

## RESEARCH ARTICLE

### Polynomial identities for the ternary cyclic sum

Murray R. Bremner<sup>a,†</sup> and Luiz A. Peresi<sup>b,‡</sup>

<sup>a</sup> Department of Mathematics and Statistics, University of Saskatchewan, Canada

<sup>b</sup> Departamento de Matemática, Universidade de São Paulo, Brazil

(Received 00 Month 200x; in final form 00 Month 200x)

We simplify the results of Bremner and Hentzel [J. Algebra 231 (2000) 387–405] on polynomial identities of degree 9 in two variables satisfied by the ternary cyclic sum  $[a, b, c] = abc + bca + cab$  in every totally associative ternary algebra. We also obtain new identities of degree 9 in three variables which do not follow from the identities in two variables. Our results depend on (i) the LLL algorithm for lattice basis reduction, and (ii) linearization operators in the group algebra of the symmetric group which permit efficient computation of the representation matrices for a nonlinear identity. Our computational methods can be applied to polynomial identities for other algebraic structures.

**Keywords:** nonassociative algebra; polynomial identities; trilinear operations; lattice basis reduction; representation theory of the symmetric group; computational linear algebra

**AMS Subject Classification:** 17A40; 11H06, 15-04, 17A50, 20C30

#### 1. Introduction

Bremner and Hentzel [2] studied polynomial identities satisfied by three ternary generalizations of the Lie and Jordan products: the symmetric, alternating, and cyclic sums. In the first two cases, there are new identities in degrees 3 and 7, and no new identities in degrees 5 and 9. (We call an identity “new” if it is not a consequence of identities of lower degree.) For the ternary cyclic sum, there are new identities in degrees 3, 7 and 9, and no new identities in degree 5. The identities in degree 9 are very complex: Table IV of [2] gives 9 identities in two variables which have up to 95 terms and whose coefficients have up to 4 digits. One expects that there are simpler equivalent identities.

In this paper we use the LLL algorithm for lattice basis reduction [8] to find simpler identities. We present 4 identities in two variables which have up to 65 terms and whose coefficients have only 1 digit and which are equivalent to the identities in [2, Table IV]. We then use the same methods to obtain simple identities in three variables which do not follow from the identities in two variables.

To demonstrate that our new two-variable identities are equivalent to [2, Table IV], we use the representation theory of the symmetric group, and for this we need multilinear identities. The obvious way to linearize an identity produces an equivalent identity with many more terms; computation with the linearized identity is often not practical. To avoid this problem we introduce linearization operators  $L$  in the group algebra  $\mathbb{Q}S_n$  which allow us to compute the representation matrices of a nonlinear identity  $I$  by evaluating the products  $L\ell(I)$  where  $\ell(I)$  is a multilinear identity with the same number of terms as  $I$ .

---

† Email: bremner@math.usask.ca    ‡ Email: peresi@ime.usp.br

## 2. Preliminaries

**Definition 2.1:** A **ternary algebra** is a vector space  $V$  over a field  $\mathbb{F}$  with a trilinear product  $V \times V \times V \rightarrow V$  denoted  $(a, b, c) \mapsto abc$ . The ternary algebra  $V$  is **totally associative** if  $(abc)de = a(bcd)e = ab(cde)$  for all  $a, b, c, d, e \in V$ .

**Definition 2.2:** A **trilinear operation** is a linear combination of permutations of  $a, b, c$ : we write  $[a, b, c] \equiv x_1 abc + x_2 acb + x_3 bac + x_4 bca + x_5 cab + x_6 cba$  where  $x_i \in \mathbb{F}$ . If we regard the permutations as products in a totally associative ternary algebra, then  $[a, b, c]$  defines a new nonassociative product on the underlying vector space. (We often omit the commas in  $[a, b, c]$ , especially when we consider monomials of higher degree.)

**Definition 2.3:** The trilinear operation  $[a, b, c] = abc + bca + cab$  is called the **ternary cyclic sum**. It satisfies the **cyclic law**  $C(a, b, c) = [a, b, c] - [b, c, a] \equiv 0$ .

**Definition 2.4:** Consider a partition  $n = n_1 + \dots + n_k$  ( $n \geq n_1 \geq \dots \geq n_k \geq 1$ ). Let  $m$  be a **nonassociative monomial** of degree  $n$  and multidegree  $(n_1, \dots, n_k)$ ; that is,  $m$  is a permutation of the associative monomial  $a_1^{n_1} \dots a_k^{n_k}$  with some valid arrangement of parentheses. (In this paper,  $m$  will be a monomial in the ternary cyclic sum.) For  $i = 1, \dots, k$  we set  $N_i = \{n_1 + \dots + n_{i-1} + 1, \dots, n_1 + \dots + n_i\}$ , so that  $N_1 \cup N_2 \cup \dots \cup N_k$  is a partition of  $N = \{1, \dots, n\}$  with  $|N_i| = n_i$ . We define  $\ell(m)$ , the **basic linearization** of  $m$ , to be the multilinear monomial obtained by first replacing the old variables  $a_1, \dots, a_k$  by temporary variables  $a'_1, \dots, a'_k$  and then replacing the  $n_i$  occurrences of  $a'_i$  by the new variables  $a_j$  for  $j \in N_i$  with  $j$  increasing from left to right. The operation  $\ell$  extends to linear combinations of monomials with the same multidegree.

*For the rest of this paper  $\mathbb{F} = \mathbb{Q}$ , the field of rational numbers.*

**Definition 2.5:** With notation as above, let  $P_i$  be the symmetric group on  $N_i$ . The **(symmetric) linearization** of  $m$  is

$$L^+(m) = \sum_{\sigma_1 \in P_1} \dots \sum_{\sigma_k \in P_k} \sigma_1 \dots \sigma_k \ell(m),$$

where  $\sigma_i$  acts on the subscripts of the  $a_j$  for  $j \in N_i$ . (This action does not change the association type.)

**Definition 2.6:** Let  $I = \sum_{t \in T} x_t m_t$  be a **multihomogeneous** nonassociative polynomial:  $T$  is an index set,  $x_t \in \mathbb{Q}$  for all  $t \in T$ , and each monomial  $m_t$  has degree  $n$  and multidegree  $(n_1, \dots, n_k)$ . The **(symmetric) linearization** of  $I$  is

$$L^+(I) = \sum_{t \in T} x_t L^+(m_t).$$

**Definition 2.7:** Let  $\lambda$  be another partition of  $n$  and let  $R_\lambda: \mathbb{Q}S_n \rightarrow M_d(\mathbb{Q})$  be the corresponding irreducible representation of  $S_n$  of dimension  $d$ . (We regard  $R_\lambda$  as projection onto a simple ideal of  $\mathbb{Q}S_n$ .) For the partition  $n = n_1 + n_2 + \dots + n_k$  we define  $P_i$  as above. The **linearization operator** for these two partitions is

$$L_\lambda^{n_1, \dots, n_k} = \sum_{\sigma \in P_1 \times \dots \times P_k} R_\lambda(\sigma).$$

*Remark:* Let  $I$  be a multihomogeneous nonassociative polynomial with multidegree  $(n_1, \dots, n_k)$ . Suppose that there are  $C$  association types in degree  $n$ . In  $I$  we collect

the terms with the same association type and write  $I = M_1 + \cdots + M_C$ . In each  $M_s$  the monomials differ only by permutation of the variables, so we can regard  $L^+(M_s)$  as an element of  $\mathbb{Q}S_n$ , and  $L^+(I) = L^+(M_1) + \cdots + L^+(M_C)$  as an element of  $(\mathbb{Q}S_n)^C$ .

**Lemma 2.8:** *With notation as above, let  $I = M_1 + \cdots + M_C$  have multidegree  $(n_1, \dots, n_k)$ . Let  $\lambda$  be a partition of  $n$  and  $R_\lambda$  the corresponding irreducible representation of  $S_n$ . The  $C$ -tuple of matrices in  $(\mathbb{Q}S_n)^C$  representing the linearization of  $I$  is*

$$\left( L_\lambda^{n_1, \dots, n_k} R_\lambda(\ell(M_1)), \dots, L_\lambda^{n_1, \dots, n_k} R_\lambda(\ell(M_C)) \right).$$

**Proof:** It suffices to check each component separately, so we write  $M = \sum_{t \in T} x_t m_t \in \mathbb{Q}S_n$ . Since  $R_\lambda$  is a linear map, for any monomial  $m = m_t$  we have

$$R_\lambda(L^+(m)) = \sum_{\sigma_1 \in P_1} \cdots \sum_{\sigma_k \in P_k} R_\lambda(\sigma_1 \cdots \sigma_k \ell(m)).$$

Since  $R_\lambda$  is a representation of  $S_n$ , and  $\ell(m)$  is a permutation of  $a_1 \cdots a_n$ , we get

$$R_\lambda(L^+(m)) = \sum_{\sigma_1 \in P_1} \cdots \sum_{\sigma_k \in P_k} R_\lambda(\sigma_1) \cdots R_\lambda(\sigma_k) R_\lambda(\ell(m)).$$

Every term has the rightmost factor  $R_\lambda(\ell(m))$ , so we obtain

$$R_\lambda(L^+(m)) = \left[ \sum_{\sigma_1 \in P_1} \cdots \sum_{\sigma_k \in P_k} R_\lambda(\sigma_1) \cdots R_\lambda(\sigma_k) \right] R_\lambda(\ell(m)).$$

The expression in brackets is the linearization operator  $L_\lambda^{n_1, \dots, n_k}$ . We now have

$$\begin{aligned} R_\lambda(L^+(M)) &= R_\lambda\left(\sum_{t \in T} x_t L^+(m_t)\right) = \sum_{t \in T} x_t R_\lambda(L^+(m_t)) \\ &= \sum_{t \in T} x_t L_\lambda^{n_1, \dots, n_k} R_\lambda(\ell(m_t)) = L_\lambda^{n_1, \dots, n_k} R_\lambda \ell(M). \end{aligned}$$

The proof is complete.  $\square$

*Remark:* Given a polynomial with multidegree  $(n_1, \dots, n_k)$ , the obvious way to linearize it is to replace each term by  $n_1! \cdots n_k!$  new terms. By contrast, the expression  $\ell(M)$  has the same number of terms as  $M$ . We therefore avoid creating a linearized identity with many more terms than the original identity. Given a family of identities with the same multidegree  $(n_1, \dots, n_k)$  we can precompute the linearization operators  $L_\lambda^{n_1, \dots, n_k}$  for all  $\lambda$  and then the representation matrices can be quickly determined.

**Definition 2.9:** The **alternating linearizations** of  $m$  and  $I = \sum_{t \in T} x_t m_t$  are

$$L^-(m) = \sum_{\sigma_1 \in S_1} \cdots \sum_{\sigma_k \in S_k} \epsilon_1(\sigma_1) \cdots \epsilon_k(\sigma_k) \sigma_1 \cdots \sigma_k \ell(m), \quad L^-(I) = \sum_{t \in T} x_t L^-(m_t),$$

where  $\epsilon_i: S_i \rightarrow \{\pm 1\}$  is the sign homomorphism.

*Remark:* An algebra over  $\mathbb{Q}$  satisfies the polynomial identity  $I$  if and only if it satisfies  $L^+(I)$ ; the same statement does not hold for  $L^-(I)$ . See Zhevlakov et al. [9, Chapter 1].

### 3. Polynomial identities in degrees 3 and 5

**Lemma 3.1:** *For the ternary cyclic sum, every multilinear polynomial identity of degree 3 follows from the cyclic law  $C$ .*

**Proof:** In  $\mathbb{Q}S_3$  let  $I = abc - bca$  (the cyclic law) and  $O = abc + bca + cab$  (the cyclic sum). One checks that  $IO = 0$ , and that if  $JO = 0$  then  $J = KI$  for some  $K \in \mathbb{Q}S_3$ . In the decomposition  $\mathbb{Q}S_3 \cong \mathbb{Q} \oplus M_2(\mathbb{Q}) \oplus \mathbb{Q}$ , the left ideal generated by  $I$  (respectively  $O$ ) is  $M_2(\mathbb{Q})$  (respectively  $\mathbb{Q} \oplus \mathbb{Q}$ ). The details are similar to the proof of Lemma 10 in Bremner and Peresi [3].  $\square$

**Lemma 3.2:** *For the ternary cyclic sum, every multilinear polynomial identity of degree 5 follows from the cyclic law  $C$ .*

**Proof:** Write  $A = \mathbb{Q}S_5$  where  $\sigma \in S_5$  represents the totally associative monomial  $a^\sigma b^\sigma c^\sigma d^\sigma e^\sigma$ . There are 3 association types in degree 5:  $[[abc]de]$ ,  $[a[bcd]e]$ ,  $[ab[cde]]$ . The cyclic law implies  $[[abc]de] = [e[abc]d] = [de[abc]]$ , so we only need the first type. Write  $B = \mathbb{Q}S_5$  where  $\sigma \in S_5$  represents the nonassociative monomial  $[[a^\sigma b^\sigma c^\sigma]d^\sigma e^\sigma]$ . Left multiplication gives an action of  $S_5$  on  $A$  and  $B$ . Let  $E \in A$  be the expansion of  $M = [[abc]de] \in B$ :

$$E = abcde + bcade + cabde + deabc + debca + decab + eabcd + ebcad + ecabd.$$

The submodule  $X \subseteq A \oplus B$  generated by  $E - M$  contains the relations between the totally associative operation and the nonassociative operation. The submodule  $X \cap B$  contains the polynomial identities for the ternary cyclic sum. The submodule  $T \subseteq B$  generated by  $[[abc]de] - [[bca]de]$  contains the consequences of the cyclic law. It suffices to show that  $X \cap B = T$ . For this we use the decomposition of  $\mathbb{Q}S_5$  as a direct sum of simple ideals: for each partition  $\lambda$  of 5, there is a simple ideal isomorphic to the algebra of  $d \times d$  matrices, as follows:

partition $\lambda$	5	41	32	$31^2$	$2^21$	$21^3$	$1^5$
dimension $d$	1	4	5	6	5	4	1

We now compute one representation at a time. For any  $\sigma \in S_5$  and any  $\lambda$ , the matrix representing  $\sigma$  in the simple ideal corresponding to  $\lambda$  can be computed efficiently following Clifton [6]. Taking linear combinations gives the matrix representing any element of  $\mathbb{Q}S_5$ . For each partition  $\lambda$  with dimension  $d$ , we represent an element of  $A \oplus B$  by a  $d \times 2d$  matrix. We give details for the largest representation ( $\lambda = 31^2$ ,  $d = 6$ ); the others are similar. The  $6 \times 12$  matrix representing  $E - M$  is

$$\left[ \begin{array}{cccccc|cccccc} 4 & 0 & 1 & 1 & 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & 1 & -2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & 1 & -2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 2 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 3 & -3 & 6 & 0 & -3 & 6 & 0 & 0 & 0 & 0 & 0 & -1 \end{array} \right]$$

Its row canonical form is

$$\left[ \begin{array}{cccccc|cccccc} 1 & 0 & 1/4 & 1/4 & 1/2 & 1/4 & -1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -7/4 & 1/4 & 3/2 & -7/4 & -1/4 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1/3 \end{array} \right]$$

Only the last 4 rows of the RCF have leading 1s in the right part (the  $B$  part): these rows represent polynomial identities  $I_1, \dots, I_4$  satisfied by the ternary cyclic sum. The  $6 \times 6$  matrix representing  $[[abc]de] - [[bca]de]$  and its RCF are

$$\left[ \begin{array}{cccccc} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \left[ \begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 & -1/3 \\ 0 & 0 & 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

The nonzero rows of the RCF agree with the rows representing  $I_1, \dots, I_4$ . □

#### 4. Polynomial identities in degree 7

**Definition 4.1:** We consider two nonassociative ternary polynomials:

$$D = [[[aaa]ab]aa] - [[aab][aaa]a], \quad E = [[[aaa]ba]aa] - [[aaa][aab]a].$$

**Example 4.2** Both  $D$  and  $E$  have multidegree  $(6, 1)$ . We consider the linearizations of  $D$ ; the linearizations of  $E$  are similar. We have

$$\ell(D) = [[[a_1a_2a_3]a_4a_7]a_5a_6] - [[a_1a_2a_7][a_3a_4a_5]a_6].$$

The occurrences of  $a$  have been replaced by  $a_1, \dots, a_6$  from left to right, and the single occurrence of  $b$  by  $a_7$ . Summing over all permutations of the  $a$  positions, we obtain the symmetric and alternating linearizations, and their sum:

$$\begin{aligned} L^+(D) &= \sum_{\sigma \in S_6} \left( [[[a_{\sigma_1}a_{\sigma_2}a_{\sigma_3}]a_{\sigma_4}a_7]a_{\sigma_5}a_{\sigma_6}] - [[a_{\sigma_1}a_{\sigma_2}a_7][a_{\sigma_3}a_{\sigma_4}a_{\sigma_5}]a_{\sigma_6}] \right), \\ L^-(D) &= \sum_{\sigma \in S_6} \epsilon(\sigma) \left( [[[a_{\sigma_1}a_{\sigma_2}a_{\sigma_3}]a_{\sigma_4}a_7]a_{\sigma_5}a_{\sigma_6}] - [[a_{\sigma_1}a_{\sigma_2}a_7][a_{\sigma_3}a_{\sigma_4}a_{\sigma_5}]a_{\sigma_6}] \right), \\ L^+(D) + L^-(D) &= \\ &= 2 \sum_{\sigma \in A_6} \left( [[[a_{\sigma_1}a_{\sigma_2}a_{\sigma_3}]a_{\sigma_4}a_7]a_{\sigma_5}a_{\sigma_6}] - [[a_{\sigma_1}a_{\sigma_2}a_7][a_{\sigma_3}a_{\sigma_4}a_{\sigma_5}]a_{\sigma_6}] \right). \end{aligned}$$

The last sum is over the alternating group  $A_6$ .

**Lemma 4.3:** For the ternary cyclic sum, every multilinear polynomial identity of degree 7 follows from the cyclic law  $C$  and the identities  $L^+(D) + L^-(D)$  and  $L^+(E) + L^-(E)$ .

**Proof:** This result was stated informally and without proof in Bremner and Hentzel [2, page 400]. Write  $A = \mathbb{Q}S_7$  where  $\sigma \in S_7$  represents the totally associative monomial  $a^\sigma b^\sigma c^\sigma d^\sigma e^\sigma f^\sigma g^\sigma$ . The cyclic law implies that we need only two

Table 1. Representation matrix for the submodule  $X$  in degree 7

8	-3	-6	-7	-6	6	-1	0	0	0	0	0	0	0	0	0	0	0
8	6	-6	-7	-6	-3	0	-1	0	0	0	0	0	0	0	0	0	0
2	0	-3	-4	0	3	0	0	-1	0	0	0	0	0	0	0	0	0
2	3	0	-4	-3	0	0	0	0	-1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0
6	-3	-3	6	-6	-6	0	0	0	0	0	0	-1	0	0	0	0	0
0	3	3	0	-3	-3	0	0	0	0	0	0	0	-1	0	0	0	0
0	3	3	0	-3	-3	0	0	0	0	0	0	0	0	-1	0	0	0
0	3	3	0	-3	-3	0	0	0	0	0	0	0	0	0	-1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1

Table 2. Row canonical form of the matrix from Table 1

1	0	0	0	-3/2	-1/2	0	1/9	0	-4/9	0	0	-1/6	0	0	1/18	0	0
0	1	0	0	0	-1	0	-2/9	0	5/9	0	0	1/9	0	0	-1/3	0	0
0	0	1	0	-1	0	0	2/9	0	-5/9	0	0	-1/9	0	0	0	0	0
0	0	0	1	0	-1	0	-1/9	0	4/9	0	0	0	0	0	-2/9	0	0
0	0	0	0	0	0	0	1	1	0	-5	0	0	-1	0	0	3	0
0	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	-1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 3. Representation matrix for the submodule  $K$  in degree 7

0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	2	-1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	-1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-20	-20	120	-20	0	0	20	0	0	60	-60	-60	-60	-60	-60	-60	-60	-60
0	0	0	0	30	30	0	-30	-30	60	-20	-20	-20	-20	-20	-20	-20	-20
0	0	0	0	30	30	0	-30	-30	60	-16	-16	-16	-16	-16	-16	-16	-16
0	0	0	0	27	27	0	-27	-27	54	-12	-12	-12	-12	-12	-12	-12	-12
0	0	0	0	21	21	0	-21	-21	42	-8	-8	-8	-8	-8	-8	-8	-8
0	0	0	0	12	12	0	-12	-12	24	-4	-4	-4	-4	-4	-4	-4	-4
-20	-20	-20	120	0	0	20	-60	-60	40	0	0	0	0	0	0	0	0
0	0	0	0	30	30	0	-20	-20	40	-30	-30	-30	-30	-30	-30	-30	-30
0	0	0	0	30	30	0	-16	-16	32	-30	-30	-30	-30	-30	-30	-30	-30
0	0	0	0	27	27	0	-12	-12	24	-27	-27	-27	-27	-27	-27	-27	-27
0	0	0	0	21	21	0	-8	-8	16	-21	-21	-21	-21	-21	-21	-21	-21
0	0	0	0	12	12	0	-4	-4	8	-12	-12	-12	-12	-12	-12	-12	-12

association types:  $[[[abc]de]fg]$ ,  $[[abc][def]g]$ . Write  $B = \mathbb{Q}S_7 \oplus \mathbb{Q}S_7$  where  $(\sigma, 0)$  represents  $[[[a^\sigma b^\sigma c^\sigma]d^\sigma e^\sigma]f^\sigma g^\sigma]$  and  $(0, \sigma)$  represents  $[[a^\sigma b^\sigma c^\sigma][d^\sigma e^\sigma f^\sigma]g^\sigma]$ . Left multiplication gives an action of  $S_7$  on  $A$  and  $B$ . Let  $E_1, E_2 \in A$  be the expansions of  $M_1 = [[[abc]de]fg]$ ,  $M_2 = [[abc][def]g] \in B$ . The submodule  $X \subseteq A \oplus B$  generated by  $E_1 - M_1, E_2 - M_2$  contains the relations between the totally associative operation and the nonassociative operation. The submodule  $X \cap B$  contains the polynomial

$$[X] = \left[ \begin{array}{c|ccccc} R_\lambda(E_1) & -I & O & O & O & O \\ R_\lambda(E_2) & O & -I & O & O & O \\ R_\lambda(E_3) & O & O & -I & O & O \\ R_\lambda(E_4) & O & O & O & -I & O \\ R_\lambda(E_5) & O & O & O & O & -I \end{array} \right] \quad \text{RCF}([X]) = \left[ \begin{array}{c|c} * & ** \\ \hline O & [Y] \end{array} \right]$$

Figure 1. Representation matrix for expansions in degree 9

$$[T] = \left[ \begin{array}{ccccc} R_\lambda(I_1) & O & O & O & O \\ O & R_\lambda(I_2) & O & O & O \\ O & R_\lambda(I_3) & O & O & O \\ O & O & R_\lambda(I_4) & O & O \\ O & O & R_\lambda(I_5) & O & O \\ O & O & O & R_\lambda(I_6) & O \\ O & O & O & R_\lambda(I_7) & O \\ O & O & O & O & R_\lambda(I_8) \\ O & O & O & O & R_\lambda(I_9) \end{array} \right]$$

Figure 2. Representation matrix for consequences of cyclic law

$$[L] = \left[ \begin{array}{ccccc} R_\lambda(J'_{1,1}) & R_\lambda(J'_{1,2}) & R_\lambda(J'_{1,3}) & R_\lambda(J'_{1,4}) & R_\lambda(J'_{1,5}) \\ R_\lambda(J''_{1,1}) & R_\lambda(J''_{1,2}) & R_\lambda(J''_{1,3}) & R_\lambda(J''_{1,4}) & R_\lambda(J''_{1,5}) \\ R_\lambda(J'''_{1,1}) & R_\lambda(J'''_{1,2}) & R_\lambda(J'''_{1,3}) & R_\lambda(J'''_{1,4}) & R_\lambda(J'''_{1,5}) \\ R_\lambda(J'_{2,1}) & R_\lambda(J'_{2,2}) & R_\lambda(J'_{2,3}) & R_\lambda(J'_{2,4}) & R_\lambda(J'_{2,5}) \\ R_\lambda(J''_{2,1}) & R_\lambda(J''_{2,2}) & R_\lambda(J''_{2,3}) & R_\lambda(J''_{2,4}) & R_\lambda(J''_{2,5}) \\ R_\lambda(J'''_{2,1}) & R_\lambda(J'''_{2,2}) & R_\lambda(J'''_{2,3}) & R_\lambda(J'''_{2,4}) & R_\lambda(J'''_{2,5}) \end{array} \right]$$

Figure 3. Representation matrix for liftings of degree 7 identities

identities for the ternary cyclic sum. The submodule  $T \subseteq B$  generated by

$$[[[abc]de]fg] - [[[bca]de]fg], [[abc][def]g] - [[bca][def]g], [[abc][def]g] - [[abc][efd]g],$$

contains the consequences of the cyclic law. Let  $K \subseteq B$  be the submodule generated by these three identities together with  $L^+(D) + L^-(D)$  and  $L^+(E) + L^-(E)$ . It suffices to show that  $X \cap B = K$ ; we do this one representation at a time. The partitions  $\lambda$  of 7 and the dimensions  $d$  of the irreducible representations of  $S_7$  are

$\lambda$	7	61	52	51 <sup>2</sup>	43	421	41 <sup>3</sup>	3 <sup>2</sup> 1	32 <sup>2</sup>	321 <sup>2</sup>	31 <sup>4</sup>	2 <sup>3</sup> 1	2 <sup>2</sup> 1 <sup>3</sup>	21 <sup>5</sup>	1 <sup>7</sup>
$d$	1	6	14	15	14	35	20	21	21	35	15	14	14	6	1

For every partition except 7, 61, 21<sup>5</sup>, 1<sup>7</sup> we have  $X \cap B = T$ ; since  $T \subseteq K \subseteq X \cap B$  we obtain  $X \cap B = K$  in these cases. For the other 4 cases we give details only for  $\lambda = 61, d = 6$ . The  $12 \times 18$  matrix representing  $E_1 - M_1$  and  $E_2 - M_2$  is in Table 1; its row canonical form is in Table 2. Only the last 8 rows of the RCF have leading 1s in the right part (the  $B$  part): these rows represent polynomial identities  $J_1, \dots, J_8$  satisfied by the ternary cyclic sum. The  $30 \times 12$  matrix representing the five generators of  $K$  is given in Table 3. Its RCF has 8 nonzero rows, which agree with the rows representing  $J_1, \dots, J_8$ . □

partitions	dimension	rank( $K$ )	rank( $Y$ )	difference
81 21 <sup>7</sup>	8	31	34	3
72 2 <sup>2</sup> 1 <sup>5</sup>	27	108	112	4
71 <sup>2</sup> 31 <sup>6</sup>	28	108	115	7
63 2 <sup>3</sup> 1 <sup>3</sup>	48	201	202	1
61 <sup>3</sup> 41 <sup>5</sup>	56	223	227	4
54 2 <sup>4</sup> 1	42	180	181	1
51 <sup>4</sup>	70	280	283	3

Figure 4. Ranks of new identities in degree 9

**5. Polynomial identities in degree 9**

Write  $A = \mathbb{Q}S_9$  where the associative monomial  $a^\sigma b^\sigma c^\sigma d^\sigma e^\sigma f^\sigma g^\sigma h^\sigma i^\sigma$  is represented by  $\sigma \in S_9$ . The cyclic law implies that we need only five association types:

$$[[[[abc]de]fg]hi], [[[[abc][def]g]hi], [[[[abc]de][fgh]i], [[[[abc]de]f[ghi]], [[abc][def][ghi]].$$

Write  $M_1, \dots, M_5$  for these nonassociative monomials, and  $B = (\mathbb{Q}S_9)^5$  where  $\sigma \in S_9$  in the  $i$ -th summand represents  $\sigma M_i$ . Left multiplication gives an action of  $S_9$  on  $A$  and  $B$ . Let  $E_1, \dots, E_5 \in A$  be the expansions of  $M_1, \dots, M_5 \in B$ . The submodule  $X \subseteq A \oplus B$  generated by  $E_i - M_i$  for  $i = 1, \dots, 5$  contains the relations between the totally associative operation and the nonassociative operation. The submodule  $X \cap B$  contains the polynomial identities for the ternary cyclic sum. Given a partition  $\lambda$  with representation of dimension  $d$ , the component of  $X$  can be represented as a matrix  $[X]$  of size  $5d \times 6d$  consisting of  $d \times d$  blocks;  $[X]$  has a  $5d \times d$  left part (for  $A$ ) and a  $5d \times 5d$  right part (for  $B$ ). For  $i = 1, \dots, 5$  the block in position  $(i, 1)$  contains  $R_\lambda(E_i)$  and the block in position  $(i+1, i)$  contains  $-I_d$  (representing  $-M_i$ ); the other blocks are zero (Figure 1). In the row canonical form, a row with leading 1 in the right part represents a polynomial identity for the ternary cyclic sum. In Figure 1 the vertical lines in  $[X]$  and its RCF are in the same position; the horizontal line in the RCF occurs between rows  $r$  and  $r+1$ , where the latter is the first row with leading 1 in the right part. In the RCF, every row in the upper left  $r \times d$  block denoted  $(*)$  is nonzero, and the lower left  $(5d-r) \times d$  block is zero. Since  $[X]$  has full row rank, every row in the lower right  $(5d-r) \times 5d$  block  $[Y]$  of the RCF is nonzero. The submodule  $T \subseteq B$  generated by

$$\begin{aligned} I_1 &= [[[[abc]de]fg]hi] - [[[[bca]de]fg]hi], \\ I_2 &= [[[[abc][def]g]hi] - [[[[bca][def]g]hi], \quad I_3 = [[[[abc][def]g]hi] - [[[[abc][efd]g]hi], \\ I_4 &= [[[[abc]de][fgh]i] - [[[[bca]de][fgh]i], \quad I_5 = [[[[abc]de][fgh]i] - [[[[abc]de][ghf]i], \\ I_6 &= [[[[abc]de]f[ghi]] - [[[[bca]de]f[ghi]], \quad I_7 = [[[[abc]de]f[ghi]] - [[[[abc]de]f[hig]], \\ I_8 &= [[[[abc][def][ghi]] - [[[[bca][def][ghi]], \quad I_9 = [[[[abc][def][ghi]] - [[def][ghi][abc]], \end{aligned}$$

contains the consequences of the cyclic law. The component of  $T$  in partition  $\lambda$  is represented by a matrix  $[T]$  of size  $9d \times 5d$  (Figure 2). Write  $J_1 = L^+(D) + L^-(D)$  and  $J_2 = L^+(E) + L^-(E)$  for the identities of Lemma 4.3. Both are unchanged by an even permutation of  $a_1, \dots, a_6$ . We therefore need only three liftings of each to degree 9, which we call  $J'_i, J''_i, J'''_i$ :

$$J_i([a, h, i], b, c, d, e, f, g), \quad J_i(a, b, c, d, e, f, [g, h, i]), \quad [J_i(a, b, c, d, e, f, g), h, i].$$

Table 4. Polynomial identities with two variables in degree 9

$-[[[aaa]ab]ba]bb$	$-[[[aaa]ba]ba]bb$	$-[[[aaa]bb]aa]bb$	$-[[[aaa]bb]ba]ab$	$-[[[aaa]bb]ba]ba$
$+[[[bbb]aa]aa]ab$	$+[[[bbb]aa]aa]ba$	$-[[[bbb]aa]ab]aa$	$-2[[[bbb]aa]ba]aa$	$+[[[bbb]ab]aa]aa$
$+[[[bbb]ba]aa]aa$	$-[[[aaa]abb]b]ba$	$-[[[aaa]bbb]b]aa$	$+2[[[aab]aaa]b]bb$	$+3[[[aab]aab]b]ab$
$+3[[[abb]aaa]b]ab$	$+2[[[abb]aaa]b]ba$	$+3[[[abb]aab]b]aa$	$+[[[aaa]aa]bbb]b$	$+[[[aaa]ab]abb]b$
$+[[[aaa]ba]abb]b$	$-[[[aaa]bb]aaa]b$	$+[[[aab]aa]bbb]a$	$+3[[[aab]ba]abb]a$	$-[[[aab]bb]aaa]b$
$-[[[abb]ba]aaa]b$	$+3[[[abb]ba]aab]a$	$-[[[bbb]aa]aaa]b$	$+[[[bbb]aa]aab]a$	$-[[[aaa]ab]a]bbb$
$-[[[aaa]ba]a]bbb$	$+2[[[aaa]bb]b]aab$	$+[[[aab]aa]a]bbb$	$-3[[[aab]ab]a]abb$	$+[[[aab]bb]b]aaa$
$-3[[[abb]ab]a]aab$	$-[[[abb]ba]b]aaa$	$-[[[bbb]aa]b]aaa$	$-[[[aaa]abb]abb]$	$-[[[aaa]bbb]aab]$
$-6[[[aab]aab]abb]$				
$[[[aaa]aa]bb]bb$	$-2[[[aaa]ab]ab]bb$	$-2[[[aaa]ab]ba]bb$	$+[[[aaa]ab]bb]ab$	$+[[[aaa]ab]bb]ba$
$+[[[aaa]ba]ab]bb$	$+[[[aaa]ba]ba]bb$	$+[[[aaa]ba]bb]ab$	$+[[[aaa]ba]bb]ba$	$-2[[[aaa]bb]aa]bb$
$-2[[[aaa]bb]ab]ab$	$-2[[[aaa]bb]ab]ba$	$+[[[aaa]bb]ba]ab$	$+[[[aaa]bb]ba]ba$	$+[[[aaa]bb]bb]aa$
$-[[[bbb]aa]aa]ab$	$-[[[bbb]aa]aa]ba$	$+2[[[bbb]aa]ab]aa$	$-[[[bbb]aa]ba]aa$	$-[[[bbb]ab]aa]aa$
$-[[[bbb]ba]aa]aa$	$-3[[[aaa]abb]b]ba$	$-2[[[aaa]bbb]a]ab$	$-2[[[aaa]bbb]a]ba$	$+[[[aaa]bbb]b]aa$
$-3[[[abb]aaa]a]ab$	$+3[[[abb]aaa]a]ab$	$+[[[bbb]aaa]a]ab$	$+[[[bbb]aaa]a]ba$	$+[[[aaa]ab]abb]b$
$-2[[[aaa]ba]abb]b$	$-2[[[aaa]bb]aab]b$	$-2[[[aaa]bb]abb]a$	$-[[[aab]aa]bbb]a$	$+[[[aab]bb]aaa]b$
$-2[[[abb]ab]aaa]b$	$-2[[[abb]ba]aaa]b$	$-2[[[abb]bb]aaa]a$	$+2[[[bbb]aa]aab]a$	$+4[[[aaa]ab]b]abb$
$+[[[aaa]ba]b]abb$	$+[[[aaa]bb]a]abb$	$+4[[[aaa]bb]b]aab$	$-[[[aab]aa]a]bbb$	$+[[[aaa]bb]b]aaa$
$+[[[abb]ab]b]aaa$	$+[[[abb]ba]b]aaa$	$+4[[[abb]bb]a]aaa$	$-[[[bbb]aa]a]aab$	$+[[[aaa]abb]bbb]$
$-[[[aaa]aa]bb]bb$	$-[[[aaa]ab]ab]bb$	$+[[[aaa]ab]ba]bb$	$-[[[aaa]ab]bb]ab$	$-[[[aaa]ab]bb]ba$
$+[[[aaa]ba]ab]bb$	$-[[[aaa]bb]ab]ab$	$-[[[aaa]bb]ab]ba$	$-[[[aaa]bb]bb]aa$	$-3[[[aab]ab]ab]ab$
$-3[[[aab]ab]ab]ba$	$+3[[[aab]ab]ba]ab$	$+3[[[aab]ab]ba]ba$	$+3[[[aab]ba]ab]ab$	$+3[[[aab]ba]ab]ba$
$+3[[[aab]ba]bb]aa$	$-3[[[aab]bb]aa]ab$	$-3[[[aab]bb]aa]ba$	$-3[[[aab]bb]ab]aa$	$-3[[[abb]ab]ab]aa$
$+3[[[abb]ab]ba]aa$	$+3[[[abb]ba]ab]aa$	$-3[[[abb]bb]aa]aa$	$+[[[bbb]aa]aa]ab$	$+[[[bbb]aa]aa]ba$
$+[[[bbb]ab]aa]aa$	$+[[[bbb]ba]aa]aa$	$+[[[aaa]abb]a]bb$	$-[[[aaa]abb]b]ba$	$-[[[aaa]bbb]b]aa$
$+[[[aab]aaa]b]bb$	$-3[[[aab]aab]a]bb$	$-3[[[aab]aab]b]ba$	$-3[[[aab]abb]a]ab$	$-3[[[aab]abb]a]ba$
$-3[[[aab]abb]b]aa$	$-2[[[aab]bbb]a]aa$	$+2[[[abb]aaa]a]bb$	$+3[[[abb]aaa]b]ab$	$+2[[[abb]aaa]b]ba$
$+3[[[abb]aaa]b]aa$	$+[[[bbb]aaa]a]ab$	$+[[[bbb]aaa]a]ba$	$+[[[bbb]aaa]b]aa$	$+[[[bbb]aaa]b]aa$
$-2[[[aaa]ba]abb]b$	$-[[[aaa]bb]aab]b$	$+[[[aab]aa]bbb]a$	$+3[[[aab]ab]aab]b$	$-3[[[abb]ba]aab]b$
$-[[[aab]bb]aaa]b$	$+[[[abb]ab]aaa]b$	$-[[[abb]ba]aaa]b$	$-[[[abb]bb]aaa]a$	$-[[[aaa]ab]b]abb$
$+[[[aaa]bb]a]abb$	$+[[[aaa]bb]b]aab$	$+[[[aab]aa]a]bbb$	$+3[[[aab]ba]b]aab$	$+3[[[aab]bb]a]aab$
$+[[[abb]bb]b]aaa$	$+[[[abb]ba]b]aaa$	$-[[[aaa]aab]bbb]$	$-[[[aaa]bbb]aab]$	
$-[[[aaa]ab]ab]bb$	$+[[[aaa]ba]ba]bb$	$+[[[aaa]bb]ba]ab$	$+[[[aaa]bb]ba]ba$	$-3[[[aab]ab]aa]bb$
$+3[[[aab]ba]bb]aa$	$-3[[[abb]aa]ba]ab$	$-3[[[abb]ab]ba]aa$	$-3[[[abb]ab]ba]aa$	$+3[[[abb]ba]ab]bb$
$-2[[[bbb]aa]ba]aa$	$+[[[bbb]ba]aa]aa$	$-[[[aaa]abb]a]bb$	$+[[[aaa]abb]b]ab$	$-2[[[aaa]abb]b]ba$
$+[[[aaa]bbb]a]ab$	$-[[[aaa]bbb]b]aa$	$+3[[[aab]aab]b]ab$	$+3[[[aab]aab]b]ba$	$-3[[[aab]abb]b]aa$
$+[[[aab]bbb]a]aa$	$-[[[abb]aaa]a]bb$	$-3[[[abb]aaa]b]ab$	$-6[[[abb]aab]a]ab$	$+3[[[abb]aab]a]ba$
$-2[[[bbb]aaa]a]ab$	$-[[[bbb]aaa]b]aa$	$+5[[[aaa]ab]abb]b$	$+[[[aaa]ab]bbb]a$	$-3[[[aaa]ba]abb]b$
$-[[[aaa]ba]bbb]a$	$+2[[[aaa]bb]aab]b$	$+2[[[aaa]bb]abb]a$	$-[[[aab]aa]bbb]a$	$+9[[[aab]ab]aab]b$
$+3[[[aab]ab]abb]a$	$-3[[[aab]ba]aab]b$	$-3[[[aab]ba]abb]a$	$+[[[aab]bb]aaa]b$	$+3[[[aab]bb]aab]a$
$-3[[[abb]aa]abb]a$	$+[[[abb]ab]aaa]b$	$-3[[[abb]ab]aab]a$	$+[[[abb]ba]aaa]b$	$+3[[[abb]ba]aab]a$
$-[[[bbb]aa]aab]a$	$-[[[bbb]ab]aaa]a$	$+[[[bbb]ba]aaa]a$	$+[[[aaa]aa]b]bbb$	$-2[[[aaa]ab]b]abb$
$+[[[aaa]ba]a]bbb$	$+2[[[aaa]ba]b]abb$	$+[[[aaa]bb]a]abb$	$+2[[[aab]aa]a]bbb$	$-3[[[aab]ab]a]abb$
$-6[[[aab]ab]b]aab$	$+3[[[abb]aa]a]abb$	$+3[[[abb]ab]a]aab$	$+[[[abb]ab]b]aaa$	$+[[[abb]ba]b]aaa$
$+2[[[abb]bb]a]aaa$	$+[[[bbb]ab]a]aaa$	$-2[[[aaa]aab]bbb]$	$-3[[[aaa]abb]abb]$	$-2[[[aaa]bbb]aab]$

Let  $L \subseteq B$  be the submodule generated by these 6 liftings. We collect terms with the same association type; for example,  $J'_1 = J'_{1,1} + \cdots + J'_{1,5}$ . The component of  $L$  in partition  $\lambda$  is represented by a matrix  $[L]$  of size  $6d \times 5d$  (Figure 3). The submodule  $K = T + L \subseteq B$  (in general not a direct sum) contains the identities in degree 9 which follow from identities in lower degrees. The component of  $K$  in partition  $\lambda$  is represented by the  $15d \times 5d$  matrix  $[K]$  obtained by stacking  $[T]$  and  $[L]$ . The row canonical form of  $[K]$  has at most  $5d-r$  nonzero rows, since this is the rank of the matrix  $[Y]$  which contains all the identities for the ternary cyclic sum in this partition. If  $\text{rank}([K]) < \text{rank}([Y])$  then there are new identities in degree 9: identities which are not consequences of the cyclic law and the lifted identities. If  $\text{rank}([K]) = \text{rank}([Y])$  then there are no new identities. The computations in Bremner and Hentzel [2] show that  $\text{rank}([K]) = \text{rank}([Y])$  except for the partitions in Figure 4: the dimensions and ranks for conjugate partitions are the same, so we put them in the same row.

In the rest of this paper we find (i) four identities in two variables which are substantially simpler than the nine identities of [2, Table IV], and (ii) seven identities in three variables which do not follow from the identities in two variables.

Table 5. Polynomial identities 1 to 6 with three variables in degree 9

$[[[aaa][aaa]b]ac]$ $+ [[[aaa]cb]a[aaa]]$	$+3[[[aac][aaa]b]aa]$ $- [[[aaa][aaa]acb]]$	$- [[[(aaa)ba][aaa]c]$ $-3[[[aaa][aab]aac]]$	$- [[[(aaa)bc][aaa]a]$	$+ [[[(aaa)ab]c[aaa]]$
$[[[(aaa)aa]ab]ac]$ $- [[[(aaa)ab]aa]ca]$ $+ [[[(aaa)ba]aa]ca]$ $- [[[(aaa)cb]aa]aa]$ $- [[[(aab)[aaa]a]ac]$ $+ [[[(aaa)ba][aac]a]$ $- [[[(aaa)ab]a[aac]]]$ $-2[[[aac]ba]a[aaa]]$	$+ [[[(aaa)aa]ab]ca]$ $- [[[(aaa)ac]aa]ab]$ $+ [[[(aaa)bc]aa]aa]$ $+ [[[(aaa)[aab]a]ac]$ $- [[[(aab)[aaa]a]ca]$ $- [[[(aaa)ca][aab]a]$ $+ [[[(aaa)ac]a[aab]]]$ $- [[[(aaa)[aab]aac]]]$	$- [[[(aaa)aa]ba]ac]$ $- [[[(aaa)ac]ab]aa]$ $- [[[(aaa)[aaa]b]ca]$ $+ [[[(aaa)ca]aa]ba]$ $+ [[[(aaa)[aab]a]ca]$ $- [[[(aac)aa][aaa]b]$ $+ [[[(aac)ba]a[aac]]]$ $+ [[[(aaa)[aac]aab]]]$	$- [[[(aaa)aa]ba]ca]$ $- [[[(aaa)ac]ba]aa]$ $+ [[[(aaa)ca]ab]aa]$ $+ [[[(aaa)[aac]b]aa]$ $- [[[(abc)[aaa]a]aa]$ $+2[[[aac]ab][aaa]a]$ $- [[[(aac)aa]b[aaa]]]$	$- [[[(aaa)ab]aa]ac]$ $+ [[[(aaa)ba]aa]ac]$ $+ [[[(aaa)ca]ba]aa]$ $+ [[[(aaa)ca]ba]aa]$ $+ [[[(aaa)cb]aa]aa]$ $+ [[[(aaa)[aac]b]aa]$ $- [[[(abc)[aaa]a]aa]$ $+ [[[(aaa)ab]a[aaa]]]$ $- [[[(aac)ab]a[aaa]]]$
$[[[(aaa)aa]ca]ab]$ $+ [[[(aaa)ac]aa]ba]$ $+ [[[(aaa)ca]ba]aa]$ $- [[[(aaa)[aac]a]ba]$ $- [[[(aac)[aaa]b]aa]$ $+ [[[(aaa)ba][aac]a]$ $+ [[[(aac)aa][aaa]b]$ $- [[[(aaa)aa]a[abc]]]$ $- [[[(aaa)bc]a[aaa]]]$ $- [[[(aac)ba]a[aaa]]]$ $- [[[(aaa)[aab]aac]]]$	$+ [[[(aaa)aa]ca]ba]$ $- [[[(aaa)ac]ab]aa]$ $- [[[(aaa)[aaa]b]ac]$ $- [[[(aaa)[aac]b]aa]$ $+2[[[abc][aaa]a]aa]$ $+ [[[(aaa)bc][aaa]a]$ $- [[[(aac)ab][aaa]a]$ $+2[[[aaa]aa]a[acb]]]$ $- [[[(aaa)ca]a[aab]]]$ $-2[[[abc]aa]a[aaa]]]$	$- [[[(aaa)ab]aa]ca]$ $+ [[[(aaa)ba]ca]aa]$ $- [[[(aaa)[aaa]b]ca]$ $-2[[[aaa][abc]a]aa]$ $+2[[[aac]aa][abc]a]$ $+ [[[(aaa)ca][aab]a]$ $+ [[[(aac)ba][aaa]a]$ $- [[[(aaa)ab]a[aac]]]$ $+ [[[(ab)aa]c[aaa]]]$ $+ [[[(cb)aa]a[aaa]]]$	$+ [[[(aaa)ab]ca]aa]$ $- [[[(aaa)[aab]a]ac]$ $+ [[[(aab)[aaa]a]ac]$ $- [[[(aaa)aa][acb]a]$ $- [[[(aaa)aa][acb]a]$ $+ [[[(aaa)ac]a[aab]]]$ $+2[[[aab]ac]a[aaa]]]$ $- [[[(aaa)[aaa]abc]]]$	$-2[[[(aaa)ac]aa]ab]$ $- [[[(aaa)[aab]c]ba]$ $+ [[[(aac)[aaa]a]ba]$ $+ [[[(aaa)ba][aaa]c]$ $+2[[[aab]ca][aaa]a]$ $- [[[(abc)aa][aaa]a]$ $- [[[(aaa)ac]b[aaa]]]$ $+ [[[(aac)aa]b[aaa]]]$ $- [[[(aaa)[aaa]abc]]]$
$[[[(aaa)aa]ba]ca]$ $+ [[[(aaa)ab]ac]aa]$ $- [[[(aaa)ba]ac]aa]$ $- [[[(aaa)cb]aa]aa]$ $- [[[(aaa)[aab]a]ca]$ $+ [[[(aac)[aaa]a]ab]$ $- [[[(aaa)ba][aaa]c]$ $+3[[[aab]ca][aaa]a]$ $- [[[(aaa)aa]a[acb]]]$ $+ [[[(aaa)ca]b[aaa]]]$ $+2[[[aac]ba]a[aaa]]]$	$- [[[(aaa)aa]ca]ba]$ $+ [[[(aaa)ab]ca]aa]$ $- [[[(aaa)ba]ca]aa]$ $+ [[[(aaa)[aaa]b]ac]$ $- [[[(aaa)[aac]b]aa]$ $+2[[[abc][aaa]a]aa]$ $-2[[[aaa]ba][aac]a]$ $- [[[(aac)aa][aaa]b]$ $-2[[[aaa]ac]a[aab]]]$ $+ [[[(aab)ac]a[aaa]]]$ $- [[[(aaa)[aaa]abc]]]$	$+ [[[(aaa)aa]cb]aa]$ $- [[[(aaa)ac]ab]aa]$ $- [[[(aaa)ca]aa]ba]$ $+ [[[(aaa)[aaa]c]ab]$ $+ [[[(aaa)[acb]a]aa]$ $+ [[[(aaa)[aaa]a]aa]$ $- [[[(aaa)bc][aaa]a]$ $- [[[(aac)ab][aaa]a]$ $+ [[[(aac)ba]a[aac]]]$ $-3[[[aab]ca]a[aaa]]]$ $+2[[[aaa][aab]aac]]]$	$- [[[(aaa)ab]aa]ac]$ $+ [[[(aaa)ca]ab]aa]$ $+ [[[(aaa)[aaa]c]ba]$ $+ [[[(aab)[aaa]a]ca]$ $- [[[(aaa)aa][aac]b]$ $- [[[(aaa)ca][aaa]b]$ $- [[[(aac)ba][aaa]a]$ $+ [[[(aaa)bc]a[aaa]]]$ $+ [[[(aac)aa]b[aaa]]]$	$+ [[[(aaa)ab]aa]ca]$ $- [[[(aaa)ba]aa]ca]$ $+ [[[(aaa)ca]ba]aa]$ $+ [[[(aaa)[aab]a]ac]$ $+ [[[(aab)[aaa]c]aa]$ $- [[[(aaa)aa][aac]b]a]$ $-2[[[aab]ac][aaa]a]$ $+ [[[(cb)aa][aaa]a]$ $- [[[(aaa)ca]a[aab]]]$ $+2[[[aac]ab]a[aaa]]]$
$- [[[(aaa)aa]ab]ca]$ $- [[[(aaa)aa]cb]aa]$ $- [[[(aaa)ba]aa]ac]$ $+2[[[(aaa)ca]ba]aa]$ $- [[[(aaa)[aaa]c]ba]$ $+ [[[(aab)[aaa]a]ac]$ $- [[[(aaa)aa][abc]a]$ $- [[[(aaa)ba][aac]a]$ $- [[[(aab)ca][aaa]a]$ $- [[[(aaa)aa]b[aac]]]$ $+ [[[(aaa)ba]c[aaa]]]$ $+ [[[(aab)ca]a[aaa]]]$ $- [[[(aaa)[aab]aac]]]$	$- [[[(aaa)aa]ac]ab]$ $+2[[[(aaa)ab]ac]aa]$ $- [[[(aaa)ba]aa]ca]$ $- [[[(aaa)cb]aa]aa]$ $+ [[[(aaa)[aab]c]aa]$ $+ [[[(aab)[aaa]a]ca]$ $- [[[(aaa)ab][aaa]c]$ $- [[[(aaa)bc][aaa]a]$ $+ [[[(aac)aa][aaa]b]$ $+ [[[(aaa)ab]a[aac]]]$ $- [[[(aaa)bc]a[aaa]]]$ $- [[[(aac)aa]b[aaa]]]$ $+ [[[(aaa)[aac]aab]]]$	$+ [[[(aaa)aa]ba]ac]$ $+ [[[(aaa)ab]ca]aa]$ $- [[[(aaa)ba]aa]ca]$ $- [[[(aaa)ca]aa]ba]$ $+ [[[(aaa)[aaa]c]ab]$ $+ [[[(aaa)[acb]a]aa]$ $- [[[(aaa)bc][aaa]a]$ $- [[[(aac)ab][aaa]a]$ $+ [[[(aac)ba]a[aac]]]$ $-3[[[aab]ca]a[aaa]]]$ $+2[[[aaa][aab]aac]]]$	$+ [[[(aaa)aa]bc]aa]$ $- [[[(aaa)ac]aa]ab]$ $-2[[[(aaa)ba]ca]aa]$ $+ [[[(aaa)aa]a]b]ca]$ $- [[[(aaa)[aac]b]aa]$ $+ [[[(aac)[aaa]b]aa]$ $+2[[[aac]ac][aaa]b]$ $+ [[[(aaa)cb][aaa]a]$ $- [[[(abc)aa][aaa]a]$ $+ [[[(aaa)ac]b[aaa]]]$ $+ [[[(aaa)cb]a[aaa]]]$ $+2[[[aaa][aaa]abc]]]$	$+ [[[(aaa)aa]ca]ba]$ $-2[[[(aaa)ac]ab]aa]$ $+ [[[(aaa)bc]aa]ab]$ $+ [[[(aaa)aa]a]c]ba]$ $+ [[[(aaa)[aab]c]aa]$ $+ [[[(aaa)aa][aac]b]$ $+ [[[(aaa)ba][aaa]c]$ $+ [[[(aaa)aa]a[acb]]]$ $+ [[[(aaa)ba]a[aac]]]$ $+ [[[(aab)ac]a[aaa]]]$ $-2[[[aaa][aaa]abc]]]$
$[[[(aaa)aa]ab]ac]$ $- [[[(aaa)ab]ca]aa]$ $+ [[[(aaa)ba]ca]aa]$ $+ [[[(aaa)cb]aa]aa]$ $+2[[[aaa][aab]c]aa]$ $- [[[(aab)[aaa]a]ac]$ $-2[[[abc][aaa]a]aa]$ $+2[[[aaa]ba][aac]a]$ $+ [[[(aab)ca][aaa]a]$ $+ [[[(aaa)aa]a[abc]]]$ $- [[[(aaa)ca]a[aab]]]$ $-2[[[aac]ab]a[aaa]]]$ $+2[[[aaa][aac]aab]]]$	$- [[[(aaa)aa]ac]ab]$ $- [[[(aaa)ac]aa]ba]$ $+ [[[(aaa)bc]aa]ca]$ $- [[[(aaa)[aaa]b]ca]$ $- [[[(aaa)[aac]a]ab]$ $+ [[[(aab)[aaa]a]ca]$ $+ [[[(aaa)aa][aab]c]$ $- [[[(aaa)bc][aaa]a]$ $+ [[[(aac)aa][aaa]b]$ $+ [[[(aaa)ab]a[aac]]]$ $- [[[(aaa)bc]a[aaa]]]$ $- [[[(aac)aa]b[aaa]]]$ $+ [[[(aaa)[aac]aab]]]$	$+ [[[(aaa)aa]ba]ca]$ $- [[[(aaa)ac]ba]aa]$ $+ [[[(aaa)ca]aa]ba]$ $+ [[[(aaa)[aaa]c]ab]$ $+2[[[aaa][aac]a]ba]$ $+ [[[(aab)[aaa]c]aa]$ $- [[[(aaa)aa][abc]a]$ $- [[[(aaa)aa][aab]a]$ $+2[[[aac]ab][aaa]a]$ $+ [[[(aac)cb][aaa]a]$ $- [[[(aaa)ab]c[aaa]]]$ $+ [[[(aab)ac]a[aaa]]]$ $- [[[(acb)aa]a[aaa]]]$	$- [[[(aaa)aa]ca]ba]$ $+ [[[(aaa)ba]aa]ac]$ $- [[[(aaa)ca]ab]aa]$ $+ [[[(aaa)[aaa]c]ba]$ $- [[[(aaa)[aac]b]aa]$ $- [[[(aac)[aaa]a]ba]$ $- [[[(aaa)ab][aaa]c]$ $+ [[[(aab)aa][aaa]c]$ $+ [[[(aac)ba][aaa]a]$ $- [[[(aaa)ba]a[aac]]]$ $-3[[[aab]ca]a[aaa]]]$ $+ [[[(aaa)[aaa]abc]]]$	$-2[[[(aaa)ab]aa]ac]$ $- [[[(aaa)ba]ac]aa]$ $+ [[[(aaa)ca]ba]aa]$ $+ [[[(aaa)[aab]a]ac]$ $+ [[[(aaa)ab]c]aa]$ $+ [[[(aac)[aaa]b]aa]$ $+ [[[(aaa)ac][aaa]b]$ $- [[[(aab)ac][aaa]a]$ $- [[[(abc)aa][aaa]a]$ $-2[[[aaa]ba]c[aaa]]]$ $- [[[(aac)aa]b[aaa]]]$ $- [[[(aaa)[aab]aac]]]$

6. Degree 9: identities with two variables

The only partitions of 9 with two parts are 81, 72, 63, 54; all occur in Figure 4. We show that it suffices to consider identities with multidegree (5, 4). The  $p = 126$  permutations of  $a^5b^4$  in lexicographical order are the **associative monomials**. Applying the five association types we get 630 monomials, many of which are equal by the cyclic law; we obtain  $q = 56 + 26 + 26 + 26 + 4 = 138$  **nonassociative**

Table 6. Polynomial identity 7 with three variables in degree 9

$[[[[[aaa]aa]ab]ac]$	$+[[[[[aaa]aa]ac]ba]$	$-[[[[[aaa]aa]ba]ac]$	$+[[[[[aaa]aa]ba]ca]$	$+[[[[[aaa]aa]ca]ab]$
$-[[[[[aaa]aa]ca]ba]$	$-[[[[[aaa]ab]aa]ac]$	$-[[[[[aaa]ab]aa]ca]$	$-2[[[[[aaa]ab]ac]aa]$	$+3[[[[[aaa]ab]ca]aa]$
$+[[[[[aaa]ac]aa]ab]$	$-[[[[[aaa]ac]aa]ba]$	$+[[[[[aaa]ac]ab]aa]$	$-2[[[[[aaa]ac]ba]aa]$	$+[[[[[aaa]ba]aa]ac]$
$+[[[[[aaa]ba]aa]ca]$	$+4[[[[[aaa]ba]ac]aa]$	$-[[[[[aaa]ca]aa]ab]$	$+[[[[[aaa]ca]aa]ba]$	$-2[[[[[aaa]ca]ab]aa]$
$+[[[[[aaa]ca]ba]aa]$	$-[[[[[aaa]aaa]b]ca]$	$-[[[[[aaa]aaa]c]ab]$	$+[[[[[aaa]aac]a]ba]$	$+[[[[[aaa]aac]a]ba]$
$-[[[[[aab]aaa]a]ac]$	$-[[[[[aab]aaa]a]ca]$	$+[[[[[aab]aaa]c]aa]$	$+[[[[[aac]aaa]a]ab]$	$-2[[[[[aac]aaa]b]aa]$
$-[[[[[aaa]aa]aac]b]$	$-[[[[[aaa]aa]abc]a]$	$-[[[[[aaa]ab]aaa]c]$	$-2[[[[[aaa]ab]aac]a]$	$-[[[[[aaa]ac]aab]a]$
$+[[[[[aaa]ba]aac]a]$	$+[[[[[aaa]bc]aaa]a]$	$+[[[[[aaa]ca]aaa]b]$	$-[[[[[aaa]cb]aaa]a]$	$+[[[[[aab]aa]aaa]c]$
$+2[[[[[aab]ac]aaa]a]$	$-[[[[[aab]ca]aaa]a]$	$-[[[[[aac]aa]aaa]b]$	$+[[[[[aaa]aa]a]abc]$	$-[[[[[aaa]aa]a]acb]$
$+[[[[[aaa]aa]c]aab]$	$+[[[[[aaa]ab]c]aaa]$	$+2[[[[[aaa]ac]a]aab]$	$-2[[[[[aaa]ba]c]aaa]$	$+2[[[[[aaa]ca]b]aaa]$
$+[[[[[aab]aa]c]aaa]$	$-2[[[[[aab]ac]a]aaa]$	$+[[[[[aab]ca]a]aaa]$	$-[[[[[aac]aa]b]aaa]$	$-[[[[[aaa]aaa]abc]$
$+[[[[[aaa]aaa]acb]$	$-[[[[[aaa]aab]aac]$	$-2[[[[[aaa]aac]aab]$		

**monomials.** Let  $E$  be the  $p \times q$  **expansion matrix**: the rows are labeled by the associative monomials, the columns by the nonassociative monomials, and  $E_{ij}$  is the coefficient of the  $i$ -th associative monomial in the expansion of the  $j$ -th nonassociative monomial. The polynomial identities for the ternary cyclic sum are the (nonzero) vectors in the **nullspace lattice**  $N(E) = \{X \in \mathbb{Z}^q \mid EX = O\}$ . To find a basis of  $N(E)$  we use the Hermite Normal Form of  $E^t$ ; see Adkins and Weintraub [1], Cohen [5]. We compute matrices  $H$  ( $q \times p$ , unique) and  $U$  ( $q \times q$ , not unique) such that  $UE^t = H$  where  $H$  is the HNF of  $E^t$  and  $U$  is an integer matrix with  $\det(U) = \pm 1$ . If  $r = \text{rank}(E)$  then the last  $q-r$  rows of  $U$  form a lattice basis of  $N(E)$ . We apply the LLL algorithm for lattice basis reduction [8] to these rows; to achieve the best reduction we use the limiting value 1 of the parameter. For further details on this method, see Bremner and Peresi [4].

**Theorem 6.1:** *The four identities for the ternary cyclic sum in Table 4 are independent and do not follow from identities of lower degree. They are equivalent modulo identities of lower degree to the nine identities of Table IV in [2].*

**Proof:** Computations with Maple show that for polynomial identities of multidegree (5, 4) the nullspace lattice has rank 41. The LLL algorithm produces 41 polynomial identities with single-digit coefficients. We sort these identities by increasing Euclidean length of the coefficient vectors. The first has 8 terms, coefficients in  $\{\pm 1, \pm 3\}$  and length  $\sqrt{24}$ ; the last has 84 terms, coefficients in  $\{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9\}$  and length  $\sqrt{760}$ . For each partition  $\lambda$ , we apply the linearization operator to each identity and determine whether the identity increases the rank. The 41 identities produce the required increase from  $\text{rank}(K)$  to  $\text{rank}(Y)$  for all four partitions  $\lambda = 81, 72, 63, 54$ . Further computations show that identities 29, 32, 37, 38 in the sorted list are sufficient; these are the identities in Table 4.  $\square$

**Corollary 6.2:** *The second identity in Table 4 produces by itself the required increase in the ranks for partitions  $\lambda = 54, 63$ .*

*Remark:* The identities of [2, Table IV] have coefficients with up to 4 digits. The LLL algorithm has found the much simpler equivalent identities of Table 4.

## 7. Degree 9: identities with three variables

In Figure 4 the only partition with three parts is  $71^2$ .

**Theorem 7.1:** *The seven identities for the ternary cyclic sum in Tables 5 and 6 are independent and do not follow from the identities of lower degree or those of Table 4.*

**Proof:** There are 72 associative monomials: the permutations of  $a^7bc$ . The five association types produce  $44 + 24 + 24 + 24 + 4 = 120$  nonassociative monomials.

The  $72 \times 120$  expansion matrix has a nullspace lattice of rank 59. The LLL algorithm produces polynomial identities with coefficients  $\leq 4$  in absolute value; we sort them by increasing Euclidean length of their coefficient vectors. The first has 2 terms, coefficients in  $\{\pm 1\}$  and length  $\sqrt{2}$ ; the last has 58 terms, coefficients in  $\{\pm 1, \pm 2, 3, 4\}$  and length  $\sqrt{114}$ . For each partition  $\lambda$ , we apply the linearization operator to each identity and determine whether the identity increases the rank. The 59 identities produce the required increase for partitions  $\lambda = 81, 72, 71^2$ . Further computations show that identities 33, 45, 51, 54, 55, 56, 59 in the sorted list are sufficient; these are the identities of Tables 5 and 6.  $\square$

**Corollary 7.2:** *Every identity in degree 9 with two or three variables for the ternary cyclic sum follows from the second identity of Table 4 and the identities of Tables 5 and 6.*

*Remark:* One expects that there ought to be polynomial identities with very few terms (say  $\leq 10$ ) and very small coefficients (say  $\pm 1$ ) that imply all the new identities in degree 9 for the ternary cyclic sum. However, to use the methods of the present paper to determine whether such identities actually exist would require extremely large amounts of computer time and memory.

### Acknowledgements

Murray Bremner thanks NSERC for financial support, the University of Saskatchewan for a sabbatical leave from January to June 2008, and the Department of Mathematics at the University of São Paulo for its hospitality during his visit from January to April 2008.

### References

- [1] W. A. Adkins and S. H. Weintraub, *Algebra: An Approach via Module Theory*, Springer, Berlin, 1992.
- [2] M. R. Bremner and I. R. Hentzel, *Identities for generalized Lie and Jordan products on totally associative triple systems*, *J. Algebra* 231 (2000), pp. 387–405.
- [3] M. R. Bremner and L. A. Peresi, *Classification of trilinear operations*, *Comm. Algebra* 35 (2007), pp. 2932–2959.
- [4] M. R. Bremner and L. A. Peresi, *An application of lattice basis reduction to polynomial identities for algebraic structures*; preprint, March 2008.
- [5] H. Cohen, *A Course in Computational Algebraic Number Theory*, Springer, Berlin, 1993.
- [6] J. M. Clifton, *A simplification of the computation of the natural representation of the symmetric group  $S_n$* , *Proc. Amer. Math. Soc.* 83 (1981), pp. 248–250.
- [7] I. R. Hentzel, *Processing identities by group representation*, in *Computers in Nonassociative Rings and Algebras*, Special Session, 82nd Annual Meeting of the American Mathematical Society, San Antonio, Texas, 1976, Academic Press, New York, 1977, pp. 13–40.
- [8] A. K. Lenstra, H. W. Lenstra Jr. and L. Lovász, *Factoring polynomials with rational coefficients*, *Math. Ann.* 261 (1982), pp. 515–534.
- [9] K. A. Zhevlakov, A. M. Slinko, I. P. Shestakov and A. I. Shirshov, *Rings that are Nearly Associative*, Academic Press, New York, 1982.