SPECIAL IDENTITIES FOR QUASI-JORDAN ALGEBRAS

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Abstract. Semispecial quasi-Jordan algebras (also called Jordan dialgebras) are defined by the polynomial identities

$$a(bc) = a(cb), \qquad (ba)a^2 = (ba^2)a, \qquad (b, a^2, c) = 2(b, a, c)a.$$

These identities are satisfied by the product $ab = a \dashv b + b \vdash a$ in an associative dialgebra. We use computer algebra to show that every identity for this product in degree ≤ 7 is a consequence of the three identities in degree ≤ 4, but that six new identities exist in degree 8. Some but not all of these new identities are noncommutative preimages of the Glennie identity.

1. Introduction

Loday [20, 21] introduced a new variety of algebras with two binary operations.

Definition 1. An **associative dialgebra** is a vector space with bilinear operations $a \dashv b$ and $a \vdash b$, the **left** and **right** products, satisfying these polynomial identities:

$$(a \vdash b) \vdash c = (a \dashv b) \vdash c, \quad a \dashv (b \dashv c) = a \dashv (b \vdash c),$$
$$(a \dashv b) \dashv c = a \dashv (b \dashv c), \quad (a \vdash b) \vdash c = a \vdash (b \vdash c), \quad (a \vdash b) \dashv c = a \vdash (b \dashv c).$$

Since $(a \dashv b) \vdash c = a \dashv (b \vdash c)$ does not hold in general, this gives a class of algebras that are "nearly associative"; see Shirshov [28], Zhevlakov et al. [32].

Definition 2. A **dialgebra monomial** on the set X of generators is a product $w = \overline{a_1 \cdots a_n}$ where $a_1, \dots, a_n \in X$ and the bar indicates some placement of parentheses and some choice of operations. We define $c(w)$, the **center** of w, inductively: If $w \in X$ then $c(w) = w$; otherwise $c(w_1 \dashv w_2) = c(w_1)$ and $c(w_1 \vdash w_2) = c(w_2)$.

Lemma 3. (Loday [21], 1.7 Theorem) *If* $w = \overline{a_1 \cdots a_n}$ *and* $c(w) = a_k$ *then*

$$w = (a_1 \vdash \cdots \vdash a_{k-1}) \vdash a_k \dashv (a_{k+1} \dashv \cdots \dashv a_n).$$

Definition 4. The **normal form** of w in Lemma 3 will be abbreviated as

$$w = a_1 \cdots a_{k-1} \widehat{a}_k a_{k+1} \cdots a_n.$$

Lemma 5. (Loday [21], 2.5 Theorem) *The monomials* $a_1 \cdots a_{k-1} \widehat{a}_k a_{k+1} \cdots a_n$ *with* $k = 1, \dots, n$ *and* $a_1, \dots, a_n \in X$ *form a basis of the free associative dialgebra on* X.

Notation 6. We write FD_n for the multilinear subspace of degree n in the free associative dialgebra on n generators. Lemma 5 implies that $\dim FD_n = n(n!)$.

Definition 7. (Velásquez and Felipe [31]) The **quasi-Jordan product** in a dialgebra over a field of characteristic $\neq 2$ is defined by

$$a \triangleleft b = \tfrac{1}{2}(a \dashv b + b \vdash a).$$

If D is a dialgebra, then its **plus algebra** D^+ has the same underlying vector space but the operation $a \lhd b$. We omit the symbol $\lhd$ and the coefficient $\frac{1}{2}$ and write

$$ab = a \dashv b + b \vdash a.$$

Definition 8. We consider **right commutativity**, the **quasi-Jordan identity**, and the **associator-derivation identity**:

$$a(bc) = a(cb), \qquad (ba)a^2 = (ba^2)a, \qquad (b, a^2, c) = 2(b, a, c)a,$$

where $(a, b, c) = (ab)c - a(bc)$. The multilinear forms of the last two identities are

$$J = (a(bc))d + (a(bd))c + (a(cd))b - (ab)(cd) - (ac)(bd) - (ad)(bc),$$
$$K = ((ab)d)c + ((ac)d)b - (a(bc))d - (a(bd))c - (a(cd))b + a((bc)d).$$

Remark 9. Equivalent identities for the opposite product appear in the work of Kolesnikov. If we replace xy by yx in equations (26) and (27) of [17], replace x_1, x_2, x_3, x_4 by a, b, c, d and apply right commutativity, then we obtain

$$L = ((ac)b)d + ((ad)b)c - (ab)(cd) - (ac)(bd) - (ad)(bc) + a((cd)b),$$
$$M = (b(cd))a + (b(ac))d + (b(ad))c - (ba)(cd) - (bd)(ac) - (bc)(ad).$$

One easily verifies that

$$J(a, b, c, d) = M(b, a, d, c), \qquad K(a, b, c, d) = L(a, d, b, c) - M(b, a, c, d),$$
$$L(a, b, c, d) = J(a, b, c, d) + K(a, c, d, b), \qquad M(a, b, c, d) = J(b, a, c, d).$$

See also the remarks on Jordan dialgebras in Pozhidaev [25], Section 3.

Lemma 10. (Velásquez and Felipe [31]) *The quasi-Jordan product in an associative dialgebra satisfies right commutativity and the quasi-Jordan identity.*

Lemma 11. (Bremner [3]) *The quasi-Jordan product in an associative dialgebra satisfies the associator-derivation identity. The identities of Definition 8 imply every identity of degree* ≤ 4 *for the quasi-Jordan product in an associative dialgebra.*

Definition 12. A **quasi-Jordan algebra** is a nonassociative algebra over a field of characteristic $\ne 2, 3$ satisfying right commutativity and the quasi-Jordan identity. A **semispecial** quasi-Jordan algebra (also called a **Jordan dialgebra**) is a quasi-Jordan algebra satisfying the associator-derivation identity.

Remark 13. Strictly speaking, a Jordan dialgebra as defined by Kolesnikov [17] and Pozhidaev [25] has two binary operations $\dashv$ and $\vdash$ which are in fact opposite as a result of the dialgebra version of commutativity, $x \dashv y = y \vdash x$. We simplify the notation by using only one operation and writing this operation as juxtaposition.

Definition 14. A quasi-Jordan algebra is **special** if it is isomorphic to a subalgebra of D^+ for some associative dialgebra D. Every special algebra is semispecial.

Glennie [7, 8, 9] (see also Hentzel [12]) discovered an identity satisfied by special Jordan algebras that is not satisfied by all Jordan algebras. In this paper we consider the analogous question for quasi-Jordan algebras. We use computer algebra to show that the identities in Definition 8 imply every identity of degree ≤ 7 for the quasi-Jordan product in an associative dialgebra. We demonstrate the existence of identities in degree 8 which do not follow from the identities of Definition 8. Some but not all of these new identities are noncommutative preimages of the Glennie identity. These new identities are special identities in the following sense.

Definition 15. A **special identity** is a polynomial identity satisfied by all special quasi-Jordan algebras but not satisfied by all semispecial quasi-Jordan algebras.

We present an explicit special identity which has three variables and is linear in one variable; this cannot happen for special Jordan algebras by a theorem of Macdonald [22]. Our methods depend on computational linear algebra with large matrices over a finite field, together with the representation theory of the symmetric group. Our computations were done with C [16], Maple [24] and Albert [13].

2. Preliminaries on free nonassociative algebras

2.1. **Free right-commutative algebras.** The simplest identity satisfied by the quasi-Jordan product is right commutativity. Our computations depend on basic facts about free right-commutative algebras. Kurosh [18] proved that every subalgebra of an (absolutely) free nonassociative algebra is also free; Shirshov [26] proved the same result for free commutative and free anticommutative algebras. The referee pointed out a simple proof that this does not hold for a free right commutative algebra R: the commutator $[R, R]$ is a subalgebra with trivial multiplication (all products are zero) and hence is certainly not free.

Lemma 16. *Let* $w = \overline{a_1 \cdots a_n}$ *be a nonassociative monomial where* $a_1, \dots, a_n \in X$ *and the bar denotes some placement of parentheses. Right-commutativity implies that in any submonomial* $x = yz$ *we may assume commutativity for* z.

Proof. By induction on n; for $n \leq 3$ the claim is immediate. The monomial w has the unique factorization $w = uv$; the inductive hypothesis implies the claim for u and v. Any right factor of a submonomial of w is either v, a right factor of a submonomial of u, or a right factor of a submonomial of v. It therefore suffices to show that we may assume commutativity for v. We have the unique factorization $v = xy$, and we may assume commutativity for y. Right-commutativity implies $uv = u(xy) = u(yx)$, and by induction we may assume commutativity for x. □

Lemma 16 implies the following algorithm for generating a complete minimal set of right-commutative association types up to a given degree n.

Algorithm 17. Assume that the right-commutative association types have been generated for degrees $\leq n-1$. Any right-commutative type in degree n has the form $w = uv$ where u is a right-commutative type in degree $n-i$ and v is a commutative type in degree i. This induces a total order on the association types.

Algorithm 17 implies the following recursive formula for the number R_n of right-commutative types in degree n which requires the number C_n of commutative types.

Lemma 18. *We have* $C_1 = R_1 = 1$*, and for* $n \geq 2$ *we have*

$$C_n = \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} C_{n-i} C_i + \binom{C_{n/2}+1}{2}, \qquad R_n = \sum_{i=1}^{n-1} R_{n-i} C_i.$$

(The binomial coefficient only appears for n *even.)*

Example 19. The following table gives the numbers C_n and R_n for $1 \leq n \leq 12$, together with the Catalan number K_n of all nonassociative types in degree n:

n	1	2	3	4	5	6	7	8	9	10	11	12
C_n	1	1	1	2	3	6	11	23	46	98	207	451
R_n	1	1	2	4	9	20	46	106	248	582	1376	3264
K_n	1	1	2	5	14	42	132	429	1430	4862	16796	58786

Lemma 20. *The generating functions of C_n and R_n are related by the equations*

$$C(x) = \sum_{n=1}^{\infty} C_n x^n, \qquad \sum_{n=1}^{\infty} R_n x^n = \frac{x}{1 - C(x)}.$$

Proof. Sloane [29] (sequences A001190 and A085748). □

Definition 21. In the **basic monomial** for an association type in degree n the variables are the first n letters of the alphabet in lexicographical order.

Example 22. For $n = 1$ (resp. $n = 2$) we have the single type a (resp. ab). For $3 \le n \le 5$ we present the basic commutative and right-commutative monomials:

n	commutative	right-commutative
3	$(ab)c$	$(ab)c, a(bc)$
4	$((ab)c)d, (ab)(cd)$	$((ab)c)d, (a(bc))d, (ab)(cd), a((bc)d)$
5	$(((ab)c)d)e, ((ab)(cd))e, ((ab)c)(de)$	$(((ab)c)d)e, ((a(bc))d)e, ((ab)(cd))e,$ $(a((bc)d))e, ((ab)c)(de), (a(bc))(de),$ $(ab)((cd)e), a(((bc)d)e), a((bc)(de))$

2.2. **Multilinear right-commutative monomials.** Throughout most of this paper we consider only multilinear identities: in degree n, the variables in each monomial are a permutation of the first n letters of the alphabet. To obtain a basis for the space of multilinear right-commutative polynomials in degree n, we need a straightening algorithm which replaces each monomial w by the first monomial (in lex order) in its equivalence class $[w]$: the set of all monomials which are equal to w as a consequence of right-commutativity. To straighten a right-commutative monomial, it suffices to determine the symmetries of its association type.

Definition 23. Let v be the basic monomial for a right-commutative association type. Suppose that $v = \cdots (xy) \cdots$ contains the submonomial xy where x and y are submonomials with the same degree and association type. Let $w = \cdots (yx) \cdots$ be the monomial obtained from v by transposing x and y. If right-commutativity implies $v = w$ then this identity will be called a **symmetry** of the association type.

Lemma 24. *If a right-commutative association type in degree n has s symmetries, then the number of multilinear monomials with this association type is $n!/2^s$.*

Proof. Each symmetry reduces the number of monomials by a factor of 2. □

Example 25. The symmetries of the right-commutative types in degree 5:

type 1: $(((ab)c)d)e$ has no symmetries
type 2: $((a(bc))d)e = ((a(cb))d)e$ type 3: $((ab)(cd))e = ((ab)(dc))e$
type 4: $(a((bc)d))e = (a((cb)d))e$ type 5: $((ab)c)(de) = ((ab)c)(ed)$
type 6: $(a(bc))(de) = (a(cb))(de) = (a(bc))(ed)$
type 7: $(ab)((cd)e) = (ab)((dc)e)$ type 8: $a(((bc)d)e) = a(((cb)d)e)$
type 9: $a((bc)(de)) = a((cb)(de)) = a((bc)(ed)) = a((de)(bc))$

These types have (respectively) 0, 1, 1, 1, 1, 2, 1, 1, 3 symmetries, and contain 120, 60, 60, 60, 60, 30, 60, 60, 15 distinct multilinear monomials, for a total of 525.

Notation 26. We write FRC_n for the multilinear subspace of degree n in the free right-commutative algebra on n generators. An ordered basis of FRC_n consists of the distinct right-commutative monomials in degree n, ordered first by association type and then by lex order of the underlying permutation.

Lemma 27. *If $s(i)$ is the number of symmetries in association type i then*

$$\dim FRC_n = \sum_{i=1}^{R_n} \frac{n!}{2^{s(i)}}.$$

Proof. This follows directly from Lemma 24. □

Algorithm 28. This algorithm to find the symmetries of a right-commutative association type (represented by a basic monomial) uses a global variable `symmetrylist`, initially empty. On input $w = uv$, the primary procedure `findsymmetry` calls itself on input u and then calls the secondary procedure `findcommutativesymmetry` on input v. Writing $v = xy$, the secondary procedure calls itself on input x and then on input y; it then checks to see if x and y have the same association type, and if so it appends the symmetry $u(xy) = u(yx)$ to `symmetrylist`. Both procedures do nothing if the input has degree 1; this is the basis of the recursion.

Example 29. The dimension $\dim FRC_n$ (the total number of multilinear right-commutative monomials) for $1 \le n \le 9$:

n	1	2	3	4	5	6	7	8	9
$\dim FRC_n$	1	2	9	60	525	5670	72765	1081080	18243225

These values satisfy the following formula from Sloane [29] (sequence A001193).

Conjecture 30. *For all $n \ge 1$ we have*

$$\dim FRC_n \overset{?}{=} \frac{n(2n-2)!}{2^{n-1}(n-1)!}.$$

2.3. **The expansion map and the expansion matrix.**

Recall that FRC_n and FD_n are the spaces of multilinear right-commutative and dialgebra polynomials.

Definition 31. We define the **expansion map** $E_n \colon FRC_n \to FD_n$ on basis monomials and extend linearly: if $\deg w = 1$ then $E_1(w) = w$; if $w = uv$ where $\deg u = n-i$ and $\deg v = i$ then

$$E_n(w) = E_{n-i}(u) \dashv E_i(v) + E_i(v) \vdash E_{n-i}(u).$$

This map is well-defined since the quasi-Jordan product is right-commutative.

Lemma 32. *The multilinear polynomial identities in degree n satisfied by the quasi-Jordan product are precisely the (nonzero) elements of the kernel of E_n.*

Some of these kernel identities may be consequences of identities of lower degree. We need to distinguish the "old" from the "new" identities.

Definition 33. With respect to the ordered bases of FRC_n and FD_n, we represent E_n by the **expansion matrix** $[E_n]$: we have $[E_n]_{ij} = 1$ if dialgebra monomial i occurs in the expansion of right-commutative monomial j, and $[E_n]_{ij} = 0$ otherwise.

The sizes of the matrices $[E_n]$ grow very rapidly. We can use `LinearAlgebra` in Maple to compute a basis for the nullspace of $[E_n]$ over $\mathbb{Q}$ for $n \le 5$; we can use `LinearAlgebra[Modular]` to compute a basis over $\mathbb{F}_p$ for $n \le 6$. For $n \ge 7$ we must make the matrices smaller; for this we use the representation theory of the symmetric group as described in Section 5.

Definition 31 gives a recursive algorithm for computing the expansion of a right-commutative monomial. To initialize $[E_n]$, we let the column index j go from left to right, compute the expansion of the corresponding right-commutative monomial, obtain 2^{n-1} dialgebra monomials, convert each dialgebra monomial to normal form and determine its row index i, and set the (i, j) entry of the matrix to 1.

2.4. **Lifting multilinear identities.** Let $I(x_1, \dots, x_n)$ be a multilinear polynomial identity in degree n; we want to find all its consequences in degree $n{+}1$.

Definition 34. In the free algebra the **T-ideal** generated by a polynomial identity I is the smallest ideal which contains I and is sent to itself by all endomorphisms. For the endomorphism condition we introduce a new variable x_{n+1} and substitute $x_i x_{n+1}$ for x_i. For the ideal condition we multiply on the left or right by x_{n+1}.

Lemma 35. *If $I(x_1, \dots, x_n)$ is a multilinear polynomial identity in degree n, then every consequence of I in degree $n{+}1$ is a linear combination of permutations of the following $n{+}2$ multilinear identities in degree $n{+}1$:*

$$\begin{gathered} I(x_1 x_{n+1}, x_2, \dots, x_n), \quad \dots, \quad I(x_1, \dots, x_{n-1}, x_n x_{n+1}). \\ I(x_1, \dots, x_n) x_{n+1}, \quad x_{n+1} I(x_1, \dots, x_n). \end{gathered}$$

Definition 36. The identities of Lemma 35 are the **liftings** of I to degree $n{+}1$.

An identity I in degree n produces $(n{+}2) \cdots (n{+}k{+}1)$ liftings in degree $n{+}k$. In general, a subset of these liftings generates all the consequences of I in degree $n{+}k$. For example, the symmetries of the right-commutative association types in degree n are the liftings of right-commutativity from degree 3. Our choice of association types eliminates most of the consequences; only the symmetries remain. In this paper, the most important examples are the liftings of the multilinear identities J and K of Definition 8 from degree 4 to degree n.

3. Nonexistence of new identities in degree 5

In this section we provide detailed examples of our methods; for higher degrees the objects we work with — polynomial identities and expansion matrices — become so large that it is impossible to include all the details of the computations.

3.1. **Old identities.** Identities J and K each have six liftings to degree 5. The terms of each lifting must be straightened to lie in the standard basis of FRC_5; see Table 1. We allocate memory for a matrix M of size 645×525 with a 525×525 upper block and a 120×525 lower block; 525 is the number of multilinear right-commutative monomials and 120 is the number of permutations of 5 variables. For each of the 12 lifted and straightened identities L in Table 1, we do the following: for permutations $\pi_1, \dots, \pi_{120}$ of a, b, c, d, e in lex order we apply π_j to L, straighten the terms, and store the resulting coefficient vector in row $525{+}j$ of M; we then compute the row canonical form of M and record the rank (the lower block of M is now zero). Using rational arithmetic we obtain the following ranks: 20, 50, 50, 50, 70, 90, 150, 210, 210, 220, 250, 250. The lifted identities which do not increase the rank are redundant, so we consider only numbers 1, 2, 5, 6, 7, 8, 10, 11. Using modular arithmetic we obtain the same ranks approximately 1000 times faster.

$$\begin{aligned}
J(ae,b,c,d) &= ((ae)(bc))d+((ae)(bd))c+((ae)(cd))b-((ae)b)(cd)-((ae)c)(bd)-((ae)d)(bc) \\
J(a,be,c,d) &= (a((be)c))d+(a((be)d))c+(a(cd))(be)-(a(be))(cd)-(ac)((be)d)-(ad)((be)c) \\
J(a,b,ce,d) &= (a(b(ce)))d+(a(bd))(ce)+(a((ce)d))b-(ab)((ce)d)-(a(ce))(bd)-(ad)(b(ce)) \\
&= (a((ce)b))d+(a(bd))(ce)+(a((ce)d))b-(ab)((ce)d)-(a(ce))(bd)-(ad)((ce)b) \\
J(a,b,c,de) &= (a(bc))(de)+(a(b(de)))c+(a(c(de)))b-(ab)(c(de))-(ac)(b(de))-(a(de))(bc) \\
&= (a(bc))(de)+(a((de)b))c+(a((de)c))b-(ab)((de)c)-(ac)((de)b)-(a(de))(bc) \\
J(a,b,c,d)e &= ((a(bc))d)e+((a(bd))c)e+((a(cd))b)e-((ab)(cd))e-((ac)(bd))e-((ad)(bc))e \\
eJ(a,b,c,d) &= e((a(bc))d)+e((a(bd))c)+e((a(cd))b)-e((ab)(cd))-e((ac)(bd))-e((ad)(bc)) \\
&= e(((bc)a)d)+e(((bd)a)c)+e(((cd)a)b)-e((ab)(cd))-e((ac)(bd))-e((ad)(bc)) \\
K(ae,b,c,d) &= (((ae)b)d)c+(((ae)c)d)b-((ae)(bc))d-((ae)(bd))c-((ae)(cd))b+(ae)((bc)d) \\
K(a,be,c,d) &= ((a(be))d)c+((ac)d)(be)-(a((be)c))d-(a((be)d))c-(a(cd))(be)+a(((be)c)d) \\
K(a,b,ce,d) &= ((ab)d)(ce)+((a(ce))d)b-(a(b(ce)))d-(a(bd))(ce)-(a((ce)d))b+a((b(ce))d) \\
&= ((ab)d)(ce)+((a(ce))d)b-(a((ce)b))d-(a(bd))(ce)-(a((ce)d))b+a(((ce)b)d) \\
K(a,b,c,de) &= ((ab)(de))c+((ac)(de))b-(a(bc))(de)-(a(b(de)))c-(a(c(de)))b+a((bc)(de)) \\
&= ((ab)(de))c+((ac)(de))b-(a(bc))(de)-(a((de)b))c-(a((de)c))b+a((bc)(de)) \\
K(a,b,c,d)e &= (((ab)d)c)e+(((ac)d)b)e-((a(bc))d)e-((a(bd))c)e-((a(cd))b)e+(a((bc)d))e \\
eK(a,b,c,d) &= e(((ab)d)c)+e(((ac)d)b)-e((a(bc))d)-e((a(bd))c)-e((a(cd))b)+e(a((bc)d)) \\
&= e(((ab)d)c)+e(((ac)d)b)-e(((bc)a)d)-e(((bd)a)c)-e(((cd)a)b)+e(((bc)d)a)
\end{aligned}$$

TABLE 1. Liftings of J and K to degree 5

Lemma 37. *The polynomial identities in degree 5 which are consequences of identities in degree ≤ 4 span a 250-dimensional subspace of FRC_5.*

3.2. **All identities.** We allocate memory for the expansion matrix $E = [E_5]$ of size 600×525. We compute the expansions of the basic monomials for the right-commutative association types; see Table 2. From these we obtain the expansions of all 525 monomials corresponding to the columns of E; we set to 1 the appropriate entries of E. We obtain a sparse 0-1 matrix in which each column has 16 nonzero entries. We find that the rank of this matrix is 275, and so the nullity is 250.

Lemma 38. *The subspace of FRC_5 consisting of polynomial identities satisfied by the quasi-Jordan product has dimension 250.*

Proposition 39. *Every polynomial identity in degree 5 for the quasi-Jordan product is a consequence of identities in degree ≤ 4.*

Proof. The subspace generated by the lifted identities is contained in the subspace of all identities; since the dimensions are equal, the subspaces are equal. $\square$

4. NONEXISTENCE OF NEW IDENTITIES IN DEGREE 6: FIRST COMPUTATION

Lemma 40. *In degree 6 there are 20 right-commutative association types:*

$$\begin{array}{lllll}
((((ab)c)d)e)f & (((a(bc))d)e)f & (((ab)(cd))e)f & ((a((bc)d))e)f & (((ab)c)(de))f \\
((a(bc))(de))f & ((ab)((cd)e))f & (a(((bc)d)e))f & (a((bc)(de)))f & (((ab)c)d)(ef) \\
((a(bc))d)(ef) & ((ab)(cd))(ef) & (a((bc)d))(ef) & ((ab)c)((de)f) & (a(bc))((de)f) \\
(ab)(((cd)e)f) & (ab)((cd)(ef)) & a((((bc)d)e)f) & a(((bc)(de))f) & a(((bc)d)(ef))
\end{array}$$

$$\begin{aligned}
(((ab)c)d)e &\longmapsto \widehat{ab}cde + e\widehat{ab}cd + d\widehat{ab}ce + ed\widehat{ab}c + c\widehat{ab}de + ec\widehat{ab}d + dc\widehat{ab}e + edc\widehat{ab} \\
&\quad + \widehat{ba}cde + e\widehat{ba}cd + d\widehat{ba}ce + ed\widehat{ba}c + c\widehat{ba}de + ec\widehat{ba}d + dc\widehat{ba}e + edc\widehat{ba} \\
((a(bc))d)e &\longmapsto \widehat{ab}cde + e\widehat{ab}cd + d\widehat{ab}ce + ed\widehat{ab}c + b\widehat{ca}de + eb\widehat{ca}d + db\widehat{ca}e + edb\widehat{ca} \\
&\quad + \widehat{ac}bde + e\widehat{ac}bd + d\widehat{ac}be + ed\widehat{ac}b + c\widehat{ba}de + ec\widehat{ba}d + dc\widehat{ba}e + edc\widehat{ba} \\
((ab)(cd))e &\longmapsto \widehat{ab}cde + e\widehat{ab}cd + cd\widehat{ab}e + ecd\widehat{ab} + \widehat{ab}dce + e\widehat{ab}dc + dc\widehat{ab}e + edc\widehat{ab} \\
&\quad + \widehat{ba}cde + e\widehat{ba}cd + cd\widehat{ba}e + ecd\widehat{ba} + \widehat{ba}dce + e\widehat{ba}dc + dc\widehat{ba}e + edc\widehat{ba} \\
(a((bc)d))e &\longmapsto \widehat{ab}cde + e\widehat{ab}cd + bc\widehat{da}e + ebc\widehat{da} + \widehat{ad}bce + e\widehat{ad}bc + db\widehat{ca}e + edb\widehat{ca} \\
&\quad + \widehat{ac}bde + e\widehat{ac}bd + cb\widehat{da}e + ecb\widehat{da} + \widehat{ad}cbe + e\widehat{ad}cb + dc\widehat{ba}e + edc\widehat{ba} \\
((ab)c)(de) &\longmapsto \widehat{ab}cde + de\widehat{ab}c + \widehat{ab}ced + ed\widehat{ab}c + c\widehat{ab}de + dec\widehat{ab} + c\widehat{ab}ed + edc\widehat{ab} \\
&\quad + \widehat{ba}cde + de\widehat{ba}c + \widehat{ba}ced + ed\widehat{ba}c + c\widehat{ba}de + dec\widehat{ba} + c\widehat{ba}ed + edc\widehat{ba} \\
(a(bc))(de) &\longmapsto \widehat{ab}cde + de\widehat{ab}c + \widehat{ab}ced + ed\widehat{ab}c + b\widehat{ca}de + deb\widehat{ca} + b\widehat{ca}ed + edb\widehat{ca} \\
&\quad + \widehat{ac}bde + de\widehat{ac}b + \widehat{ac}bed + ed\widehat{ac}b + c\widehat{ba}de + dec\widehat{ba} + c\widehat{ba}ed + edc\widehat{ba} \\
(ab)((cd)e) &\longmapsto \widehat{ab}cde + cde\widehat{ab} + \widehat{ab}ecd + ecd\widehat{ab} + \widehat{ab}dce + dce\widehat{ab} + \widehat{ab}edc + edc\widehat{ab} \\
&\quad + \widehat{ba}cde + cde\widehat{ba} + \widehat{ba}ecd + ecd\widehat{ba} + \widehat{ba}dce + dce\widehat{ba} + \widehat{ba}edc + edc\widehat{ba} \\
a(((bc)d)e) &\longmapsto \widehat{ab}cde + bcd\widehat{ea} + \widehat{ae}bcd + ebc\widehat{da} + \widehat{ad}bce + dbc\widehat{ea} + \widehat{ae}dbc + edb\widehat{ca} \\
&\quad + \widehat{ac}bde + cbd\widehat{ea} + \widehat{ae}cbd + ecb\widehat{da} + \widehat{ad}cbe + dcb\widehat{ea} + \widehat{ae}dcb + edc\widehat{ba} \\
a((bc)(de)) &\longmapsto \widehat{ab}cde + bcd\widehat{ea} + \widehat{ad}ebc + deb\widehat{ca} + \widehat{ab}ced + bce\widehat{da} + \widehat{ae}dbc + edb\widehat{ca} \\
&\quad + \widehat{ac}bde + cbd\widehat{ea} + \widehat{ad}ecb + dec\widehat{ba} + \widehat{ac}bed + cbe\widehat{da} + \widehat{ae}dcb + edc\widehat{ba}
\end{aligned}$$

TABLE 2. Expansions of the basic monomials in degree 5

Each type has (respectively) 720, 360, 360, 360, 360, 180, 360, 360, 90, 360, 180, 180, 180, 360, 180, 360, 90, 360, 90, 180 monomials, for a total of 5670.

Proof. This follows directly from Lemmas 18 and 24. □

For a matrix with 5670 columns, it is not practical to use rational arithmetic to compute the row canonical form. Instead we use modular arithmetic (with $p = 101$) to compute the dimensions of the subspaces of lifted identities and all identities.

4.1. **Old identities.** Our computations in degree 5 showed that we needed only 8 liftings to generate all consequences of J and K in degree 5:

$$\begin{array}{llll}
J(ae,b,c,d), & J(a,be,c,d), & J(a,b,c,d)e, & eJ(a,b,c,d), \\
K(ae,b,c,d), & K(a,be,c,d), & K(a,b,c,de), & K(a,b,c,d)e.
\end{array}$$

Each of these identities produces 7 liftings in degree 6, and so we obtain an ordered list of 56 liftings in degree 6. We follow the same algorithm as for degree 5, except that now the matrix M has size 6390×5670 with a 5670×5670 upper block and a 720×5670 lower block. To each of the 56 liftings, we apply all 720 permutations of the 6 variables and straighten the terms to obtain monomials in the standard basis of FRC_6; we store the coefficient vectors of the permuted liftings in the lower block, and compute the row canonical form. We obtain the following ranks: 120, 300, 300, 300, 360, 480, 540, 540, 720, 810, 810, 810, 990, 1170, 1170, 1170, 1170, 1170, 1230, 1350, 1410, 1410, 1410, 1410, 1410, 1530, 1626, 1626, 1986, 2346, 2346, 2406, 2586, 2766, 2766, 2766, 3126, 3210, 3330, 3330, 3510, 3510, 3510, 3510, 3510, 3510, 3510, 3570, 3570, 3570, 3570, 3570, 3570, 3570, 3690, 3690. Only 25 liftings in the ordered list produce an increase in the rank: numbers 1, 2, 5, 6, 7, 9, 10, 13, 14, 19, 20, 21, 26, 27, 29, 30, 32, 33, 34, 37, 38, 39, 41, 48, 55.

Lemma 41. *The polynomial identities in degree 6 which are consequences of identities in degree* ≤ 5 *span a 3690-dimensional subspace of* FRC_6.

4.2. **All identities.** The expansion matrix $E = [E_6]$ has size 4320 × 5670. As for degree 5, we compute the expansions of the basic monomials, determine the normal forms of the dialgebra monomials, obtain the expansions of all the multilinear right-commutative monomials, and store the results in the columns of E. We obtain a very sparse 0-1 matrix in which each column has 32 nonzero entries. We find that the rank of this matrix is 1980, and so the nullity is 3690.

Lemma 42. *The subspace of* FRC_6 *consisting of polynomial identities satisfied by the quasi-Jordan product has dimension 3690.*

Proposition 43. *Every polynomial identity in degree 6 for the quasi-Jordan product (over the field* $\mathbb{F}_{101}$*) is a consequence of identities in degree* ≤ 5.

We next show how to obtain the same results using much smaller matrices.

5. Preliminaries on representation theory

5.1. **Representations of semisimple algebras.** Let A be a finite-dimensional semisimple associative algebra over a field F; then A is the direct sum of simple two-sided ideals which are orthogonal as subalgebras:

$$A = A_1 \oplus \cdots \oplus A_r, \qquad A_i A_j = \{0\} \; (1 \le i \ne j \le r). \tag{1}$$

Each A_i is isomorphic to the algebra of $d_i \times d_i$ matrices with entries in a division algebra D_i over F. The action of A on the left regular representation ${}^*A$ is $a \cdot b = ab$ for $a \in A$, $b \in {}^*A$, and (1) gives the decomposition of ${}^*A$ into isotypic components:

$${}^*A = {}^*A_1 \oplus \cdots \oplus {}^*A_r. \tag{2}$$

Each ${}^*A_i$ is the direct sum of d_i isomorphic simple submodules, and each of these is a d_i-dimensional minimal left ideal in A (a column in the matrix algebra). We consider the direct sum of t copies of ${}^*A$ with the diagonal action:

$$({}^*A)^t = ({}^*A)^{[1]} \oplus \cdots \oplus ({}^*A)^{[t]}, \qquad a \cdot (b_1, \dots, b_t) = (ab_1, \dots, ab_t). \tag{3}$$

If $U \subseteq ({}^*A)^t$ is a submodule then usually U is not homogeneous with respect to (3):

$$U \ne \sum_{k=1}^{t} \oplus \Big(U \cap ({}^*A)^{[k]} \Big). \tag{4}$$

We combine (2) and (3) to obtain a finer decomposition of $({}^*A)^t$:

$$({}^*A)^t = \sum_{k=1}^{t} \oplus \big({}^*A\big)^{[k]} = \sum_{k=1}^{t} \oplus \sum_{i=1}^{r} \oplus \big({}^*A\big)_i^{[k]} = \sum_{i=1}^{r} \oplus \sum_{k=1}^{t} \oplus \big({}^*A\big)_i^{[k]}. \tag{5}$$

This gives a direct sum decomposition of $({}^*A)^t$ into components R_i:

$$({}^*A)^t = \sum_{i=1}^{r} \oplus R_i, \qquad R_i = \sum_{k=1}^{t} \oplus \big({}^*A\big)_i^{[k]}. \tag{6}$$

Every submodule $U \subseteq ({}^*A)^t$ is homogeneous with respect to (6).

Lemma 44. *If* $U \subseteq ({}^*A)^t$ *is a submodule then*

$$U = \sum_{i=1}^{r} \oplus \Big(U \cap R_i \Big).$$

Proof. For any $u \in U$ we have $u = u_1 + \cdots + u_r$ where $u_i \in R_i$. If $I_i \in A_i$ corresponds to the identity matrix then (1), (3), (6) imply $I_i \cdot u = u_i$; hence $u_i \in U$. □

5.2. **Irreducible representations of the symmetric group.** We apply the general construction to the group algebra FS_n over F of the symmetric group S_n. We assume that $F = \mathbb{Q}$ or $F = \mathbb{F}_p$ for $p > n$; then FS_n is semisimple by Maschke's theorem. We recall the structure theory from James and Kerber [15]. The irreducible representations of S_n are in bijection with the partitions of n. Let $\lambda = (n_1, \dots, n_\ell)$ be a partition: $n = n_1 + \cdots + n_\ell$ with $n_1 \geq \cdots \geq n_\ell \geq 1$. The frame $[\lambda]$ consists of n empty boxes in ℓ left-justified rows with n_i boxes in row i. A tableau for λ is a bijection from $1, \dots, n$ to the boxes of $[\lambda]$. In a standard tableau the numbers increase in each row from left to right and in each column from top to bottom. The number d_λ of standard tableaux with frame $[\lambda]$ is the dimension of the corresponding irreducible representation. We have the following direct sum decomposition of FS_n into orthogonal two-sided ideals isomorphic to simple matrix algebras over F:

$$FS_n \approx \sum_{\lambda} \oplus A_\lambda, \qquad A_\lambda = M_{d_\lambda}(F). \tag{7}$$

For us the most important problem is: Given a permutation π and a partition λ, compute the $d_\lambda \times d_\lambda$ matrix representing π; that is, compute the projection of π onto the summand A_λ in (7). A simple algorithm was found by Clifton [5] (see also Bergdolt [2]). Let $T_1, \dots, T_d$ ($d = d_\lambda$) be the standard tableaux for λ ordered in some fixed way. Let R^λ_π be the matrix defined as follows: *Apply* π *to the tableau* T_j*. If there exist two numbers that appear together in a column of* T_i *and a row of* πT_j*, then* $(R^\lambda_\pi)_{ij} = 0$*. If not, then* $(R^\lambda_\pi)_{ij}$ *equals the sign of the vertical permutation for* T_i *which leaves the columns of* T_i *fixed as sets and takes the numbers of* T_i *into the correct rows they occupy in* πT_j. The matrix R^λ_{id} corresponding to the identity permutation is not necessarily the identity matrix, but it is always invertible.

Lemma 45. [5] *The matrix representing* π *in partition* λ *equals* $(R^\lambda_{\mathrm{id}})^{-1} R^\lambda_\pi$.

Since Clifton's algorithm is very important for us, we present it formally in Figure 1, following an idea of Hentzel: the algorithm tries to compute the vertical permutation whose sign gives $(R^\lambda_\pi)_{ij}$, and returns 0 if it fails.

5.3. **Polynomial identities and representation theory.** The application of the representation theory of the symmetric group to polynomial identities was initiated independently by Malcev [23] and Specht [30] in 1950. The implementation of this theory in computer algebra was initiated by Hentzel [10, 11] in the 1970's. We first recall that any polynomial identity (not necessarily multilinear or even homogeneous) of degree $\leq n$ over a field F of characteristic 0 or $p > n$ is equivalent to a finite set of multilinear identities; see Zhevlakov et al. [32, Chapter 1]. We consider a multilinear nonassociative identity $I(x_1, \dots, x_n)$ of degree n. We collect the terms which have the same association type: $I = I_1 + \cdots + I_t$. In each summand I_k every monomial has association type k: the monomials differ only by the permutation of $x_1, \dots, x_n$. We can therefore regard each I_k as an element of the group algebra FS_n, and the identity I as an element of the direct sum of t copies

```
• Input: A permutation $\pi \in S_n$ and a partition $\lambda = (n_1, \dots, n_\ell)$ of $n$.
• Output: The Clifton matrix $R^\lambda_\pi$.
(1) Compute the standard tableaux $T_1, \dots, T_d$ for $\lambda$ where $d = d_\lambda$.
(2) For $j$ from 1 to $d$ do:
    (a) Compute $\pi T_j$.
    (b) For $i$ from 1 to $d$ do:
          (i) Set ijentry $\leftarrow 1$, number $\leftarrow 1$, finished $\leftarrow$ false.
         (ii) While number $\leq n$ and not finished do:
                • Set irow, icol $\leftarrow$ row, column indices of number in $T_i$.
                • Set jrow, jcol $\leftarrow$ row, column indices of number in $\pi T_j$.
                • If irow $\neq$ jrow then [number is not in the correct row]
                  – If icol $> n_{\texttt{jrow}}$ then
                        [the required position does not exist]
                        set ijentry $\leftarrow 0$, finished $\leftarrow$ true
                    else if $(T_i)_{\texttt{jrow,icol}} < (T_i)_{\texttt{irow,icol}}$ then
                        [the required position is already occupied]
                        set ijentry $\leftarrow 0$, finished $\leftarrow$ true
                    else
                        [transpose number into the required position]
                        set ijentry $\leftarrow -$ijentry,
                        interchange $(T_i)_{\texttt{irow,icol}}$ and $(T_i)_{\texttt{jrow,icol}}$
                • Set number $\leftarrow$ number $+ 1$
        (iii) Set $(R^\lambda_\pi)_{ij} \leftarrow$ ijentry
(3) Return $R^\lambda_\pi$.
```

FIGURE 1. Hentzel's algorithm to compute the Clifton matrix R^λ_π

of FS_n. Following the previous two subsections, let U be the submodule of $(FS_n)^t$ generated by I. Every element of U is a linear combination of permutations of I, and hence is an identity implied by I. By Lemma 44 we know that U is the direct sum of its components corresponding to the irreducible representations of S_n. This allows us to study I and its consequences one representation at a time, so we can break down a large computational problem into much smaller pieces.

Example 46. Consider the Jordan identity $(a^2b)a - a^2(ba)$ in a free commutative nonassociative algebra. The multilinear form of this identity (divided by 2) is

$$u = ((ac)b)d + ((ad)b)c + ((cd)b)a - (ac)(bd) - (ad)(bc) - (cd)(ba).$$

In degree 4 there are two association types $((ab)c)d$ and $(ab)(cd)$ for a commutative nonassociative operation. We regard $u = u_1 + u_2$ as an element of the direct sum of two copies of the group algebra $\mathbb{Q}S_4$ corresponding to the two association types:

$$u_1 = ((ac)b)d + ((ad)b)c + ((cd)b)a, \qquad u_2 = -(ac)(bd) - (ad)(bc) - (cd)(ba).$$

To illustrate the inequality of equation (4) we note that the two components u_1 and u_2 are identities which are not consequences of the Jordan identity. To illustrate the equality of Lemma 44 we decompose the submodule $U \subseteq (\mathbb{Q}S_4)^2$ generated by u into components corresponding to the irreducible representations of S_4. We obtain the linearization of fourth-power associativity (for $\lambda = 4$) and the identity which says that the commutator of multiplications is a derivation (for $\lambda = 31$).

$$\left[\begin{array}{ccccc|ccccc}
\rho_\lambda(E_1^1) & \rho_\lambda(E_2^1) & \cdots & \rho_\lambda(E_{n-1}^1) & \rho_\lambda(E_n^1) & -I_d & O & \cdots & O & O \\
\rho_\lambda(E_1^2) & \rho_\lambda(E_2^2) & \cdots & \rho_\lambda(E_{n-1}^2) & \rho_\lambda(E_n^2) & O & -I_d & \cdots & O & O \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\rho_\lambda(E_1^{t-1}) & \rho_\lambda(E_2^{t-1}) & \cdots & \rho_\lambda(E_{n-1}^{t-1}) & \rho_\lambda(E_n^{t-1}) & O & O & \cdots & -I_d & O \\
\rho_\lambda(E_1^t) & \rho_\lambda(E_2^t) & \cdots & \rho_\lambda(E_{n-1}^t) & \rho_\lambda(E_n^t) & O & O & \cdots & O & -I_d
\end{array}\right]$$

Table 3. Representation matrix in partition λ for the dialgebra expansions of the right-commutative association types in degree n

5.4. **Ranks and multiplicities.** Let $u^{[1]}, \dots, u^{[g]}$ be a set of multilinear polynomial identities of degree n over a field F of characteristic 0 or $p > n$. Suppose that t association types occur in the terms of the $u^{[i]}$, and let $U \subseteq (FS_n)^t$ be the submodule generated by $u^{[1]}, \dots, u^{[g]}$. We fix a partition λ of n and write $d = d_\lambda$ for the dimension of the corresponding irreducible representation of S_n. To determine the corresponding component of U we construct a $dg \times dt$ matrix M_λ with g rows and t columns of $d \times d$ blocks. In block (i, j) we put the representation matrix for the terms of $u^{[i]}$ in association type j, which can be computed by repeated application of Lemma 45. We compute $\mathrm{RCF}(M_\lambda)$, the row canonical form of M_λ.

Definition 47. The number of nonzero rows in $\mathrm{RCF}(M_\lambda)$ is the **rank of the submodule U in partition λ.**

Lemma 48. *The number of nonzero rows in* $\mathrm{RCF}(M_\lambda)$ *is the multiplicity of the irreducible representation corresponding to* λ *in the submodule* U.

We use a modification of this procedure to determine the structure of the kernel of the expansion map for the quasi-Jordan product. In degree n there are $t = R_n$ right-commutative association types (Lemma 18) and n dialgebra association types (corresponding to the position of the center). We fix a partition λ and write $d = d_\lambda$. We create a $td \times (n{+}t)d$ matrix X_λ with t rows and $n{+}t$ columns of $d \times d$ blocks; see Table 3. In the right side of X_λ, in block $(i, n{+}i)$ for $1 \le i \le t$, we put $-I_d$, the negative of the identity matrix; the other blocks of the right side are zero. In the left side of X_λ, in block (i, j) for $1 \le i \le t$ and $1 \le j \le n$, we put $\rho_\lambda(E_j^i)$, the representation matrix of the terms with dialgebra association type j in the expansion of the basic monomial with right-commutative association type i. The i-th row of blocks states that the basic monomial for the i-th right-commutative association type equals its expansion in the free associative dialgebra. Since the right side of X_λ is the negative of the identity matrix, X_λ has full row rank. We compute $\mathrm{RCF}(X_\lambda)$; there are no zero rows. We distinguish upper and lower parts of $\mathrm{RCF}(X_\lambda)$: the upper part contains the rows with leading ones in the left side, and the lower part contains the rows with leading ones in the right side. The lower left part is zero; the lower right part represents polynomial identities which are satisfied by the right-commutative association types as a result of dependence relations among the dialgebra expansions of the basic right-commutative monomials.

Definition 49. The number of (nonzero) rows in the lower right block of $\mathrm{RCF}(X_\lambda)$ will be called the **rank of all identities satisfied by the quasi-Jordan product in partition λ.**

			old identities			all identities			
	λ	d	rows	cols	rank	rows	cols	rank	new
1	6	1	21	20	17	20	26	17	0
2	51	5	105	100	85	100	130	85	0
3	42	9	189	180	153	180	234	153	0
4	411	10	210	200	172	200	260	172	0
5	33	5	105	100	85	100	130	85	0
6	321	16	336	320	274	320	416	274	0
7	3111	10	210	200	176	200	260	176	0
8	222	5	105	100	85	100	130	85	0
9	2211	9	189	180	157	180	234	157	0
10	21111	5	105	100	91	100	130	91	0
11	111111	1	21	20	19	20	26	19	0

TABLE 4. Degree 6: matrix ranks for all representations

Lemma 50. *The number of rows in the lower right block of* $\mathrm{RCF}(X_\lambda)$ *is the multiplicity of the irreducible representation corresponding to* λ *in the kernel of* E_n*.*

Lemma 51. *Let the submodule* U *of lifted identities in degree* n *for the quasi-Jordan product be generated by* $u^{[i]}, \dots, u^{[g]}$*. Let* λ *be a partition of* n*, let* $\mathrm{oldrank}(\lambda)$ *be the rank of* U *in partition* λ *(Definition 47), and let* $\mathrm{allrank}(\lambda)$ *be the rank of all identities in partition* λ *(Definition 49). Then* $\mathrm{oldrank}(\lambda) \leq \mathrm{allrank}(\lambda)$ *with equality if and only if there are no new identities corresponding to partition* λ*.*

5.5. **Rational arithmetic and modular arithmetic.** We prefer to use rational arithmetic, but this is impractical when the matrices are large: during the computation of the RCF, the numerators and denominators of the entries can become extremely large, even if the original matrix has small integer entries. To control the amount of memory required, we use modular arithmetic with a prime p greater than the degree n of the identities; this guarantees that $\mathbb{F}_p S_n$ is semisimple. The structure theory of $\mathbb{Q}S_n$, in particular isomorphism (7), shows that the elements of $\mathbb{Q}S_n$ which represent the matrix units in the simple ideals A_λ have coefficients in which the denominators are divisors of $n!$. It follows that the S_n-module $(FS_n)^t$ has the "same" structure over $\mathbb{Q}$ and over $\mathbb{F}_p$ when $p > n$. Therefore the ranks we obtain using modular arithmetic will be the same as the ranks we would have obtained using rational arithmetic. This leaves the problem of reconstructing rational results from modular results. In some cases, as in this paper, modular arithmetic produces coefficients for which the corresponding rational coefficients are easy to recover: 1, 2, 3, 49, 50, 51, 52, 98, 99, 100 in $\mathbb{F}_{101}$ represent 1, 2, 3, $-3/2$, $-1/2$, $1/2$, $3/2$, -3, -2, -1 in $\mathbb{Q}$. In other cases, we have to use many different primes and the Chinese Remainder Theorem; see Bremner and Peresi [4].

6. NONEXISTENCE OF NEW IDENTITIES IN DEGREE 6: SECOND COMPUTATION

Table 4 gives the matrix ranks for $p = 101$. There are no new identities, confirming our earlier computations without the representation theory of S_n.

When we use representation theory, we have two kinds of lifted identities: the 31 symmetries of the right-commutative association types and the 56 liftings of the identities J and K. In degree 6, we have 20 association types. For partition λ with

			old identities			all identities			
	λ	d	rows	cols	rank	rows	cols	rank	new
1	7	1	47	46	42	46	53	42	0
2	61	6	282	276	255	276	318	255	0
3	52	14	658	644	594	644	742	594	0
4	511	15	705	690	641	690	795	641	0
5	43	14	658	644	595	644	742	595	0
6	421	35	1645	1610	1490	1610	1855	1490	0
7	4111	20	940	920	859	920	1060	859	0
8	331	21	987	966	895	966	1113	895	0
9	322	21	987	966	892	966	1113	892	0
10	3211	35	1645	1610	1499	1610	1855	1499	0
11	31111	15	705	690	651	690	795	651	0
12	2221	14	658	644	598	644	742	598	0
13	22111	14	658	644	607	644	742	607	0
14	211111	6	282	276	265	276	318	265	0
15	1111111	1	47	46	45	46	53	45	0

Table 5. Degree 7: matrix ranks for all representations

dimension $d = d_\lambda$, we create a matrix with $20d$ columns and $21d$ rows, initialized to zero. For each lifted identity we put the representation matrices in the bottom d rows and compute the RCF. In Table 4 under "old identities" the columns "rows" and "cols" contain $21d$ and $20d$; the column "rank" contains the final rank of the matrix. From the complete list of 56 identities obtained by lifting J and K to degree 6, we retain only those identities which increase the rank for at least one partition. We recover the same 25 generators that we obtained earlier.

To compute all the identities for partition λ, we create a matrix with $20d$ rows and $26d$ columns, with a $20d \times 6d$ left block corresponding to the dialgebra expansions, and a $20d \times 20d$ right block corresponding to the basic right-commutative monomials. In Table 4 under "all identities" the columns "rows" and "cols" contain $20d$ and $26d$; the column "rank" contains the number of rows in the RCF which have leading ones in the lower right block of the matrix: these rows represent polynomial identities satisfied by the quasi-Jordan product.

When the two ranks are the same for partition λ, there are no new identities for this representation (Lemma 51). We checked this by verifying that the two matrices are in fact equal. Let r be the common rank for partition λ. The first matrix has size $r \times 20d$; these are the nonzero rows of the RCF of the matrix for the lifted identities. The second matrix has the same size; it contains the rows of the RCF of the matrix for the expansion identities with leading ones in the lower right block.

7. Nonexistence of new identities in degree 7

Table 5 gives the ranks for $p = 101$: there are no new identities in degree 7. The computations are similar to those for degree 6, except that the matrices are larger. The lifted ("old") identities consist of 89 symmetries of the 46 right-commutative association types and 200 liftings of J and K. The matrix of "old identities" has size $47d \times 46d$, and the matrix of "all identities" has size $46d \times 53d$. A subset of 55 identities suffices to generate all the lifted identities.

8. Special identities

The previous sections show that there are no special identities of degree ≤ 7 for quasi-Jordan algebras. The referee suggested the basic ideas for the following algorithm to construct certain special identities.

8.1. **Construction of noncommutative preimages of the Glennie identity.** Recall the Jordan triple product $\{abc\} = (ab)c + (cb)a - (ac)b$ and the Glennie identity of degree 8 from [32, page 79]:

$$\begin{aligned} G = {} & 2\{\{b\{aca\}b\}c(ab)\} - \{b\{a\{c(ab)c\}a\}b\} \\ & - 2\{(ab)c\{a\{bcb\}a\}\} + \{a\{b\{c(ab)c\}b\}a\}. \end{aligned}$$

We have the following algorithm for constructing noncommutative preimages of the Glennie identity:

1. Start with the Glennie identity G, a commutative nonassociative polynomial, homogeneous of degree 8, with variables $aaabbbcc$. Choose either $x = a$ or $x = b$ or $x = c$ as the variable to linearize.
2. Partially linearize G by replacing each occurrence of x in each term by the new variable d in every possible position; that is, apply the operator $\Delta_x^1(d)$ from [32, Chapter 1]. For example, if $x = a$ and a term has the form $-a-a-a-$ then the term becomes the sum of the three new terms
$$-d-a-a-, \quad -a-d-a-, \quad -a-a-d-.$$
We obtain a special Jordan identity H which is linear in d.
3. Apply the algorithm of Pozhidaev [25, Section 3]: convert each algebra monomial into a dialgebra monomial by making d the center. Operations to the left of d become $\vdash$ and operations to the right of d become $\dashv$. We obtain a nonassociative dialgebra polynomial I.
4. Replace the right dialgebra operation $\vdash$ in I by the opposite of the left dialgebra operation $\dashv$; that is, replace every occurrence of $y \vdash z$ by $z \dashv y$. (Recall that $y \vdash z = z \dashv y$ is the dialgebra consequence of commutativity.) Remove the symbols $\dashv$ from I. We obtain a noncommutative nonassociative polynomial J in one binary operation written as juxtaposition.
5. Replace every occurrence of d in J by the original variable x. We obtain a nonassociative polynomial K, homogeneous of degree 8, with variables $aaabbbcc$; this is a noncommutative preimage of the Glennie identity G.
6. Expand K into the free associative dialgebra using the quasi-Jordan product: that is, each product yz in K becomes $y \dashv z + z \vdash y$. We convert each term in the expansion to its normal form, collect similar terms, and verify that the result collapses to zero in all cases: $x = a$, $x = b$, and $x = c$.

For $x = a, b$ the identity K has 100 terms. For $x = c$ the identity K has 72 terms; see Table 6. (The monomials have been sorted and the coefficients divided by -2.) It is not clear *a priori* that the preimages obtained by this algorithm hold in any special quasi-Jordan algebra, but we have verified this by direct computation using Maple in step (6) of the algorithm.

8.2. **A new special identity.** The preceding arguments prove that special identities exist and that some of them are noncommutative preimages of the Glennie identity. We have discovered a new special identity, homogeneous of degree 8, with variables $aaaabbbc$; this identity has three variables and is linear in one variable. If

$$\begin{array}{rrr}
4(((((c(ab))c)a)a)b)b & -4(((((c(ab))c)b)b)a)a & -4(((((cc)(ab))a)a)b)b \\
+4(((((cc)(ab))b)b)a)a & +4((((c((ba)c))a)a)b)b & -4((((c((ba)c))b)b)a)a \\
-2((((c(ab))c)(aa))b)b & +2((((c(ab))c)(bb))a)a & +2((((cc)(ab))(aa))b)b \\
-2((((cc)(ab))(bb))a)a & -2(((c((ba)c))(aa))b)b & +2(((c((ba)c))(bb))a)a \\
+4(((((ca)a)b)b)(ab))c & -4(((((cb)b)a)a)(ba))c & -2((((c(aa))b)b)(ab))c \\
+2((((c(bb))a)a)(ba))c & -2((((ca)a)(bb))(ab))c & +2((((cb)b)(aa))(ba))c \\
+(((c(aa))(bb))(ab))c & -(((c(bb))(aa))(ba))c & -4(((((ca)a)b)b)c)(ab) \\
+4(((((cb)b)a)a)c)(ab) & +2((((c(aa))b)b)c)(ab) & -2((((c(ab))c)a)a)(bb) \\
+2((((c(ab))c)b)b)(aa) & -2((((c(bb))a)a)c)(ab) & +2((((cc)(ab))a)a)(bb) \\
-2((((cc)(ab))b)b)(aa) & -2(((c((ba)c))a)a)(bb) & +2(((c((ba)c))b)b)(aa) \\
+2((((ca)a)(bb))c)(ab) & -2((((cb)b)(aa))c)(ab) & -(((c(aa))(bb))c)(ab) \\
+(((c(bb))(aa))c)(ab) & +(((c(ab))c)(aa))(bb) & -(((c(ab))c)(bb))(aa) \\
-(((cc)(ab))(aa))(bb) & +(((cc)(ab))(bb))(aa) & +((c((ba)c))(aa))(bb) \\
-((c((ba)c))(bb))(aa) & +2(c((a(ca))(bb)))(ab) & -2(c((b(cb))(aa)))(ab) \\
-(c((c(aa))(bb)))(ab) & +(c((c(bb))(aa)))(ab) & +4(c(a((b(cb))a)))(ab) \\
-2(c(a((c(bb))a)))(ab) & -4(c(b((a(ca))b)))(ab) & +2(c(b((c(aa))b)))(ab) \\
-4((((ca)a)b)b)(c(ba)) & +4((((cb)b)a)a)(c(ba)) & +2(((c(aa))b)b)(c(ba)) \\
-2(((c(bb))a)a)(c(ba)) & +2(((ca)a)(bb))(c(ba)) & -2(((cb)b)(aa))(c(ba)) \\
-((c(aa))(bb))(c(ba)) & +((c(bb))(aa))(c(ba)) & -2(c(ba))((a((bb)c))a) \\
+4(c(ba))((a((bc)b))a) & +2(c(ba))((b((aa)c))b) & -4(c(ba))((b((ac)a))b) \\
+(c(ba))((aa)((bb)c)) & -2(c(ba))((aa)((bc)b)) & -(c(ba))((bb)((aa)c)) \\
+2(c(ba))((bb)((ac)a)) & +2c(((b(cb))(aa))(ba)) & -c(((c(bb))(aa))(ba)) \\
-4c((a((b(cb))a))(ba)) & +2c((a((c(bb))a))(ba)) & -2c((ba)((a(ca))(bb))) \\
+c((ba)((c(aa))(bb))) & +4c((ba)(b((a(ca))b))) & -2c((ba)(b((c(aa))b)))
\end{array}$$

TABLE 6. Noncommutative preimage of the Glennie identity ($x = c$)

it was a noncommutative preimage of the Glennie identity, then its commutative version would be satisfied by all special Jordan algebras and hence by all Jordan algebras (Macdonald's theorem); but its commutative version would be the Glennie identity, and this is a contradiction.

9. NEW IDENTITIES IN DEGREE 8

The computations in degree 8 are similar to degree 7, except that the matrices are larger; see Table 7. There are no new identities except in the representations corresponding to $\lambda = 431, 422, 332, 3311, 3221$ where the difference between allrank(λ) and oldrank(λ) is 1, 1, 2, 1, 1 respectively. The lifted identities consist of 242 symmetries of the 106 association types together with 495 liftings of J and K. The matrix of "old identities" has size $107d \times 106d$, and the matrix of "all identities" has size $106d \times 114d$. The ranks were computed using modular arithmetic with $p = 101$. We can recover rational results from modular results, and use rational arithmetic to verify the results; see the next section for the case $\lambda = 431$.

Definition 52. We say that a polynomial identity in degree n is **irreducible** if its complete linearization generates an irreducible representation of S_n.

Theorem 53. *There are six new irreducible identities for the quasi-Jordan product in degree 8: one each for $\lambda = 431, 422, 3311, 3221$ and two for $\lambda = 332$.*

			old identities			all identities				
	λ	d	rows	cols	rank	rows	cols	rank	new	
1	8	1	107	106	102	106	114	102	0	
2	71	7	749	742	714	742	798	714	0	
3	62	20	2140	2120	2040	2120	2280	2040	0	
4	611	21	2247	2226	2145	2226	2394	2145	0	
5	53	28	2996	2968	2856	2968	3192	2856	0	
6	521	64	6848	6784	6532	6784	7296	6532	0	
7	5111	35	3745	3710	3582	3710	3990	3582	0	
8	44	14	1498	1484	1428	1484	1596	1428	0	
9	431	70	7490	7420	7142	7420	7980	7143	1	$\leftarrow$
10	422	56	5992	5936	5712	5936	6384	5713	1	$\leftarrow$
11	4211	90	9630	9540	9199	9540	10260	9199	0	
12	41111	35	3745	3710	3594	3710	3990	3594	0	
13	332	42	4494	4452	4284	4452	4788	4286	2	$\leftarrow$
14	3311	56	5992	5936	5722	5936	6384	5723	1	$\leftarrow$
15	3221	70	7490	7420	7149	7420	7980	7150	1	$\leftarrow$
16	32111	64	6848	6784	6565	6784	7296	6565	0	
17	311111	21	2247	2226	2169	2226	2394	2169	0	
18	2222	14	1498	1484	1429	1484	1596	1429	0	
19	22211	28	2996	2968	2870	2968	3192	2870	0	
20	221111	20	2140	2120	2065	2120	2280	2065	0	
21	2111111	7	749	742	729	742	798	729	0	
22	11111111	1	107	106	105	106	114	105	0	

TABLE 7. Degree 8: matrix ranks for all representations

From this we obtain the following example of an exceptional (non-special) quasi-Jordan algebra.

Corollary 54. *There exists a noncommutative nilpotent exceptional quasi-Jordan algebra.*

Proof. Let $\mathcal{N}$ be the variety of quasi-Jordan algebras defined by the identities $x_1 \cdots x_9 = 0$ with any placement of parentheses. Let X be the free algebra in $\mathcal{N}$ on the generators a, b, c. The quasi-Jordan polynomial discussed in the next section is nonzero in X but is zero in every special quasi-Jordan algebra. $\square$

10. A SPECIAL IDENTITY FOR PARTITION 431

Since the rank has increased by 1 for partition 431, we expect there to be a new identity in which every monomial consists of a right-commutative association type applied to a permutation of *aaaabbbc*. There are 106 association types and 280 permutations, so the number of monomials is at most 29680; the other partitions with new identities give larger upper bounds. Right-commutativity implies that many of these monomials are equal; if we count only those which equal their own straightened forms then we obtain 12131 distinct monomials. For dialgebra monomials we have 8 association types giving 2240 distinct monomials. The expansion of each right-commutative monomial is a linear combination of 128 dialgebra monomials.

$$
\begin{array}{rrr}
2((((((aa)a)a)b)b)c)b & -2((((((aa)a)b)a)b)c)b & +2((((((aa)a)b)b)a)b)c \\
-2((((((aa)a)b)b)b)a)c & +2((((((aa)a)b)b)b)c)a & -2((((((aa)a)b)b)c)a)b \\
-2((((((aa)a)b)b)c)b)a & +4((((((aa)a)b)c)b)b)a & +2((((((aa)a)c)b)a)b)b \\
-4((((((aa)a)c)b)b)a)b & -2((((((aa)b)a)a)b)b)c & +4((((((aa)b)a)b)a)b)c \\
-2((((((aa)b)a)b)b)a)c & -2((((((aa)b)a)b)c)a)b & +2((((((aa)b)a)b)c)b)a \\
+2((((((aa)b)a)c)a)b)b & -2((((((aa)b)a)c)b)a)b & +2((((((aa)b)b)c)b)a)a \\
-2((((((aa)b)c)b)a)a)b & +2((((((aa)b)c)b)a)b)a & -2((((((aa)b)c)b)b)a)a \\
+2((((((ab)a)a)a)b)b)c & -2((((((ab)a)a)b)a)b)c & -2((((((ab)a)a)b)a)c)b \\
-2((((((ab)a)a)b)c)b)a & -2((((((ab)a)a)c)a)b)b & +2((((((ab)a)a)c)b)a)b \\
-4((((((ab)a)b)a)a)b)c & +2((((((ab)a)b)a)a)c)b & +4((((((ab)a)b)a)b)a)c \\
+4((((((ab)a)b)a)c)a)b & -2((((((ab)a)b)b)a)c)a & +2((((((ab)a)b)c)a)a)b \\
-6((((((ab)a)b)c)a)b)a & -2((((((ab)a)c)a)a)b)b & +4((((((ab)a)c)a)b)a)b \\
-2((((((ab)a)c)a)b)b)a & +2((((((ab)a)c)b)a)a)b & +2((((((ab)a)c)b)b)a)a \\
+2((((((ab)b)a)b)c)a)a & -2((((((ab)b)a)c)a)a)b & +2((((((ab)b)a)c)a)b)a \\
-2((((((ab)b)a)c)b)a)a & +2((((((ab)b)c)a)b)a)a & -2((((((ab)b)c)b)a)a)a \\
-2((((((ab)c)a)b)a)a)b & +2((((((ab)c)a)b)a)b)a & -2((((((ab)c)a)b)b)a)a \\
+2((((((ab)c)b)a)a)a)b & -2((((((ab)c)b)a)a)b)a & +2((((((ab)c)b)b)a)a)a \\
+2((((((ac)a)a)b)a)b)b & -2((((((ac)a)a)b)b)a)b & +2((((((ac)a)a)b)b)b)a \\
-2((((((ac)a)b)a)a)b)b & +2((((((ac)a)b)a)b)a)b & -4((((((ac)a)b)a)b)b)a \\
+2((((((ac)a)b)b)a)a)b & +2((((((ac)a)b)b)a)b)a & -2((((((ac)a)b)b)b)a)a \\
-2((((((ac)b)a)b)a)a)b & +2((((((ac)b)a)b)a)b)a & +2((((((ac)b)a)b)b)a)a \\
-2((((((ac)b)b)a)b)a)a & -2(((((a(aa))a)b)b)c)b & +2(((((a(aa))b)a)b)c)b \\
-2(((((a(aa))b)b)a)b)c & +2(((((a(aa))b)b)b)a)c & -2(((((a(aa))b)b)b)c)a \\
+2(((((a(aa))b)b)c)a)b & +2(((((a(aa))b)b)c)b)a & -4(((((a(aa))b)c)b)b)a \\
-2(((((a(aa))c)b)a)b)b & +4(((((a(aa))c)b)b)a)b & -2(((((a(ab))a)a)b)b)c \\
-2(((((a(ab))a)a)c)b)b & +2(((((a(ab))a)b)a)b)c & +4(((((a(ab))a)b)a)c)b \\
-2(((((a(ab))a)b)c)a)b & +2(((((a(ab))a)b)c)b)a & +4(((((a(ab))a)c)a)b)b \\
-4(((((a(ab))a)c)b)a)b & +2(((((a(ab))a)c)b)b)a & +4(((((a(ab))b)a)a)b)c \\
-6(((((a(ab))b)a)b)a)c & +2(((((a(ab))b)a)b)c)a & -4(((((a(ab))b)a)c)a)b \\
-2(((((a(ab))b)a)c)b)a & +2(((((a(ab))b)b)a)c)a & -2(((((a(ab))b)b)c)a)a \\
-2(((((a(ab))b)c)a)a)b & +6(((((a(ab))b)c)a)b)a & +2(((((a(ab))c)a)a)b)b \\
-2(((((a(ab))c)a)b)a)b & -2(((((a(ab))c)b)a)b)a & -4(((((a(ac))a)b)a)b)b \\
+4(((((a(ac))a)b)b)a)b & -2(((((a(ac))a)b)b)b)a & +2(((((a(ac))b)a)a)b)b \\
-2(((((a(ac))b)b)a)b)a & +2(((((a(ac))b)b)b)a)a & +(((((a(bb))a)a)a)c)b \\
-(((((a(bb))a)a)c)a)b & -(((((a(bb))a)b)a)a)c & +(((((a(bb))a)b)a)c)a \\
+(((((a(bb))a)b)c)a)a & -(((((a(bb))a)c)b)a)a & -(((((a(bb))b)c)a)a)a \\
+(((((a(bb))c)b)a)a)a & +2(((((a(bc))a)b)a)a)b & -2(((((a(bc))a)b)a)b)a \\
+2(((((a(bc))b)a)b)a)a & +2(((((aa)(ab))a)b)b)c & -2(((((aa)(ab))a)b)c)b \\
+4(((((aa)(ab))a)c)b)b & -2(((((aa)(ab))b)a)b)c & -4(((((aa)(ab))b)a)c)b \\
+2(((((aa)(ab))b)b)a)c & -2(((((aa)(ab))b)b)c)a & +6(((((aa)(ab))b)c)a)b \\
-4(((((aa)(ab))c)a)b)b & +4(((((aa)(ab))c)b)a)b & -2(((((aa)(ab))c)b)b)a \\
-3(((((aa)(bb))a)a)c)b & +2(((((aa)(bb))a)b)a)c & -2(((((aa)(bb))a)b)c)a \\
+2(((((aa)(bb))a)c)a)b & +2(((((aa)(bb))a)c)b)a & -(((((aa)(bb))b)a)c)a \\
+2(((((aa)(bb))c)a)a)b & -2(((((aa)(bb))c)a)b)a & +(((((aa)(bb))c)b)a)a \\
+2(((((ab)(ac))a)a)b)b & -2(((((ab)(ac))a)b)a)b & +2(((((ab)(ac))a)b)b)a \\
-2(((((ab)(ac))b)a)a)b & +2(((((ab)(ac))b)a)b)a & -2(((((ab)(ac))b)b)a)a \\
-2(((((ab)(bc))a)a)a)b & +2(((((ab)(bc))a)a)b)a & -2(((((ab)(bc))a)b)a)a \\
-2(((((ab)(bc))b)a)a)a & +2(((((aa)a)(ab))b)c)b & -2(((((aa)a)(ab))c)b)b \\
-3(((((aa)a)(bb))a)b)c & +4(((((aa)a)(bb))a)c)b & +2(((((aa)a)(bb))b)a)c \\
+(((((aa)a)(bb))c)a)b & -2(((((aa)a)(bb))c)b)a & -2(((((aa)a)(bc))a)b)b
\end{array}
$$

Table 8. Special identity for partition 431 (terms 1 to 150)

$+4(((((aa)a)(bc))b)a)b$	$+4(((((aa)b)(ac))a)b)b$	$+2(((((aa)b)(ac))b)a)b$
$+2(((((aa)b)(ac))b)b)a$	$+2(((((aa)b)(bc))a)a)b$	$+2(((((aa)b)(bc))a)b)a$
$+2(((((ab)a)(ab))a)b)c$	$-2(((((ab)a)(ab))a)c)b$	$-2(((((ab)a)(ab))b)a)c$
$+2(((((ab)a)(ab))b)c)a$	$-6(((((ab)a)(ab))c)a)b$	$-4(((((ab)a)(ac))a)b)b$
$-6(((((ab)a)(ac))b)a)b$	$-2(((((ab)a)(ac))b)b)a$	$+(((((ab)a)(bb))a)c)a$
$-2(((((ab)a)(bb))c)a)a$	$-4(((((ab)a)(bc))a)a)b$	$-2(((((ab)a)(bc))a)b)a$
$-2(((((ab)a)(bc))b)a)a$	$-2(((((ab)b)(ac))a)b)a$	$+2(((((ab)b)(ac))b)a)a$
$+2(((((ab)b)(bc))a)a)a$	$-4(((((ac)a)(ab))b)a)b$	$+2(((((ac)a)(ab))b)b)a$
$-(((((ac)a)(bb))a)a)b$	$-(((((ac)a)(bb))a)b)a$	$+(((((ac)a)(bb))b)a)a$
$+2((((a(aa))(bb))a)b)c$	$-((((a(aa))(bb))a)c)b$	$-2((((a(aa))(bb))b)a)c$
$+2((((a(aa))(bb))b)c)a$	$-2((((a(aa))(bb))c)a)b$	$+2((((a(aa))(bc))a)b)b$
$-4((((a(aa))(bc))b)a)b$	$-2((((a(ab))(ab))a)b)c$	$+4((((a(ab))(ab))b)a)c$
$-4((((a(ab))(ab))b)c)a$	$+4((((a(ab))(ab))c)a)b$	$+4((((a(ab))(ac))b)a)b$
$+((((a(ab))(bb))a)a)c$	$-2((((a(ab))(bb))a)c)a$	$+((((a(ab))(bb))c)a)a$
$+2((((a(ab))(bc))a)a)b$	$+2((((a(ab))(bc))b)a)a$	$+((((a(ac))(bb))a)b)a$
$-((((a(ac))(bb))b)a)a$	$+(((((aa)a)a)(bb))b)c$	$-2(((((aa)a)a)(bb))c)b$
$+2(((((aa)a)b)(ac))b)b$	$-2(((((aa)a)b)(bc))b)a$	$-2(((((aa)b)a)(ab))b)c$
$+2(((((aa)b)a)(ab))c)b$	$-4(((((aa)b)a)(ac))b)b$	$-(((((aa)b)a)(bb))a)c$
$-4(((((aa)b)a)(bc))a)b$	$-2(((((aa)b)a)(bc))b)a$	$-6(((((aa)b)b)(ac))a)b$
$-4(((((aa)b)b)(ac))b)a$	$-2(((((aa)b)b)(bc))a)a$	$-2(((((aa)c)a)(bb))a)b$
$+(((((aa)c)a)(bb))b)a$	$+2(((((ab)a)a)(ac))b)b$	$+(((((ab)a)a)(bb))a)c$
$-2(((((ab)a)a)(bb))c)a$	$+4(((((ab)a)a)(bc))a)b$	$+6(((((ab)a)b)(ac))a)b$
$+6(((((ab)a)b)(ac))b)a$	$+4(((((ab)a)b)(bc))a)a$	$+2(((((ab)b)a)(ab))c)a$
$+4(((((ab)b)a)(ac))a)b$	$+2(((((ab)b)a)(ac))b)a$	$+2(((((ab)b)a)(bc))a)a$
$+4(((((ab)c)a)(ab))a)b$	$+2(((((ab)c)a)(bb))a)a$	$+(((((ac)a)a)(bb))a)b$
$-(((((ac)a)a)(bb))b)a$	$-(((((ac)b)a)(bb))a)a$	$+((((a(aa))a)(bb))c)b$
$+4((((a(aa))b)(bc))b)a$	$-2((((a(ab))a)(bb))a)c$	$+3((((a(ab))a)(bb))c)a$
$-4((((a(ab))b)(ac))a)b$	$-4((((a(ab))b)(ac))b)a$	$-4((((a(ab))b)(bc))a)a$
$+2(((((aa)a)a)b)(bc))b$	$-2(((((aa)a)b)a)(bb))c$	$-4(((((aa)a)b)a)(bc))b$
$-2(((((aa)a)b)b)(ac))b$	$+2(((((aa)b)a)a)(bb))c$	$+2(((((aa)b)a)a)(bc))b$
$+2(((((aa)b)a)b)(bc))a$	$+4(((((aa)b)b)a)(ac))b$	$+2(((((aa)b)b)a)(bc))a$
$+2(((((aa)b)b)b)(ac))a$	$-2(((((aa)b)c)a)(ab))b$	$-(((((ab)a)a)a)(bb))c$
$+2(((((ab)a)a)b)(ac))b$	$+4(((((ab)a)a)b)(bc))a$	$+2(((((ab)a)b)a)(ab))c$
$-2(((((ab)a)b)a)(ac))b$	$-4(((((ab)a)b)a)(bc))a$	$+4(((((ab)a)c)a)(ab))b$
$+2(((((ab)a)c)a)(bb))a$	$-2(((((ab)b)a)a)(ac))b$	$-2(((((ab)b)a)a)(bc))a$
$-6(((((ab)b)a)b)(ac))a$	$-2(((((ab)b)c)a)(ab))a$	$-2(((((ab)c)a)a)(ab))b$
$-(((((ab)c)a)a)(bb))a$	$+2(((((ac)a)b)a)(ab))b$	$+(((((ac)a)b)a)(bb))a$
$-2((((a(aa))b)b)(bc))a$	$+((((a(ab))a)a)(bb))c$	$-2((((a(ab))a)b)(ac))b$
$-4((((a(ab))a)b)(bc))a$	$-2((((a(ab))b)a)(ab))c$	$+2((((a(ab))b)a)(ac))b$
$+4((((a(ab))b)a)(bc))a$	$+4((((a(ab))b)b)(ac))a$	$-2((((a(ab))c)a)(ab))b$
$-((((a(ab))c)a)(bb))a$	$-2(((((aa)a)a)b)b)(bc)$	$+4(((((aa)a)b)a)b)(bc)$
$-2(((((aa)b)a)b)a)(bc)$	$+2(((((aa)b)a)b)b)(ac)$	$-2(((((aa)b)b)a)b)(ac)$
$+2(((((aa)b)b)c)a)(ab)$	$-(((((aa)b)c)a)a)(bb)$	$-2(((((aa)b)c)b)a)(ab)$
$+(((((aa)c)a)b)a)(bb)$	$+(((((ab)a)a)c)a)(bb)$	$-2(((((ab)a)b)a)b)(ac)$
$-2(((((ab)a)b)c)a)(ab)$	$+2(((((ab)a)c)b)a)(ab)$	$+2(((((ab)b)a)b)a)(ac)$
$-2(((((ab)b)a)c)a)(ab)$	$+2(((((ab)b)c)a)a)(ab)$	$-2(((((ab)c)a)b)a)(ab)$
$-((((a(ab))a)c)a)(bb)$	$+2((((a(ab))b)a)b)(ac)$	$-2((((a(ab))b)b)a)(ac)$
$+2((((a(ab))b)c)a)(ab)$	$+((((a(ac))a)b)a)(bb)$	$-((((a(ac))b)a)a)(bb)$
$-2((((aa)(ab))a)b)(bc)$	$+2((((aa)(ab))b)a)(bc)$	

Table 9. Special identity for partition 431 (terms 151 to 296)

In step 1, we create a 12411×12131 matrix with a 12131×12131 upper block and a 280×12131 lower block. For each lifted identity in degree 8, we apply all 280 substitutions of $aaaabbbc$ for the variables, store these nonlinear identities in the lower block, and compute the RCF using arithmetic modulo $p = 101$. After this process is complete, the rank of the matrix is 11020.

In step 2, we create a 2240×12131 matrix, initialize the columns with the coefficients of the expansions of the nonlinear right-commutative monomials, and compute the RCF using arithmetic modulo $p = 101$. The rank of the matrix is 1110 and so the nullspace has dimension 11021.

The nullity from step 2 is exactly one more than the rank from step 2, as expected from row 9 of Table 7. The row space from step 1 is a subspace of the nullspace from step 2. We need to find a nullspace vector which is not in the row space.

In step 3, we compute the canonical basis of the nullspace from step 2. We sort the basis vectors by increasing size; we define the "size" of a vector over a finite field to be the number of distinct coefficients. We include the basis vectors one at a time as a new bottom row of the matrix from step 1 until we find the first basis vector that increases the rank. We multiply this basis vector by 2 and reduce the coefficients modulo 101 using symmetric representatives so that all the coefficients become small integers. We obtain the 296-term identity in Tables 8 and 9. We expand this identity using rational arithmetic and verify that it collapses to zero in the free associative dialgebra; this verifies that it is a special identity over $\mathbb{Q}$.

Our new special identity involves three variables and is linear in one variable. Therefore the obvious generalization of Macdonald's theorem [22] to quasi-Jordan algebras is not true, since our new identity is satisfied by all special quasi-Jordan algebras but not by all quasi-Jordan algebras. If we assume commutativity and collect terms, we obtain a polynomial with 191 terms in the free commutative nonassociative algebra. This commutative identity involves three variables and is linear in one variable; it is satisfied by all special Jordan algebras since every special Jordan algebra is a special quasi-Jordan algebra (corresponding to an associative dialgebra in which the two operations coincide). Therefore, by Macdonald's theorem, this commutative identity is satisfied by all Jordan algebras, and hence must be satisfied by the Albert algebra $H_3(\mathbb{C})$.

It follows that we cannot use the Albert algebra to give a direct proof that our new identity is not satisfied by all semispecial quasi-Jordan algebras.

11. CONCLUSION

Semispecial quasi-Jordan algebras are a natural generalization of Jordan algebras to a noncommutative setting. An important open problem is to generalize classical results on free (special) Jordan algebras to semispecial quasi-Jordan algebras. For example:

- *i)* the criterion of Cohn [6] for a quotient of a free special Jordan algebra to be special, which implies that special Jordan algebras do not form a variety;
- *ii)* the characterization by Cohn [6] of the free special Jordan algebra on ≤ 3 generators as the symmetric elements in a free associative algebra;
- *iii)* the theorem of Macdonald [22] on special Jordan identities in 3 variables;
- *iv)* the theorem of Shirshov [27] that the free Jordan algebra on two generators is special (see also Jacobson and Paige [14]).

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References

[1] A. A. Albert and L. J. Paige: On a homomorphism property of certain Jordan algebras. *Trans. Amer. Math. Soc.* 93 (1959) 20–29. MR0108524 (21 #7240)

[2] G. Bergdolt: Tilted irreducible representations of the permutation group. *Comput. Phys. Comm.* 86 (1995), no. 1-2, 97–104. MR1327568 (96e:20015)

[3] M. R. Bremner: On the definition of quasi-Jordan algebra. *Comm. Algebra* (to appear, accepted 23 October 2009).

[4] M. R. Bremner and L. A. Peresi: Nonhomogeneous subalgebras of Lie and special Jordan superalgebras. *J. Algebra* 322, 6 (2009) 2000-2026. MR2542829

[5] J. M. Clifton: A simplification of the computation of the natural representation of the symmetric group S_n. *Proc. Amer. Math. Soc.* 83 (1981), no. 2, 248–250. MR0624907 (82j:20024)

[6] P. M. Cohn: On homomorphic images of special Jordan algebras. *Canadian J. Math.* 6 (1954) 253–264. MR0060496 (15,678c)

[7] C. M. Glennie: *Identities in Jordan Algebras*. Ph.D. dissertation, Yale University, 1963.

[8] C. M. Glennie: Some identities valid in special Jordan algebras but not valid in all Jordan algebras. *Pacific J. Math.* 16 (1966) 47–59. MR0186708 (32 #4166)

[9] C. M. Glennie: Identities in Jordan algebras. *Computational Problems in Abstract Algebra.* Proc. Conf., Oxford, 1967, Pergamon, Oxford, 1970, pages 307–313. MR0255629 (41 #289)

[10] I. R. Hentzel: Processing identities by group representation. *Computers in Nonassociative Rings and Algebras*. Special session, 82nd Annual Meeting Amer. Math. Soc., San Antonio, 1976, pp. 13–40. Academic Press, New York, 1977. MR0463251 (57 #3204)

[11] I. R. Hentzel: Applying group representation to nonassociative algebras. *Ring Theory.* Proc. Conf., Ohio Univ., Athens, Ohio, 1976, pp. 133–141. Lecture Notes in Pure and Appl. Math., Vol. 25, Dekker, New York, 1977. MR0435159 (55 #8120)

[12] I. R. Hentzel: Special Jordan identities. *Comm. Algebra* 7 (1979) no. 16, 1759–1793. MR0546197 (81a:17008)

[13] D. P. Jacobs: Albert: an Interactive Program to Assist the Specialist in the Study of Nonassociative Algebra. `http://www.cs.clemson.edu/~dpj/albertstuff/albert.html`

[14] N. Jacobson and L. J. Paige: On Jordan algebras with two generators. *J. Math. Mech.* 6 (1957) 895–906. MR0092780 (19,1157b)

[15] G. James and A. Kerber: *The Representation Theory of the Symmetric Group*. Encyclopedia of Mathematics and its Applications, 16. Addison-Wesley Publishing Co., Reading, Mass., 1981. MR0644144 (83k:20003)

[16] B. W. Kernighan and D. Ritchie: *The C Programming Language.* Second edition. Prentice-Hall, 1988.

[17] P. S. Kolesnikov: Varieties of dialgebras and conformal algebras. *Sibirsk. Mat. Zh.* 49 (2008), no. 2, 322–339; translation in *Sib. Math. J.* 49 (2008), no. 2, 257–272. MR2419658 (2009b:17002)

[18] A. G. Kurosh: Nonassociative free sums of algebras. *Mat. Sb. N.S.* 37(79) (1955) 251–264. MR0078356 (17,1180i)

[19] J.-L. Loday: Une version non commutative des algèbres de Lie: les algèbres de Leibniz. *Enseign. Math.* (2) 39 (1993), no. 3-4, 269–293. MR1252069 (95a:19004)

[20] J.-L. Loday: Algèbres ayant deux opérations associatives (digèbres). *C. R. Acad. Sci. Paris Sér. I Math.* 321 (1995), no. 2, 141–146. MR1345436 (96f:16013)

[21] J.-L. Loday: Dialgebras. *Dialgebras and Related Operads*, 7–66, Lecture Notes in Math., 1763, Springer, Berlin, 2001. MR1860994 (2002i:17004)

[22] I. G. Macdonald: Jordan algebras with three generators. *Proc. London Math. Soc.* (3) 10 (1960) 395–408. MR0126473 (23 #A3769)

[23] A. I. Malcev: On algebras defined by identities. *Mat. Sbornik N.S.* 26(68) (1950) 19–33. MR0033280 (11,414d)

[24] Maple: Maple 13 – The Essential Tool for Mathematics and Modeling. *Maplesoft, a Division of Waterloo Maple Inc.* `http://www.maplesoft.com`

[25] A. P. Pozhidaev: 0-dialgebras with bar unity, Rota-Baxter and 3-Leibniz algebras. Pages 245–256 of *Groups, Rings and Group Rings*, edited by A. Giambruno, C. Polcino Milies and S. K. Sehgal, Contemporary Mathematics 499, American Mathematical Society, 2009.

[26] A. I. Shirshov: Subalgebras of free commutative and free anticommutative algebras. *Mat. Sbornik N.S.* 34(76) (1954) 81–88. MR0062112 (15,929b)

[27] A. I. Shirshov: On special J-rings. *Mat. Sb. N.S.* 38(80) (1956) 149–166. MR0075936 (17,822e)

[28] A. I. Shirshov: Some problems in the theory of rings that are nearly associative. *Uspekhi Mat. Nauk* 13 (1958), no. 6(84), 3–20. MR0102532 (21 #1323); translation in *Non-associative Algebra and its Applications*, 441–459, Lect. Notes Pure Appl. Math., 246, Chapman & Hall/CRC, Boca Raton, 2006.

[29] N. J. A. Sloane: *The On-Line Encyclopedia of Integer Sequences.* AT&T Labs - Research. `http://www.research.att.com/~njas/sequences`

[30] W. Specht: Gesetze in Ringen, I. *Math. Z.* 52 (1950) 557–589. MR0035274 (11,711i)

[31] R. Velásquez and R. Felipe: Quasi-Jordan algebras. *Comm. Algebra* 36 (2008), no. 4, 1580–1602. MR2410352 (2009b:17004)

[32] K. A. Zhevlakov, A. M. Slinko, I. P. Shestakov and A. I. Shirshov: *Rings That Are Nearly Associative.* Translated from the Russian by Harry F. Smith. Pure and Applied Mathematics, 104. Academic Press, Inc., New York, 1982. MR0668355 (83i:17001)

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