity for partition* 2^21.

Proof. For each λ, we stack together L_λ and the representation matrix for the expansion of identity (5), and compute the RCF. We obtain the rows with leading 1s in the right part of RCF(E_λ) except for $\lambda = 2^21$. ☐

Theorem 7.5. *The four irreducible identities of Lemma 7.3 are equivalent (collectively) to the linearization of the identity*

$$
\begin{aligned}
&(((ca)b)a)b - (((ca)b)b)a - (((cb)a)a)b + (((cb)a)b)a + ((c(ba))a)b - ((c(ba))b)a \\
&+ (a(a(bb)))c - ((aa)c)(bb) + ((ab)c)(ab) - ((bb)a)(ac) - (a(bc))(ab) + (b(ac))(ab) \\
&+ a(((bb)a)c) \equiv 0.
\end{aligned}
$$

Proof. Since $\mu = (2, 2, 1)$ we consider identities in which each term is one of the 30 permutations of a^2b^2c in one of the 14 association types; this gives 420 nonlinear nonassociative monomials. Each term in the expansion of each basis monomial is one of the 30 permutations of a^2b^2c with one of the 32 parity vectors; this gives 960 nonlinear associative monomials. The polynomial identities are the nonzero elements of the nullspace lattice N of the combinatorial expansion matrix X_μ. We find that rank$(X_\mu) = 90$ and hence N has rank 330. The LLL algorithm gives lattice basis vectors with components in $\{0, 1, -1\}$ and Euclidean norms from $\sqrt{2}$ to $\sqrt{35}$. We linearize the identities and then use the representation theory of the symmetric group to process them. The first identity which increases the rank is number 267 in the sorted list; its original nonlinear form is the identity above, and it implies all the new identities. ☐

Corollary 7.6. *For nonhomogeneous subalgebras of Lie superalgebras, every identity of degree* ≤ 5 *follows from identities (1)–(4) of Theorem 2.5 and the identity of Theorem 7.5.* ☐

Remark 7.7. The expanded form of identity (5) from Theorem 2.5 has 144 terms, and the new identity in degree 5 from Baranov [1] has 49 terms. Both identities are multilinear, and together they are equivalent to the 4 irreducible identities of Lemma 7.3. The linearization of the identity of Theorem 7.5 has 52 terms, and is also equivalent to the identities of Lemma 7.3. We have achieved a substantial simplification of the identities in degree 5.

8. LIE SUPERBRACKET: DEGREE 6

For identities in degrees ≤ 5 we use rational arithmetic, since the matrices are small enough to permit efficient computation in characteristic 0. However, in degrees ≥ 6 we need modular arithmetic to control the size of the matrix entries. We use the Maple package `LinearAlgebra[Modular]` with $p = 101$.

Theorem 8.1. *For nonhomogeneous subalgebras of Lie superalgebras, there are no new identities in degree 6: every identity is a consequence of the identities in degrees* ≤ 5.

Proof. For each partition λ, the following table gives the dimension of the irreducible representation R_λ, and the rank of the identities from the right part of $\mathrm{RCF}(E_\lambda)$, using modular arithmetic with $p = 101$:

λ	6	51	42	41^2	33	321	31^3	2^3	2^21^2	21^4	1^6
$\dim R_\lambda$	1	5	9	10	5	16	10	5	9	5	1
e-rank(λ)	39	188	334	372	184	589	369	185	330	184	37

For each λ, we compute $\mathrm{RCF}(L_\lambda)$ and find that ℓ-rank$(\lambda) = e$-rank(λ). □

9. LIE SUPERBRACKET: DEGREE 7

Theorem 9.1. *For nonhomogeneous subalgebras of Lie superalgebras, there are no new identities in degree 7: every identity is a consequence of the identities in degrees* ≤ 5.

Proof. Similar to the proof of Theorem 8.1. We obtain the following table:

λ	$\dim R_\lambda$	e-rank(λ)	λ	$\dim R_\lambda$	e-rank(λ)	λ	$\dim R_\lambda$	e-rank(λ)
7	1	126	421	35	4322	31^4	15	1848
61	6	750	41^3	20	2470	2^31	14	1723
52	14	1736	3^21	21	2588	2^21^3	14	1723
51^2	15	1862	32^2	21	2587	21^5	6	739
43	14	1729	321^2	35	4311	1^7	1	124

For each λ, we compute $\mathrm{RCF}(L_\lambda)$ and find that ℓ-rank$(\lambda) = e$-rank(λ). □

10. JORDAN SUPERPRODUCT: DEGREE 3

In §§10–14 we study polynomial identities for nonhomogeneous subalgebras of special Jordan superalgebras. In the Lie case we knew the identities from the work of Volichenko and Baranov; in the Jordan case we have to find the identities.

Theorem 10.1. *For nonhomogeneous subalgebras of special Jordan superalgebras, there are* 0, 1, 1 *irreducible identities in degree 3 corresponding to partitions* 3, 21, 1^3 *respectively. The two irreducible identities are equivalent (collectively) to the metabelian identity* $[[a,b],c] \equiv 0$.

Proof. The expansions of the association types $t_1 = (c_1 \bullet c_2) \bullet c_3$ and $t_2 = c_1 \bullet (c_2 \bullet c_3)$ have the form $e_i(x) = \sum_{\gamma=000}^{111} f_i^\gamma$ $(i = 1, 2)$ where

$$
\begin{aligned}
f_1^{000} &= a_1a_2a_3 + a_2a_1a_3 + a_3a_1a_2 + a_3a_2a_1, & f_1^{001} &= a_1a_2b_3 + a_2a_1b_3 + a_3a_1b_2 + a_3a_2b_1,\\
f_1^{010} &= a_1b_2a_3 + a_2b_1a_3 + a_3b_1a_2 + a_3b_2a_1, & f_1^{011} &= a_1b_2b_3 + a_2b_1b_3 + a_3b_1b_2 - a_3b_2b_1,\\
f_1^{100} &= b_1a_2a_3 + b_2a_1a_3 + b_3a_1a_2 + b_3a_2a_1, & f_1^{101} &= b_1a_2b_3 + b_2a_1b_3 - b_3a_1b_2 - b_3a_2b_1,\\
f_1^{110} &= b_1b_2a_3 - b_2b_1a_3 - b_3b_1a_2 - b_3b_2a_1, & f_1^{111} &= b_1b_2b_3 - b_2b_1b_3 + b_3b_1b_2 - b_3b_2b_1,\\
f_2^{000} &= a_1a_2a_3 + a_1a_3a_2 + a_2a_3a_1 + a_3a_2a_1, & f_2^{001} &= a_1a_2b_3 + a_1a_3b_2 + a_2a_3b_1 + a_3a_2b_1,\\
f_2^{010} &= a_1b_2a_3 + a_1b_3a_2 + a_2b_3a_1 + a_3b_2a_1, & f_2^{011} &= a_1b_2b_3 - a_1b_3b_2 - a_2b_3b_1 - a_3b_2b_1,\\
f_2^{100} &= b_1a_2a_3 + b_1a_3a_2 + b_2a_3a_1 + b_3a_2a_1, & f_2^{101} &= b_1a_2b_3 + b_1a_3b_2 - b_2a_3b_1 - b_3a_2b_1,\\
f_2^{110} &= b_1b_2a_3 + b_1b_3a_2 + b_2b_3a_1 - b_3b_2a_1, & f_2^{111} &= b_1b_2b_3 - b_1b_3b_2 + b_2b_3b_1 - b_3b_2b_1.
\end{aligned}
$$

We store the expansions of the nonassociative monomials in the combinatorial expansion matrix X_μ for $\mu = (1, 1, 1)$. The nonzero rows of $\mathrm{RCF}(X_\mu)$ are

$$
\begin{bmatrix}
1 & . & . & . & . & . & . & . & . & . & . & -1\\
. & 1 & . & . & . & . & . & . & . & -1 & . & .\\
. & . & 1 & . & . & . & . & . & . & . & . & 1\\
. & . & . & 1 & . & . & . & -1 & . & . & . & .\\
. & . & . & . & 1 & . & . & . & . & 1 & . & .\\
. & . & . & . & . & 1 & . & 1 & . & . & . & .\\
. & . & . & . & . & . & 1 & 1 & . & . & . & .\\
. & . & . & . & . & . & . & . & 1 & 1 & . & .\\
. & . & . & . & . & . & . & . & . & . & 1 & 1
\end{bmatrix}
$$

The canonical basis of the nullspace is

$$
\begin{bmatrix}
. & . & . & 1 & . & -1 & -1 & 1 & . & . & . & .\\
. & 1 & . & . & -1 & . & . & . & -1 & 1 & . & .\\
1 & . & -1 & . & . & . & . & . & . & . & -1 & 1
\end{bmatrix}
$$

The nullspace basis vectors represent these identities (writing a, b, c for c_1, c_2, c_3):

$$
(bc)a - (cb)a - a(bc) + a(cb), \ (ac)b - (ca)b - b(ac) + b(ca), \ (ab)c - (ba)c - c(ab) + c(ba).
$$

The proof of the second claim is similar to that of Theorem 5.2. □

11. Jordan superproduct: degree 4

We have to find identities in degree 4 which do not follow from the identity in degree 3.

Lemma 11.1. *For nonhomogeneous subalgebras of special Jordan superalgebras, there are* 2*,* 6*,* 3*,* 8*,* 3 *irreducible identities in degree 4 corresponding to partitions* $4, 31, 2^2, 21^2, 1^4$ *respectively.*

Proof. For each partition λ, we compute $\mathrm{RCF}(E_\lambda)$ and identify the rows which have leading 1s in the right part of the matrix; these rows are displayed in Table 3. □

Lemma 11.2. *For nonhomogeneous subalgebras of special Jordan superalgebras, there are* 1*,* 5*,* 3*,* 7*,* 2 *irreducible identities which follow from the metabelian identity and correspond to partitions* $4, 31, 2^2, 21^2, 1^4$ *respectively.*

$$\lambda = 4\colon \quad \begin{bmatrix} 1 & . & -2 & . & 1 \\ . & 1 & -2 & 1 & . \end{bmatrix}$$

$$\lambda = 31\colon \quad \begin{bmatrix} 1 & . & . & . & 1 & 2 & . & . & . & -3 & . & 3 & 1 & -2 & . \\ . & 1 & . & . & . & 1 & . & . & . & -1 & . & 1 & . & -1 & 1 \\ . & . & 1 & . & -1 & 1 & . & . & . & . & . & . & . & . & . \\ . & . & . & 1 & 1 & 1 & . & . & . & -3 & . & 2 & 1 & -1 & . \\ . & . & . & . & . & . & 1 & -1 & -1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & 1 & -1 & -1 & 1 & . \end{bmatrix}$$

$$\lambda = 2^2\colon \quad \begin{bmatrix} 1 & . & . & 1 & . & . & . & 1 & 1 & . \\ . & 1 & -1 & 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & -1 & . & -1 \end{bmatrix}$$

$$\lambda = 21^2\colon \quad \begin{bmatrix} 1 & . & . & . & 1 & . & . & . & . & . & -1 & . & . & . & -1 \\ . & 1 & . & . & . & . & . & . & . & . & . & . & -1 & . & . \\ . & . & 1 & . & . & -1 & . & . & . & . & . & . & . & . & . \\ . & . & . & 1 & -1 & 1 & . & . & . & . & . & . & -1 & . & . \\ . & . & . & . & . & . & 1 & . & -1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & 1 & -1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & 1 & -1 & . & . & 1 & -1 \\ . & . & . & . & . & . & . & . & . & . & . & 1 & -1 & . & . \end{bmatrix}$$

$$\lambda = 1^4\colon \quad \begin{bmatrix} 1 & . & -2 & . & 1 \\ . & 1 & -2 & . & 1 \\ . & . & . & 1 & -1 \end{bmatrix}$$

TABLE 3. Matrices for the proof of Lemma 11.1

$$\lambda = 4\colon \quad \begin{bmatrix} 1 & -1 & . & -1 & 1 \end{bmatrix}$$

$$\lambda = 31\colon \quad \begin{bmatrix} 1 & . & . & -1 & . & 1 & . & . & . & . & . & 1 & . & -1 & . \\ . & 1 & . & . & . & 1 & . & . & . & -1 & . & 1 & . & -1 & 1 \\ . & . & 1 & . & -1 & 1 & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & -1 & -1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & 1 & -1 & -1 & 1 & . \end{bmatrix}$$

$$\lambda = 2^2\colon \quad \begin{bmatrix} 1 & . & . & 1 & . & . & . & 1 & 1 & . \\ . & 1 & -1 & 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & -1 & . & -1 \end{bmatrix}$$

$$\lambda = 21^2\colon \quad \begin{bmatrix} 1 & . & . & . & 1 & . & . & . & . & . & -1 & . & . & . & -1 \\ . & 1 & . & -1 & 1 & -1 & . & . & . & . & . & . & . & . & . \\ . & . & 1 & . & . & -1 & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . & -1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & 1 & -1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & 1 & -1 & . & . & 1 & -1 \\ . & . & . & . & . & . & . & . & . & . & . & 1 & -1 & . & . \end{bmatrix}$$

$$\lambda = 1^4\colon \quad \begin{bmatrix} 1 & -1 & . & . & . \\ . & . & . & 1 & -1 \end{bmatrix}$$

TABLE 4. Matrices for the proof of Lemma 11.2

Proof. Since $I(a,b,c) = [[a,b],c]$ alternates in a and b, it gives only four lifted identities: $I(ad,b,c)$, $I(a,b,cd)$, $I(a,b,c)d$, $dI(a,b,c)$. For each λ, the nonzero rows of $\mathrm{RCF}(L_\lambda)$ are displayed in Table 4. □

Theorem 11.3. *For nonhomogeneous subalgebras of special Jordan superalgebras, there are* 1, 1, 0, 1, 1 *new irreducible identities in degree 4 corresponding to partitions* 4, 31, 2^2, 21^2, 1^4 *respectively.*

Proof. For each partition we compare the matrices in the proofs of Lemmas 11.1 and 11.2, and identify the rows from Lemma 11.1 which have a leading 1 in a column for which Lemma 11.2 does not have a leading 1. ☐

Corollary 11.4. *The four irreducible identities of Theorem 11.3 are equivalent (collectively) to this identity:*

$$
\begin{aligned}
&((bc)d)a + (d(ab))c + (d(ca))b - (ab)(dc) - (ac)(bd) - (cb)(ad) - (cd)(ba) - (da)(bc) \\
&- (db)(ca) + b((ac)d) + c((ba)d) + a(d(cb)) \equiv 0.
\end{aligned}
$$

Proof. The 5 association types and 24 permutations give 120 nonassociative monomials; the 16 parity vectors give 384 associative monomials. In the combinatorial expansion matrix X_μ for $\mu = (1,1,1,1)$, the (i,j) entry is the coefficient of the i-th associative monomial in the expansion of the j-th nonassociative monomial. We use the LLL algorithm: the Maple procedure `HermiteForm` with `method='integer[reduced]'` produces a reduced basis; the norms are 2 (41 times), $2\sqrt{2}$ (twice), $2\sqrt{3}$ (9 times) and $\sqrt{14}$ (once). We then use our own Maple program to further reduced this basis with the limiting value 1 of the reduction parameter. We obtain norms 2 (45 times) and $2\sqrt{3}$ (8 times). We expect that the first 45 vectors represent liftings of the metabelian identity, and the last 8 vectors represent new identities. To verify this, for each partition λ let L_λ be the lifting matrix from the proof of Lemma 11.2. For $k = 1, \dots 53$ let J_k be the k-th identity in the list of nullspace basis vectors, sorted by increasing norm. Let $J_{k,\lambda}$ be the matrix with one row of $d \times d$ blocks in which block j is the representation matrix of the terms in J_k with association type j. We stack L_λ on top of $J_{k,\lambda}$ and compute the RCF. If the rank does not increase, then the component of J_k for representation λ is a consequence of the metabelian identity. For partitions $\lambda = 4, 31, 21^2, 1^4$ the smallest k for which $J_{k,\lambda}$ increases the rank is $k = 46$. The identity J_{46} is the identity displayed above. ☐

Remark 11.5. Assuming commutativity, and then setting $a = b = c$, the identity of Corollary 11.4 simplifies to $6(a^2d)a - 6a^2(ad) \equiv 0$. Therefore, in a commutative algebra over a field of characteristic $\neq 2, 3$, this identity is equivalent to the Jordan identity.

Corollary 11.6. *The identity of Corollary 11.4 is equivalent (collectively) to the following four irreducible identities corresponding to partitions* $\lambda = 4, 31, 21^2, 1^4$, *where* Σ *indicates an alternating sum over the indicated symbols:*

$$
\begin{aligned}
J_1 &= (aa^2)a - 2(a^2)^2 + a(a^2a), \\
J_2 &= (ba^2)a - a^2(ba) - (ab)a^2 + a(a^2b), \\
J_3 &= \sum_{b,c}^{\text{alt}} \big[((ab)a)c - ((ba)a)c + ((bc)a)a + b((ac)a) - b((ca)a) - a(a(bc))\big], \\
J_4 &= \sum_{a,b,c,d}^{\text{alt}} \big[((ab)c)d - 2(ab)(cd) + a(b(cd))\big].
\end{aligned}
$$

Proof. The structure theory for $\mathbb{Q}S_n$ (see Remark 4.4) implies that

(i) for partition $\lambda = (n_1, \dots, n_k)$ we need only k distinct symbols: in each term the underlying associative monomial is a permutation of $c_1^{n_1} \cdots c_k^{n_k}$; and

(ii) if λ has $\ell \geq 2$ parts equal to 1, then each monomial represents the alternating sum over the last ℓ symbols.

We give details only for $\lambda = 4$; there are 5 nonassociative monomials: $(c^2c)c$, $(cc^2)c$, $(c^2)^2$, $c(c^2c)$, $c(cc^2)$. We set $c = a + b$, expand each monomial, and store this data in the combinatorial expansion matrix X_μ for $\mu = (4)$. Computing the HNF of X_μ^t gives the matrix equation $UX_\mu^t = H$ where

$$U = \begin{bmatrix} . & . & 1 & . & . \\ . & 1 & -1 & . & . \\ . & 1 & . & -2 & 1 \\ . & 1 & -2 & 1 & . \\ 1 & -1 & . & -1 & 1 \end{bmatrix},$$

$$X_\mu^t = \begin{bmatrix} 8 & 8 & 8 & 6 & 8 & 4 & . & 2 & 8 & . & -4 & 2 & -6 & -2 & -2 & . \\ 8 & 8 & 8 & 2 & 8 & 4 & . & -2 & 8 & . & -4 & -2 & -2 & 2 & 2 & . \\ 8 & 8 & 8 & . & 8 & . & . & . & 8 & . & . & . & . & . & . & . \\ 8 & 8 & 8 & -2 & 8 & -4 & . & 2 & 8 & . & 4 & 2 & 2 & -2 & -2 & . \\ 8 & 8 & 8 & -6 & 8 & -4 & . & -2 & 8 & . & 4 & -2 & 6 & 2 & 2 & . \end{bmatrix},$$

$$H = \begin{bmatrix} 8 & 8 & 8 & . & 8 & . & . & . & 8 & . & . & . & . & . & . & . \\ . & . & . & 2 & . & 4 & . & -2 & . & . & -4 & -2 & -2 & 2 & 2 & . \\ . & . & . & . & . & 8 & . & -8 & . & . & -8 & -8 & . & 8 & 8 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \end{bmatrix}.$$

The rank of X_μ^t is 3; the last two rows of U form a basis for the nullspace lattice of X_μ. We linearize these two identities, sort the list by increasing Euclidean norm, and process it as in the proof of Corollary 11.4. The identity corresponding to row 4 is the new identity J_1; the identity corresponding to row 5 is a consequence of the lifted identities. ☐

Remark 11.7. If we set $a = b$ in J_2 we obtain J_1, so J_1 is redundant. Identities J_2, J_3, J_4 are independent: no two imply the third.

12. JORDAN SUPERPRODUCT: DEGREE 5

Lemma 12.1. *For nonhomogeneous subalgebras of special Jordan superalgebras, the ranks of identities in degree 5 are as follows:*

λ	5	41	32	31^2	2^21	21^3	1^5
e-rank(λ)	8	32	38	53	42	39	11

Proof. Since $2^5 + C_5 = 46$, for each partition λ the matrix E_λ has size $14d \times 46d$. For the largest representation we have $\lambda = 31^2$ and $d = 6$, giving a matrix of size 84×276. Except for the smallest representations, the matrices are too large to display. For $\lambda = 5$, there are 8 rows of RCF(E_λ) which have leading 1s in the right part:

$$\begin{bmatrix} 1 & . & . & . & . & . & -1/2 & -1/2 & -1 & . & . & -2 & 2 & 1 \\ . & 1 & . & . & . & . & -3/2 & -3/2 & 1 & . & . & 2 & -1 & . \\ . & . & 1 & . & . & . & -1 & -1 & . & . & . & 1 & . & . \\ . & . & . & 1 & . & . & -1/2 & -1/2 & -1 & . & . & . & 1 & . \\ . & . & . & . & 1 & . & -3/2 & -3/2 & 1 & . & . & 4 & -2 & -1 \\ . & . & . & . & . & 1 & -1 & -1 & 1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & 1 & . & -2 & . & 1 \\ . & . & . & . & . & . & . & . & . & . & 1 & -2 & 1 & . \end{bmatrix}$$

The other cases are similar. ☐

Lemma 12.2. *In degree 5, the ranks of the lifted identities are as follows:*

λ	5	41	32	31^2	2^21	21^3	1^5
ℓ-rank(λ)	7	32	37	53	41	38	10

Proof. In degree 4, there are four lifted identities and one new identity, for a total of five, so there are 30 lifted identities in degree 5. For $\lambda = 5$, there are 7 nonzero rows in RCF(λ):

$$\left[\begin{array}{cccccccccccccc}
1 & . & . & . & 1 & . & -2 & -2 & . & . & . & 2 & . & . \\
. & 1 & . & . & -1 & . & . & . & . & . & . & -2 & 1 & 1 \\
. & . & 1 & . & . & . & -1 & -1 & . & . & . & 1 & . & . \\
. & . & . & 1 & 1 & . & -2 & -2 & . & . & . & 4 & -1 & -1 \\
. & . & . & . & . & 1 & -1 & -1 & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & -2 & . & 1 \\
. & . & . & . & . & . & . & . & . & . & 1 & -2 & 1 & .
\end{array}\right]$$

The other cases are similar. ☐

Theorem 12.3. *For nonhomogeneous subalgebras of special Jordan superalgebras, the ranks of the new identities in degree 5 are as follows:*

partition λ	5	41	32	31^2	2^21	21^3	1^5
new identities	1	0	1	0	1	1	1

Proof. This follows from Lemmas 12.1 and 12.2. For $\lambda = 5$ there is only one column which has a leading 1 in the right part of RCF(E_λ), but does not have a leading 1 in RCF(L_λ), namely column 5. This leading 1 occurs in row 5, and so this row represents a new identity. The other cases are similar. ☐

We would like to follow the method of proof of Corollary 11.4 to find one identity which implies all the new identities in degree 5. To use the LLL algorithm requires computing the HNF of a 3840×1680 matrix. Instead, we follow the proof of Corollary 11.6 to find five new identities, one for each of the nonzero ranks in Theorem 12.3.

Corollary 12.4. *We have the following identities for partitions $\mu = 5, 32, 2^21, 21^3, 1^5$:*

$$K_1 = (a, a^2, a) \circ a - (a^2, a, a^2), \qquad K_2 = [[a, b]^2, a],$$

$$\begin{aligned}
K_3 = {} & ((ab)(ba))c - ((ab)(ca))b - ((bb)(ac))a - ((bc)(aa))b - ((cb)(ba))a + (a((cb)b))a \\
& + (b((ac)b))a + (a(a(bc)))b + (b(a(ca)))b + (c(a(ab)))b - (c(a(ba)))b - (a(ab))(cb) \\
& + (a(ba))(cb) - (b(ab))(ac) + (b(ba))(ac) - (c(ab))(ba) - (ba)((ab)c) + b(((ac)a)b) \\
& + b(((cb)a)a) + a((b(ab))c) - a((b(ba))c) + a((b(bc))a) + a((b(ca))b) - a((ab)(bc)) \\
& - a((ca)(bb)) - b((aa)(cb)) - b((ac)(ba)) + c((ab)(ba)),
\end{aligned}$$

$$\begin{aligned}
K_4 = \sum_{b,c,d}^{\text{alt}} \Big[& ((aa)(bc))d + ((bc)(da))a + (b((ac)a))d - (b((ca)a))d - (a(a(bc)))d \\
& - ((ab)a)(cd) + ((ab)c)(ad) - ((ab)c)(da) + a(((ab)c)d) - a(((ba)c)d) \\
& - a(((bc)d)a) + a((ba)(cd)) \Big],
\end{aligned}$$

$$K_5 = \sum_{a,b,c,d,e}^{\text{alt}} \Big[(((ab)c)d)e + ((ab)(cd))e - (a(bc))(de) - a(((bc)d)e) \Big].$$

Proof. Similar to the proof of Corollary 11.6: for each μ, we initialize the combinatorial expansion matrix X_μ and use `HermiteForm` with `method='integer[reduced]'` to compute a reduced basis of the nullspace. We then use our own Maple program with reduction parameter 1 to reduce the basis further. For $\mu = 5$, the matrix X_μ has size 32×14 and rank 6. The last 8 rows of the matrix U form a basis of the nullspace:

$$\begin{bmatrix}
1 & -1 & . & -1 & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & -1 & -1 & 1 & . & . & . & . & . \\
. & . & 1 & . & . & . & -1 & -1 & . & . & . & 1 & . & . \\
1 & . & . & -1 & . & . & . & . & . & -1 & . & . & 1 & . \\
1 & . & . & -1 & . & . & . & . & . & . & -1 & . & . & 1 \\
. & 1 & -1 & 1 & . & . & -1 & -1 & . & . & . & 1 & . & . \\
. & . & 1 & . & . & . & -1 & -1 & . & . & 1 & -1 & 1 & . \\
. & -1 & . & 1 & . & 1 & . & . & -1 & . & -1 & . & 1 & .
\end{bmatrix}$$

Only the (linearization of) the identity represented by the last row makes the rank increase:

$$-((a(aa))a)a + (a((aa)a))a + ((aa)a)(aa) - (aa)(a(aa)) - a((a(aa))a) + a(a((aa)a)).$$

This can be written more compactly as K_1. The other cases are similar. ☐

13. Jordan superproduct: degree 6

In degree 5 we have 35 identities (30 lifted and 5 new) so in degree 6 we have 245 lifted identities. For the largest representation ($\lambda = 321$, $d = 16$) we see that E_{321} has size 672×1696, and L_{321} has size 3920×672. We use modular arithmetic with $p = 101$.

Theorem 13.1. *For nonhomogeneous subalgebras of special Jordan superalgebras, the ranks of the identities, lifted identities, and new identities in degree 6 are as follows:*

partition λ	6	51	42	41^2	3^2	321	31^3	2^3	2^21^2	21^4	1^6
e-rank(λ)	30	148	260	306	145	485	324	152	290	173	37
ℓ-rank(λ)	30	148	258	305	143	482	323	152	289	173	37
new	0	0	2	1	2	3	1	0	1	0	0

Proof. Similar to the proofs of Lemmas 12.1 and 12.2 and Theorem 12.3. ☐

Corollary 13.2. ($\mu = 42$) *There are two new identities in degree 6 with symbols a^4b^2:*

$$\begin{aligned}
L_1 = {} & ((a(ab))(ba))a + ((ab)((ba)a))a - (a(((ab)b)a))a - (b(((ba)a)a))a - (a(a(b(ba))))a \\
& - (b(a(a(ab))))a + (a((ba)b))a^2 + (a(b(ab)))a^2 + a^2(((ba)b)a) + a^2((b(ab))a) \\
& - a((((ab)a)a)b) - a((((ba)b)a)a) - a((a(a(ba)))b) - a((a(b(ab)))a) + a(((ba)a)(ba)) \\
& + a((ab)(a(ab))).
\end{aligned}$$

$$\begin{aligned}
L_2 = {} & (a((ab)(ba)))a - (a((ba)(ba)))a + (b(a^2(ab)))a - (b(a^2(ba)))a - ((a(ab))a)(ba) \\
& - ((b(ab))a)a^2 + (a^2(ba))(ab) - (a^2(ba))(ba) + ((ab)a^2)(ab) - ((ab)a^2)(ba) \\
& - (a((ab)a))(ab) + (a((ba)a))(ab) + (a(a(ab)))(ba) + (a(b(ba)))a^2 + a^2(((ba)b)a) \\
& + (ba)(((ba)a)a) - (ab)((a(ba))a) + (ba)((a(ba))a) - a^2(a((ab)b)) - (ab)(a((ab)a)) \\
& + a((a^2(ab))b) - a((a^2(ba))b) + a(((ab)(ab))a) - a(((ba)(ab))a).
\end{aligned}$$

Proof. Similar to the proof of Corollary 12.4. For each of the 15 permutations of a^4b^2 we have 64 parity vectors and 42 association types. The matrix X_μ has size 960×630; each column has 2048 nonzero entries. We use `HermiteForm` with `method='integer[reduced]'` to compute

H (the HNF of X_μ^t) and U (with $UX_\mu^t = H$). We find that $\operatorname{rank}(X_\mu) = 192$ and so the last 438 rows of U form a basis of the nullspace lattice of X_μ. We sort this basis by increasing norm; the norms range from $\sqrt{2}$ to $2\sqrt{38}$. We next apply our own implementation of the LLL algorithm with parameter 1 to reduce this basis further. We obtain a new lattice basis with norms from $\sqrt{2}$ to $\sqrt{102}$, and many more short vectors. We linearize the corresponding identities and process them with the representation theory of the symmetric group. Only identities 395 and 406 in the sorted list increase the rank beyond that of the lifted identities for representation $\lambda = 42$. These (in original nonlinear form) are L_1 and L_2. ☐

Corollary 13.3. ($\mu = 41^2$) *There is one new identity in degree 6 with symbols a^4bc:*

$$\begin{aligned}
L_3 = \sum_{b,c}^{\text{alt}} \big[&((a(ba))(ca))a - ((ab)((ca)a))a - ((ba)((ac)a))a + ((ba)((ca)a))a \\
&+ (b(((ac)a)a))a + (a((b(ca))a))a - (a(a((ab)c)))a + (b(a(a(ca))))a \\
&- ((b(ac))a)a^2 + (a^2(ba))(ca) + ((ab)a^2)(ca) + (a((ba)c))a^2 + a^2(((ba)c)a) \\
&- (ab)(a^2(ac)) - (ab)((ca)a^2) - a^2(a(b(ac))) - a((((ba)a)a)c) + a(((b(ac))a)a) \\
&- a((a((ba)c))a) - a((a(a(ab)))c) + a((a(ab))(ac)) - a((ba)((ca)a)) \big].
\end{aligned}$$

Proof. Similar to the proof of Corollary 13.2, except that each nonassociative monomial represents the alternating sum over b and c. ☐

Corollary 13.4. ($\mu = 33$) *There are two new identities in degree 6 with symbols a^3b^3:*

$$\begin{aligned}
L_4 = {} &(a((ab)(ba)))b + (((ab)b)a)(ab) - (((ba)b)a)(ab) - ((b(ab))a)(ba) + ((b(ba))a)(ba) \\
&- ((ab)(ab))(ba) + ((ab)(ba))(ba) - ((ba)(ba))(ba) + (a((ab)b))(ba) + (a(b(ba)))(ab) \\
&- ((ab)a)((ba)b) + ((ab)b)((ba)a) - ((ba)a)((ab)b) - ((ba)b)(a(ba)) - (b(ab))((ab)a) \\
&+ (b(ba))((ab)a) - (b(ba))((ba)a) - (b(ab))(a(ab)) + (ab)(((ab)b)a) + (ba)((b(ba))a) \\
&- (ab)((ba)(ab)) + b(((ab)(ab))a) - b(((ab)(ba))a) + b(((ba)(ba))a),
\end{aligned}$$

$$\begin{aligned}
L_5 = {} &(((ab)b)(ba))a - (((ba)b)(ba))a - ((a(ba))(ba))b + ((b(ba))(ab))a - (b(((ba)a)b))a \\
&- (a((b(ab))b))a - (b((ab)(ba)))a - (a(b((ba)b)))a - (b(b(a(ab))))a + ((ab)(ba))(ab) \\
&+ (b((ab)a))(ab) + (b((ab)b))a^2 + (b(a(ba)))(ba) + (b(b(ba)))a^2 - ((ab)b)(a^2b) \\
&+ (b^2a)((ab)a) + ((ba)a)(ab^2) + ((ba)a)(b(ba)) + ((ba)b)(a(ba)) + (b^2a)(a(ba)) \\
&- (a(ab))((ab)b) - (ba^2)((ba)b) - (b(ab))((ab)a) - (b(ba))(a^2b) + (a(ab))(ab^2) \\
&- (ba^2)(b(ab)) + a^2(((ab)b)b) + (ba)(((ba)a)b) + a^2((b(ba))b) + (ab)((a(ab))b) \\
&+ (ab)((ba)(ba)) - a((((ba)b)b)a) - a(((a(ab))b)b) - a(((ab)(ba))b) - a((b((ba)a))b) \\
&- a((b(b(ab)))a) + a((ab)((ab)b)) - b((ba)((ba)a)).
\end{aligned}$$

Proof. Similar to the proof of Corollary 13.2. ☐

Corollary 13.5. ($\mu = 321$) *There are three new identities in degree 6 with symbols a^3b^2c:*

$$\begin{aligned}
L_6 = {} &(((ab)(bc))a)a - (((ab)(cb))a)a - (((ba)(bc))a)a + (((ba)(cb))a)a - ((ab)(bc))a^2 \\
&+ ((ab)(cb))a^2 + ((ba)(bc))a^2 - ((ba)(cb))a^2 + (a((ab)a))(bc) - (a((ab)a))(cb) \\
&- (a((ba)a))(bc) + (a((ba)a))(cb) + (a((bc)a))(ab) - (a((bc)a))(ba) - (a((cb)a))(ab) \\
&+ (a((cb)a))(ba) - ((ab)a)((bc)a) + ((ab)a)((cb)a) - ((ba)a)((cb)a) - ((bc)a)((ab)a) \\
&+ ((bc)a)((ba)a) + ((cb)a)((ab)a) + ((ba)a)(a(bc)) - (a(cb))((ba)a),
\end{aligned}$$

$$\begin{aligned}
L_7 = {} &(((ab)(ca))a)b - (((ac)(ab))a)b + (((ac)(ba))a)b - (((ba)(ca))a)b - ((ab)(ca))(ab) \\
&+ ((ac)(ab))(ab) - ((ac)(ba))(ab) + ((ba)(ca))(ab) - (a((ab)b))(ac) + (a((ab)b))(ca)
\end{aligned}$$

$$
\begin{aligned}
&- (a((ac)b))(ab) + (a((ac)b))(ba) + (a((ba)b))(ac) - (a((ba)b))(ca) + (a((ca)b))(ab) \\
&- (a((ca)b))(ba) + ((ab)a)((ac)b) - ((ab)a)((ca)b) + ((ac)a)((ab)b) - ((ac)a)((ba)b) \\
&- ((ba)a)((ac)b) + ((ba)a)((ca)b) - ((ca)a)(b(ab)) + ((ca)a)(b(ba)),
\end{aligned}
$$

$$
\begin{aligned}
L_8 = {} & 2((((ab)b)a)c)a - 2((((ab)c)a)b)a - 2((((ba)b)a)c)a - (((ab)(ab))a)c \\
&+ (((ab)(ab))c)a + 2(((ab)(ba))a)c - 2(((ab)(ba))c)a + (((ab)(bc))a)a \\
&- (((ab)(ca))a)b + (((ab)(ca))b)a - (((ab)(cb))a)a + (((ac)(ab))a)b - (((ac)(ab))b)a \\
&- (((ac)(ba))a)b + (((ac)(ba))b)a - (((ba)(ba))a)c + (((ba)(ba))c)a - (((ba)(bc))a)a \\
&+ (((ba)(ca))a)b - (((ba)(ca))b)a + (((ba)(cb))a)a + 4((a((ab)c))b)a \\
&- 2((a((ba)c))b)a - 2(((ab)a)(cb))a - 2(((ab)b)(ca))a - 2(((ab)c)(ba))a \\
&+ 2(((ba)a)(cb))a + 2(((ba)b)(ca))a + 2(((ba)c)(ba))a - 2(((bc)a)(ab))a \\
&+ 2(((bc)a)(ba))a + 4(((cb)a)(ab))a - 4(((cb)a)(ba))a + 2((a(bc))(ab))a \\
&- 2((a(bc))(ba))a - 2((ab)((ab)c))a + 2((ab)((ba)c))a - 2((ac)((ab)b))a \\
&+ 2((ac)((ba)b))a - 2((bc)((ab)a))a + 2((bc)((ba)a))a + 4(b(((ab)c)a))a \\
&+ (((ab)a)a)(bc) - (((ab)a)a)(cb) - (((ab)a)b)(ac) + (((ab)a)b)(ca) \\
&+ 2(((ab)a)c)(ab) - 2(((ab)a)c)(ba) - (((ab)b)a)(ac) + (((ab)b)a)(ca) \\
&+ (((ab)b)c)a^2 + (((ab)c)a)(ab) - (((ab)c)a)(ba) - (((ab)c)b)a^2 - (((ac)a)b)(ab) \\
&+ (((ac)a)b)(ba) - (((ac)b)b)a^2 - (((ba)a)a)(bc) + (((ba)a)a)(cb) + (((ba)a)b)(ac) \\
&- (((ba)a)b)(ca) - 2(((ba)a)c)(ab) + 2(((ba)a)c)(ba) + (((ba)b)a)(ac) \\
&- (((ba)b)a)(ca) - (((ba)b)c)a^2 + (((ba)c)a)(ab) - (((ba)c)a)(ba) + (((ba)c)b)a^2 \\
&+ (((bc)a)a)(ab) - (((bc)a)a)(ba) + (((bc)b)a)a^2 + (((ca)a)b)(ab) - (((ca)a)b)(ba) \\
&+ (((ca)b)b)a^2 - (((cb)a)a)(ab) + (((cb)a)a)(ba) - (((cb)b)a)a^2 + 2((c(ab))b)a^2 \\
&+ ((c(bb))a)a^2 - (a^2(ab))(bc) + (a^2(ab))(cb) + (a^2(ba))(bc) - (a^2(ba))(cb) \\
&+ ((ab)(bc))a^2 - ((ab)(cb))a^2 - ((ac)(bb))a^2 - ((ba)(bc))a^2 - ((ba)(cb))a^2 \\
&- ((bb)(ca))a^2 - 2((bc)(ab))a^2 + 2(a((ab)b))(ac) - 3(a((ab)c))(ab) \\
&+ (a((ab)c))(ba) + (a((ac)b))(ab) - (a((ac)b))(ba) - 2(a((ba)b))(ac) \\
&+ (a((ba)c))(ab) - 3(a((ba)c))(ba) + (a((bb)c))a^2 - (a((bc)b))a^2 \\
&- (a((ca)b))(ab) + (a((ca)b))(ba) + (a((cb)b))a^2 + (b((ac)b))a^2 - (b((ca)b))a^2 \\
&+ (a^2b)((ab)c) + (a^2b)((ba)c) - (a^2c)((ab)b) + (a^2c)((ba)b) - ((ab)b)(ca^2) \\
&+ ((ab)c)(ba^2) + ((ba)b)(ca^2) + ((ba)c)(ba^2) + a^2(((ab)b)c) - a^2(((ab)c)b) \\
&- a^2(((ba)b)c) - a^2(((ba)c)b) - 2(ab)(((ab)c)a) - 2(ab)(((ba)c)a) \\
&+ 2(ac)(((ab)b)a) - 2(ac)(((ba)b)a) - 2a((((ab)b)a)c) + 2a((((ab)c)a)b) \\
&+ 2a((((ba)b)a)c) + 2a((a((ba)c))b).
\end{aligned}
$$

Proof. There are 60 permutations of a^3b^2c. The expansion matrix X_μ has size 3840×2520; each column has 2048 nonzero entries. We use modular arithmetic and the Chinese remainder theorem. The first 10 primes > 100, and their product, are

$$
\begin{aligned}
&p_1,\, p_2,\, \dots,\, p_{10} = 101,\ 103,\ 107,\ 109,\ 113,\ 127,\ 131,\ 137,\ 139,\ 149; \\
&P = 647\,208\,138\,850\,831\,221\,463.
\end{aligned}
$$

For each $k = 1, \dots, 10$ we compute $\mathrm{RCF}(X_\mu)$ over the field $\mathbb{F}_p$ with $p = p_k$ elements. In every case the leading 1s in the RCF occur in the same positions; this will be true for any prime $p > 6$. In every case, the rank is 738, and the nullspace lattice has dimension 1782.

From the RCF we extract the canonical basis vectors of the nullspace over $\mathbb{F}_p$:

$$V_i^{(p_k)} = \big(x_{i1}^{(p_k)}, \dots, x_{i,2520}^{(p_k)} \big) \text{ for } i = 1, \dots, 1782.$$

Now consider RCF(X_μ) over $\mathbb{Q}$. For $i = 1, \dots, 1782$ let $V_i = (x_{i,1}, \dots, x_{i,2520}) \in \mathbb{Q}^{2520}$ be the canonical basis vectors for the nullspace over $\mathbb{Q}$. The representation theory of the symmetric group tells us to expect that the denominators of the components of these vectors will be divisors of 6!. For each of the unknown vectors V_i, and for each $k = 1, \dots, 10$, we have already computed $V_i^{(p_k)}$, the reduction of V_i modulo p_k. We use the Chinese remainder theorem to determine, for $i = 1, \dots, 1782$ and $j = 1, \dots, 2520$, the unique integer $x'_{i,j}$ such that

$$0 \leq x'_{i,j} < P, \qquad x'_{i,j} \equiv x_{i,j}^{(p_k)} \pmod{p_k}, \; k = 1, \dots, 10.$$

We now have $x_{i,j} \equiv x'_{i,j} \pmod{P}$, where $x_{i,j}$ is a rational number whose denominator is a divisor of 6!. We multiply by 6! to clear denominators, obtaining $6! x_{i,j} \equiv 6! x'_{i,j} \pmod{P}$, in which both sides are integers. We therefore compute $6!\, x'_{i,j}$ for all i, j and reduce again modulo P, this time using symmetric representatives (between $-P/2$ and $+P/2$). We expect that the resulting vectors will equal the vectors $6! V_i$, and will therefore be an integral basis for the nullspace of X over $\mathbb{Q}$. The components of these vectors have ≤ 5 digits; this is good evidence that our assumptions are correct, since any remaining denominator such as 2 or 3 would produce a component such as

$$323604069425415610732 \equiv \frac{1}{2} \pmod{P}, \qquad 431472092567220814309 \equiv \frac{1}{3} \pmod{P}.$$

To verify these identities we linearize them and process them with the representation theory of the symmetric group. Only identities 704, 706 and 1005 in the sorted list increase the rank beyond the lifted identities for representation $\lambda = 321$. These identities (in their original nonlinear form) are L_6, L_7 and L_8. ☐

Corollary 13.6. ($\mu = 31^3$) *There is one new identity in degree 6 with symbols a^3bcd:*

$$\begin{aligned} L_9 = \sum_{b,c,d}^{\text{alt}} \big[& (((bc)a)(da))a - ((a(bc))(ad))a + (b(((ac)d)a))a - (b(((ca)d)a))a \\ & + (a((ab)(cd)))a - (a((bc)(da)))a + (b(a(c(ad))))a - (b(a(c(da))))a \\ & + a(((b(ac))a)d) - a(((b(ca))a)d) - a(((ab)(cd))a) + a(((ba)(cd))a) \\ & + a((a((ab)c))d) - a((a((ba)c))d) + a(((bc)a)(ad)) - a(((bc)a)(da)) \big]. \end{aligned}$$

Proof. Similar to the proof of Corollary 13.2, except that each nonassociative monomial represents the alternating sum over b, c, d. ☐

Corollary 13.7. ($\mu = 2^21^2$) *There is one new identity in degree 6 with symbols a^2b^2cd:*

$$\begin{aligned} L_{10} = \sum_{c,d}^{\text{alt}} \big[& (((ab)(cd))a)b - (((ba)(cd))a)b - ((ab)(cd))(ab) + ((ba)(cd))(ab) \\ & + (a((ab)b))(cd) - (a((ba)b))(cd) + (a((cd)b))(ab) - (a((cd)b))(ba) \\ & - ((ab)a)((cd)b) + ((ba)a)((cd)b) - ((cd)a)(b(ab)) + (a(cd))(b(ba)). \end{aligned}$$

Proof. Similar to that of Corollary 13.5, except that each nonassociative monomial represents the alternating sum over c, d. ☐

14. Jordan superproduct: degree 7

For each partition λ of 7, corresponding to an irreducible representation of dimension $d = d_\lambda$, the expansion matrix has 132 rows of $d \times d$ blocks and $128 + 132 = 260$ columns of $d \times d$ blocks. For the largest representations ($\lambda = 421$, 321^2 with $d = 35$) this gives a matrix of size 4620×9100, in which each column has 8192 nonzero entries.

Theorem 14.1. *For nonhomogeneous subalgebras of special Jordan superalgebras, there are 80 new irreducible identities in degree 7.*

Proof. For each λ we compute $\mathrm{RCF}(E_\lambda)$ and find e-rank(λ). In degree 6, a subset of 60 identities (from the 245 lifted identities) generates all the lifted identities. Together with the 10 new identities in degree 6, this gives 70 identities that must be lifted to degree 7, so we have 560 lifted identities in degree 7. We compute $\mathrm{RCF}(L_\lambda)$ and find ℓ-rank(λ). The difference between the two ranks is the number of new identities; see Table 5. ☐

λ	$\dim R_\lambda$	all	lifted	new
7	1	108	108	–
61	6	639	637	2
52	14	1470	1463	7
51^2	15	1610	1609	1
43	14	1465	1455	10
421	35	3736	3727	9
41^3	20	2205	2202	3
3^21	21	2244	2235	9
32^2	21	2255	2246	9
321^2	35	3853	3840	13
31^4	15	1712	1708	4
2^31	14	1546	1539	7
2^21^3	14	1598	1593	5
21^5	6	711	710	1
1^7	1	123	123	–

Table 5. Jordan superproduct: ranks of identities in degree 7

15. Conclusion

Every identity in degrees ≤ 7 for nonhomogeneous subalgebras of Lie superalgebras follows from 3, 4, 4, 0, 0 irreducible identities in degrees 3, 4, 5, 6, 7 respectively. On the other hand, every identity in degrees ≤ 7 for nonhomogeneous subalgebras of special Jordan superalgebras follows from 2, 4, 5, 10, 80 irreducible identities in degrees 3, 4, 5, 6, 7 respectively. These contrasting results suggest two conjectures.

Conjecture 15.1. The polynomial identities satisfied by all nonhomogeneous subalgebras of Lie superalgebras are finitely generated as a T-ideal.

Conjecture 15.2. The polynomial identities satisfied by all nonhomogeneous subalgebras of special Jordan superalgebras are not finitely generated as a T-ideal.

Acknowledgements. Murray Bremner thanks NSERC for financial support, the University of Saskatchewan for a sabbatical leave from January to June 2008, and the Department of Mathematics at the University of São Paulo for its hospitality from January to April 2008.

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