Universal central extensions of elliptic affine Lie algebras

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Let $\mathfrak{g}$ be a simple complex (finite dimensional) Lie algebra, and let R be the ring of regular functions on a compact complex algebraic curve with a finite number of points removed. Lie algebras of the form $\mathfrak{g}\otimes_{\mathbb{C}}R$ are considered; these generalize Kac–Moody loop algebras since for a curve of genus zero with two punctures $R\simeq\mathbb{C}[t,t^{-1}]$. The universal central extension of $\mathfrak{g}\otimes R$ is analogous to an untwisted affine Kac–Moody algebra. By Kassel's theorem the kernel of the universal central extension is linearly isomorphic to the Kähler differentials of R modulo exact differentials. The dimension of the kernel for any R is determined first. Restricting to hyperelliptic curves with 2, 3, or 4 special points removed, a basis for the kernel is determined. Restricting further to an elliptic curve with punctures at two points (of orders one and two in the group law) we explicitly determine the cocycles which give the commutation relations for the universal central extension. The results involve Pollaczek polynomials, which are a genus-one generalization of ultraspherical (Gegenbauer) polynomials. © *1994 American Institute of Physics.*

I. INTRODUCTION

The rapid development of the theory of affine Kac–Moody Lie algebras and their representations[1–4] has led recently to a search for related families of infinite-dimensional complex Lie algebras. Recall that an untwisted affine Lie algebra is the universal central extension of the Lie algebra $\mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[t,t^{-1}]$, where $\mathfrak{g}$ is a simple complex finite-dimensional Lie algebra; the commutation relations for $\mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[t,t^{-1}]$ are $[x\otimes f,y\otimes g]=[xy]\otimes fg$. In this construction we may replace the Laurent polynomial algebra $\mathbb{C}[t,t^{-1}]$ by any other commutative associative $\mathbb{C}$-algebra, say R, and consider the universal extension of $\mathfrak{g}\otimes_{\mathbb{C}}R$. Among the $\mathbb{C}$-algebras that have been considered are two natural generalizations of the Laurent polynomials: the ring $\mathbb{C}[t_1,t_1^{-1},\ldots t_n,t_n^{-1}]$ of Laurent polynomials in more than one variable which gives the toroidal Lie algebras,[5–12] and the ring $\mathbb{C}[t,(t-a_1)^{-1},\ldots(t-a_{n-1})^{-1}]$ of rational functions with restricted denominators which gives the n-point affine Lie algebras.[13,14]

Another generalization[15] may be obtained by observing that the ring of Laurent polynomials is isomorphic to the ring of regular functions on the variety $\mathbb{C}-\{0\}$ (the Riemann sphere with the points 0 and ∞ removed). Since the Riemann sphere is a compact algebraic curve of genus zero, we may instead consider the ring R of regular functions on a compact algebraic curve of any genus with an arbitrary finite set of points removed. If we write $\mathcal{G}=\mathfrak{g}\otimes_{\mathbb{C}}R$ for the "loop" algebra, and $\hat{\mathcal{G}}$ for the universal central extension of $\mathcal{G}$, then as vector spaces we have $\hat{\mathcal{G}}=\mathcal{G}\oplus C$, where C is the kernel of the surjective homomorphism from $\hat{\mathcal{G}}$ onto $\mathcal{G}$; in other words, C is the center of $\hat{\mathcal{G}}$. (The notation can be misleading: recall that $\mathcal{G}$ is a quotient, not a subalgebra, of $\hat{\mathcal{G}}$.)

By Kassel's theorem[16] we know that C is linearly isomorphic to Ω_R^1/dR, the space of Kähler differentials of R modulo exact differentials. In fact this result applies to Lie algebras of the form $\mathfrak{g}_{\mathbb{Z}}\otimes_{\mathbb{Z}}R$ where $\mathfrak{g}_{\mathbb{Z}}$ is the Chevalley $\mathbb{Z}$-form of $\mathfrak{g}$ and R is any commutative algebra over an arbitrary commutative ring k. (By the work of Berman and Krylyuk[17] we know that the same result holds when $\mathfrak{g}$ is any finite-dimensional central simple Lie algebra over a field of characteristic 0; in fact

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they give a generalization of Kassel's theorem to any perfect Lie algebra over a field of characteristic not 2 satisfying $H^1(\mathfrak{g},\mathfrak{g}^*)=\{0\}$.)

Remarkably Kassel's result depends only on R, so determining the center C is essentially a problem in commutative algebra. Among the first questions that arise are:

(1) describe C, and in particular determine its dimension,
(2) find a basis for C, and
(3) calculate the universal cocycle $\hat{\mathcal{G}}\times\hat{\mathcal{G}}\rightarrow C$ explicitly.

In this paper, we first answer question (1) for any algebraic curve with any number of points removed, using a cohomological argument. We then specialize to the case in which R is a quadratic extension of the ring of Laurent polynomials; that is, the ring of regular functions on a hyperelliptic curve with 2, 3, or 4 special points removed. In this special case we answer question (2). Finally we consider an elliptic curve (i.e., a curve of genus one) and remove two points which differ by an element of order two (in the group law on the curve). For the resulting Lie algebras, we answer question (3).

The method we use parallels that of a previous paper,[14] which studied the case $R=\mathbb{C}[s,s^{-1},(s-1)^{-1},(s-a)^{-1}]$. This ring is isomorphic to $\mathbb{C}[t,t^{-1},u]$ where $u^2=t^2-2bt+1$ and $b=(a+1)/(a-1)$. To answer question (3) in that case, it was necessary to derive a recursion formula for the coefficients of the central elements from the structure of the space of Kähler differentials of R. A family of polynomials in the parameter b was obtained, and the recursion formula made it possible to recognize these polynomials as ultrapherical (Gegenbauer) polynomials.[18] In the current paper, the ring of elliptic functions which we consider is isomorphic to $\mathbb{C}[t,t^{-1},u]$ where $u^2=t^3-2bt^2+t$. Again we obtain a family of polynomials in b defined by recursion; these polynomials turn out to belong to the Pollaczek family of orthogonal polynomials.[19] The Pollaczek polynomials thus appear as a "genus-one" generalization of ultraspherical polynomials.

The rings of functions which arise from an elliptic curve with two points removed, and the corresponding Lie algebras, have also been considered by Sheinman.[20,21] These papers assume that the difference between the two points (in the group law on the curve) is a point of infinite order,[22] and consider the unique (up to coboundary) cocycle which is local with respect to the seminatural quasigrading in this case.[15] Sheinman was not looking for the universal central extension—which is in fact 3 dimensional, as we will see below.

The rings which arise from higher-genus curves with any number of points removed have been considered by Schlichenmaier.[23–27] These papers however deal exclusively with the derivation algebras Der R rather than the "loop" algebras $\mathfrak{g}\otimes R$.

II. THE GENERAL CASE: DIMENSION OF THE CENTER

Let R be the $\mathbb{C}$-algebra of global meromorphic functions, on a compact Riemann surface Σ of genus g, which are holomorphic outside some specified finite (nonempty) set $S=\{z_1,\ldots z_n\}$ of punctures. Note that the universal central extension of $\mathfrak{g}\otimes_{\mathbb{C}}R$ clearly exists, since $\mathfrak{g}\otimes_{\mathbb{C}}R$ is a perfect Lie algebra:[28] this is easily seen to hold for any simple complex Lie algebra $\mathfrak{g}$ and any commutative ring R with identity.

Theorem 2.1: *The dimension of the kernel Ω_R^1/dR of the universal central extension of $\mathfrak{g}\otimes_{\mathbb{C}}R$ is $2g+n-1$, where g is the genus and n is the number of punctures.*

Proof: Write $\Sigma'=\Sigma-S$. Since $n\geqslant 1$, Σ' is an affine curve, and so Ω_R^1/dR is the first cohomology group of the complex $\{\Omega_i\}$ of meromorphic forms on Σ that are holomorphic on Σ'. Grothendieck's algebraic deRham theorem[29] states that in this case (i.e., when Σ' is an affine variety) the cohomology of this complex is isomorphic to usual complex cohomology (the topological singular cohomology) of Σ'. We can use the Euler characteristic to compute the dimension of the first cohomology group. We have $\chi(\Sigma)=2-2g$, and when n points are removed

$\chi(\Sigma')=2-2g-n$. Let d_i $(i\geqslant 0)$ be the dimension of the ith cohomology group of Σ'. We have $d_0=1$ (since Σ' is connected), $d_2=0$ (since Σ' is noncompact), and $d_i=0$ for $i\geqslant 3$ (since Σ' is a surface). Since $\chi(\Sigma')=\Sigma_{i=0}^{\infty}(-1)^i d_i$ we find $2-2g-n=1-d_1$ and so $d_1=2g+n-1$. □

As a corollary we see that Sheinman's one-dimensional central extension[20,21] is not universal. In this case we have $g=1$, $n=2$, so $2g+n-1=3$; hence the kernel of the universal central extension is in fact 3 dimensional.

III. THE HYPERELLIPTIC CASE: A BASIS FOR THE CENTER

We now restrict consideration to rings R of the form $\mathbb{C}[t,t^{-1},u]$ where $u^2\in\mathbb{C}[t,t^{-1}]$; thus R has a basis consisting of t^i, t^iu for $i\in\mathbb{Z}$. Replacing u by t^iu for some $i\in\mathbb{Z}$ we may assume that $u^2=k(t)\in\mathbb{C}[t]$ and that 0 has multiplicity $\leqslant 1$ as a root of $k(t)$. We write $k(t)=\Sigma_{i=0}^{d}a_it^i$ where $a_d=1$ and a_0, a_1 are not both 0. The equation $u^2=k(t)$ defines a hyperelliptic curve.[30] The ring R has a nontrivial automorphism ρ given by $\rho(t)=t$, $\rho(u)=-u$. If we write R^α $(\alpha=0,\ 1)$ for the $(-1)^\alpha$-eigenspace of ρ, then we see that $R=R^0\oplus R^1$ is a $\mathbb{Z}/2$-graded ring, with $R^0=\mathbb{C}[t,t^{-1}]$ and $R^1=\mathbb{C}[t,t^{-1}]u$.

We call $\mathcal{G}=\mathfrak{g}\otimes R$ an *(untwisted) hyperelliptic loop algebra*. The $\mathbb{Z}/2$-grading on R induces the structure of a $\mathbb{Z}/2$-graded Lie algebra on $\mathcal{G}$ by setting $\mathcal{G}^0=\mathfrak{g}\otimes R^0$, $\mathcal{G}^1=\mathfrak{g}\otimes R^1$. Clearly $\mathcal{G}$ is linearly isomorphic to the direct sum of the ordinary loop algebra $\mathcal{G}^0$ and a copy of its adjoint representation $\mathcal{G}^1$.

We recall the definition of a quasi-graded algebra.[15] Let I be a subgroup of the additive group Q, and let $\mathcal{A}$ be an algebra. For each $i\in I$ let $\mathcal{A}_i$ be a finite-dimensional subspace of $\mathcal{A}$, and assume that $\mathcal{A}=\oplus_{i\in I}\mathcal{A}_i$. Let $l\in I, l\geqslant 0$. If we have

$$x\in\mathcal{A}_i,y\in\mathcal{A}_j\Rightarrow xy\in\bigoplus_{|k-(i+j)|\leqslant l}\mathcal{A}_k,$$

then we call $\mathcal{A}$ an *l-quasigraded algebra*, or a *quasigraded Lie algebra of spread l*. If $l=0$, then we recover the usual definition of a graded algebra. If $x\in\mathcal{A}_i$, $x\neq 0$ then (by a slight abuse of terminology) we say that x is homogeneous of degree i and we write $\deg x=i$.

We set $\deg t^i=i$. We regard u^2 as having quasidegree equal to the average of the highest and lowest degrees appearing in $f(t)$. Thus we set $\deg u=\frac{1}{4}d$ if $f(0)\neq 0$, and $\deg u=\frac{1}{4}(d+1)$ if $f(0)=0$, and so $\deg t^iu=i+\deg u$. This makes R into a quasigraded (commutative associative) algebra of spread $\frac{1}{2}d$ if $f(0)\neq 0$, and $\frac{1}{2}(d-1)$ if $f(0)=0$. We then make $\mathcal{G}$ into a quasigraded Lie algebra with the same I and l by setting $\deg(x\otimes t^i)=\deg t^i$ and $\deg(x\otimes t^iu)=\deg t^iu$ for any $x\in\mathfrak{g}$, $x\neq 0$. (Other quasigradings may be defined geometrically.[25–27])

Proposition 3.1: With the grading given above, the hyperelliptic loop algebra $\mathcal{G}$ is a quasigraded Lie algebra of spread $\frac{1}{2}d$ if $f(0)\neq 0$ and $\frac{1}{2}(d-1)$ if $f(0)=0$.

Proof: Obvious. □

Our goal is to determine a spanning set for Ω_R^1/dR. Let $F=R\otimes R$ be the left R-module with action $f(g\otimes h)=fg\otimes h$ for $f,g,h\in R$. Let K be the submodule generated by the elements $1\otimes fg-f\otimes g-g\otimes f$. Let $\Omega_R^1=F/K$ be the quotient module; this is the *module of (Kähler) differentials* of R. We denote the element $f\otimes g+K$ of Ω_R^1 by fdg. We define a map $d:R\to\Omega_R^1$ by $d(f)=df=1\otimes f+K$. The elements of the subspace $dR=d(R)$ are *exact* differentials. We denote the coset of fdg modulo dR by $\overline{fdg}$. The commutation relations for $\hat{\mathcal{G}}$, the universal central extension of $\mathcal{G}$, are

$$[x\otimes f,y\otimes g]=[xy]\otimes fg+(x,y)\overline{fdg},\quad [x\otimes f,\omega]=0,\quad [\omega,\omega']=0,$$

where x, $y\in\mathfrak{g}$, f, $g\in R$, and ω, $\omega'\in\Omega_R^1/dR$; here (x,y) denotes the Killing form on $\mathfrak{g}$.

The construction of Ω_R^1/dR may be expressed by the composition

$$R\times R\to R\otimes R\to(R\otimes R)/K=\Omega_R^1\to\Omega_R^1/dR.$$

All of these objects have a $\mathbb{Z}/2$-grading induced by that on R. As usual we will call elements of degree 0 *even* and elements of degree 1 *odd.*

The elements $t^i \otimes t^j$, $t^i \otimes t^j u$, $t^i u \otimes t^j$ and $t^i u \otimes t^j u$ form a basis of $R \otimes R$. The next three results are generalizations of previous results.[14] Observe that the assertions $f \otimes g \equiv 0 \pmod K$ and $f dg = 0$ in Ω^1_R are equivalent.

Lemma 3.2: Ω^1_R is spanned by the differentials $t^i\,dt$, $t^i u\,dt$ and $t^i\,du$ for $i \in \mathbb{Z}$.

Proof: We have to show that any basis element of $R \otimes R$ is congruent modulo K to an element in the span of $t^i \otimes t$, $t^i \otimes u$, and $t^i u \otimes t$.

We easily show by induction that $d(t^j) = jt^{j-1}\,dt$ and that $d(t^j u) = jt^{j-1}\,dt + t^j\,du$. Since K is a submodule of $R \otimes R$, we can multiply each of these equations by t^i or $t^i u$. This shows that any basis element of $R \otimes R$ reduces to an element in the span of $t^i\,dt$, $t^i\,du$, $t^i u\,dt$ and $t^i u\,du$.

To complete the proof, we use the fact that $u\,du = \frac{1}{2} d(u^2)$. Since $u^2 = t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + a_0$ we find that $u\,du = \frac{1}{2} dt^{d-1} dt + \frac{1}{2}(d-1)a_{d-1}t^{d-2}\,dt + \cdots + \frac{1}{2}a_1\,dt$. We now multiply this equation by t^i to get the result. □

Lemma 3.3: Ω^1_R is spanned by the differentials $t^i\,dt$ and $t^i u\,dt$ for $i \in \mathbb{Z}$, together with $t^{d-1}du,\ldots,t\,du$, du (where we omit du if $a_0=0$).

Proof: We have $\frac{1}{2}ud(u^2) - u\,du = 0$. Since $u^2 = \Sigma_{k=0}^d a_k t^k$ we find that

$$\sum_{k=1}^{d} \frac{1}{2} k a_k t^{k-1} u\,dt - \sum_{k=0}^{d} a_k t^k\,du = 0.$$

We multiply this equation by t^i to get

$$\sum_{k=1}^{d} \frac{1}{2} k a_k t^{i+k-1} u\,dt - \sum_{k=0}^{d} a_k t^{i+k}\,du = 0. \tag{3.3.1}$$

First assume that $a_0 \neq 0$. For $i \geq 0$, formula (3.3.1) shows (since $a_d = 1$) that $t^{i+d}\,du$ is equal to a linear combination of $t^{i+d-1}\,du,\ldots,t^i\,du$ and elements of the form $t^j u\,dt$. For $i \leq -1$ it shows (since $a_0 \neq 0$) that $t^i\,du$ is equal to a linear combination of $t^{i+1}\,du,\ldots,t^{i+d}\,du$ and elements of the form $t^j u\,dt$. From this we easily show by induction that any element of the form $t^i\,du$ is equal to a linear combination of $du,\ldots,t^{d-1}du$ and elements of the form $t^j u\,dt$. Now Lemma 3.2 completes the proof.

If $a_0 = 0$ then for $i \geq 0$, $t^{i+d}\,du$ is equal to a linear combination of $t^{i+d-1}\,du,\ldots,t^{i+1}\,du$ and elements of the form $t^j u\,dt$, and for $i \leq -1$, since $a_1 \neq 0$, $t^{i+1}\,du$ is equal to a linear combination of $t^{i+2}\,du,\ldots,t^{i+d}\,du$ and elements of the form $t^j u\,dt$. The rest of the argument is similar. □

Theorem 3.4: *A basis of Ω^1_R/dR is given by $\overline{t^{-1}\,dt}$ together with $\overline{t^{-1}u\,dt},\ldots,\overline{t^{-d}u\,dt}$ (where we omit $\overline{t^{-d}u\,dt}$ if $a_0=0$).*

Proof: We first show that the listed elements span Ω^1_R/dR. The $\mathbb{Z}/2$-grading of Ω^1_R and dR gives $\Omega^1_R/dR = (\Omega^1_R)^0/d(R^0) \oplus (\Omega^1_R)^1/d(R^1)$.

We first consider the even subspace. We have $d(t^i) = it^{i-1}\,dt$ for all $i \in \mathbb{Z}$. From this we immediately see that $t^{i-1}\,dt \equiv 0 \pmod{dR}$ for $i \neq 0$. Therefore $(\Omega^1_R)^0/d(R^0)$ is spanned by $\overline{t^{-1}\,dt}$.

Next, we consider the odd subspace. By Lemma 3.3, the odd subspace is spanned by $t^i u\,dt$ together with $t^{d-1}\,du,\ldots,t\,du$ (and du if $a_0 \neq 0$). We have $d(t^i u) = it^{i-1}u\,dt + t^i\,du$, and so

$$t^i\,du \equiv -it^{i-1}u\,dt \pmod{dR}. \tag{3.4.1}$$

Thus we only need to consider the elements $t^i u\, dt$. We will show that modulo dR each of these elements is congruent to a linear combination of the finite set listed in the statement of Theorem 3.4.

First suppose that $a_0 \neq 0$. We have $t^{i-1}u\, dt \equiv -(1/i)t^i\, du \pmod{dR}$ for $i\neq 0$. By formula (3.3.1) we know that $t^i\, du$ is a linear combination of $t^{i+1}\, du,\ldots,t^{i+d}\, du$ and $t^i u\, dt,\ldots,t^{i+d-1}u\, dt$. Using (3.4.1) we see that $t^i\, du$, and hence also $t^{i-1}u\, dt$, is congruent modulo dR to an element in the span of $t^i u\, dt,\ldots,t^{i+d-1}u\, dt$. Now using induction, we see that for $i \leq -d-1$, the element $t^{i-1}u\, dt$ is congruent modulo dR to a linear combination of $t^{-d}u\, dt,\ldots,t^{-1}u\, dt$.

We also have $t^{i+d-1}u\, dt \equiv -[1/(i+d)]t^{i+d}\, du \pmod{dR}$ for $i \neq -d$. By formula (3.3.1) again we know that $t^{i+d}\, du$ is a linear combination of $t^{i+d-1}\, du,\ldots,t^i\, du$ and $t^{i+d-1}u\, dt,\ldots,t^i u\, dt$. The coefficient of $t^{i+d-1}u\, dt$ in this linear combination is $\frac{1}{2}d$; hence we can solve for $t^{i+d-1}u\, dt$ (assuming that $i\neq -\frac{3}{2}d$), showing that it is congruent modulo dR to a linear combination of the same elements (excluding $t^{i+d-1}u\, dt$). By (3.4.1) we see that $t^{i+d-1}u\, dt$ is congruent modulo dR to an element in the span of $t^{i+d-2}u\, dt,\ldots,t^{i-1}u\, dt$. Now setting $j=i+d-1$ and using induction, we see that for $j\geq 0$ (that is $i\geq -d+1$), the element $t^j u\, dt$ is congruent modulo dR to a linear combination of $t^{-1}u\, dt,\ldots,t^{-d}u\, dt$.

The proof in the case $a_0=0$ (and $a_1\neq 0$) is similar.

We now show that the listed elements are linearly independent. The ring R is the coordinate ring of a curve Σ' of genus g with n points removed. The genus g depends on the defining polynomial $k(t)$ according to the formula

$$g=\left[\frac{d-1}{2}\right], \quad \text{equivalently} \quad 2g=\begin{cases} d-2 & \text{if } d \text{ is even} \\ d-1 & \text{if } d \text{ is odd,}\end{cases}$$

where d is the degree of $k(t)$ and $[x]$ denotes the integral part of x. The number n of points where poles are allowed depends on $k(t)$ according to the formula $n=4-r$, where r is the number of ramified points in $\{0,\infty\}$: 0 is ramified exactly when the constant term $a_0=0$, and ∞ is ramified exactly when the degree d is odd. Combining this information gives

$$2g+n-1=\begin{cases} d+1 & \text{if } a_0\neq 0 \\ d & \text{if } a_0=0.\end{cases}$$

Therefore, the number of elements listed in the statement of Theorem 3.4 equals the dimension of Ω_R^1/dR (using Theorem 2.1). □

IV. THE ELLIPTIC CASE: EXPLICIT COCYCLES

Let Σ be a nonsingular compact complex algebraic curve of genus 1. We represent Σ as the quotient of the complex plane $\mathbb{C}$ by the lattice $\Lambda=\mathbb{Z}\oplus\mathbb{Z}\lambda$ with basis $\{1,\lambda\}$ where $\operatorname{Im}\lambda>0$. For convenience we identify an element $z\in\mathbb{C}$ and the corresponding point $z+\Lambda$ on Σ. We let R denote the ring of all meromorphic functions on Σ which are holomorphic outside the set $\{0,\mu\}$ where $\mu=\frac{1}{2}(1+\lambda)$; then R is a ring of elliptic functions.[31]

Let $\wp(z)$ denote the Weierstrass $\wp$ function for the lattice Λ, and set $m=\wp(\mu)$. Now $\wp(z)-m$ has a double pole at $z=0$, a double zero at $z=\mu$ [since $\wp'(\mu)=0$], and no other zeros or poles; and $\wp'(z)$ has a triple pole at $z=0$. Therefore, R contains the functions $(\wp(z)-m)^n$ and $(\wp(z)-m)^n\wp'(z)$ for all $n\in\mathbb{Z}$, and a simple application of the Riemann–Roch Theorem[30] shows that these functions form a basis of R over $\mathbb{C}$. (A similar calculation can be done for vector fields.[22])

The $\wp$-function satisfies the differential equation

$$\wp'(z)^2=4\wp(z)^3-g_2\wp(z)-g_3, \quad g_2=60\sum_{\xi\in\Lambda-\{0\}}\xi^{-4}, \quad g_3=140\sum_{\xi\in\Lambda-\{0\}}\xi^{-6}.$$

Proposition 4.1: We have the isomorphism $R \simeq \mathbb{C}[t,t^{-1},u]$ *where* $u^2=t^3-2bt^2+t$ *and* $b=-6m/(12m^2-g_2)^2$.

Proof: If we rewrite the right side of the differential equation in powers of $\wp(z)-m$ we obtain

$$\wp'(z)^2=4(\wp(z)-m)^3+12m(\wp(z)-m)^2+(12m^2-g_2)(\wp(z)-m),$$

since $4m^3-g_2m-g_3=\wp(\mu)=0$. Setting

$$T(z)=\tfrac{1}{4}(12m^2-g_2)^2(\wp(z)-m), \quad U(z)=\tfrac{1}{4}(12m^2-g_2)^3\wp'(z),$$

we obtain

$$U(z)^2=T(z)^3-2bT(z)^2+T(z),$$

where b is given above. Now identifying t,u with $T(z),U(z)$ we get the result. The nonsingularity of the curve guarantees that $12m^2-g_2\neq 0$. □

To make the commutation relations for $\hat{\mathscr{G}}$ explicit we need to compute $\overline{f\,dg}$ for any basis elements $f,g\in R$. We deal separately with the two cases fg even and fg odd. Note that $\overline{f\,dg}$ is always the linear combination of the spanning elements for Ω_R^1/dR which gives the congruence class of $f\,dg$ modulo dR. By Theorem 3.4 we know that the elements $\overline{t^{-1}\,dt},\overline{t^{-1}u\,dt},\overline{t^{-2}u\,dt}$ are a basis.

Proposition 4.2: (The even case). For $i,j\in\mathbb{Z}$ *we have*

$$t^i d(t^j)\equiv\begin{cases} jt^{-1}dt, & \text{for } i+j=0\\ 0, & \text{for } i+j\neq 0\end{cases} \quad (\text{mod } dR),$$

and

$$t^{i-1}ud(t^{j-1}u)\equiv\begin{cases} \left(j+\frac{1}{2}\right)t^{-1}dt, & \text{for } i+j=-1\\ -2jbt^{-1}dt, & \text{for } i+j=0\\ \left(j-\frac{1}{2}\right)t^{-1}dt, & \text{for } i+j=1\\ 0, & \text{for } |i+j|\geqslant 2\end{cases} \quad (\text{mod } dR).$$

Proof: The first congruence follows from the relation $t^id(t^j)=jt^{i+j-1}\,dt$. The relation

$$t^{i-1}ud(t^{j-1}u)=(j+\tfrac{1}{2})t^{i+j}dt-2jbt^{i+j-1}\,dt+(j-\tfrac{1}{2})t^{i+j-2}\,dt,$$

implies the second congruence. □

We next consider the odd case. We define two sequences of polynomials $p_k(b)$, $q_k(b)$ for $k\in\mathbb{Z}$ by

$$\overline{t^{k-2}u\,dt}=p_k(b)\overline{t^{-1}u\,dt}+q_k(b)\overline{t^{-2}u\,dt}.$$

The 4-parameter Pollaczek polynomials $P_k(b)=P_k^\lambda(b;\alpha,\beta,\gamma)$ satisfy the recursion formula[19]

$$(k+\gamma)P_k(b)=2[(k+\lambda+\alpha+\gamma-1)b+\beta]P_{k-1}(b)-(k+2\lambda+\gamma-2)P_{k-2}(b). \quad (4.3)$$

Lemma 4.4: The polynomials $p_k(b),q_k(b)$ *are Pollaczek polynomials for the parameter values* $\lambda=-1/2$, $\alpha=0$, $\beta=-1$, $\gamma=1/2$. *The initial conditions are*

$$p_0(b)=0, \quad p_1(b)=1, \quad q_0(b)=1, \quad q_1(b)=0.$$

Proof: We make explicit the recursion formula implicit in the inductive argument in the proof of Theorem 3.4. With $u^2=t^3-2bt+t$, formula (3.3.1) becomes

$$\tfrac{1}{2}t^iu\,dt-2bt^{i+1}u\,dt+\tfrac{3}{2}t^{i+2}u\,dt-t^{i+1}\,du+2bt^{i+2}\,du-t^{i+3}\,du=0,$$

which holds in Ω_R^1. Using formula (3.4.1) this equation reduces to

$$(i+\tfrac{3}{2})\overline{t^iu\,dt}-2b(i+3)\overline{t^{i+1}u\,dt}+(i+\tfrac{9}{2})\overline{t^{i+2}u\,dt}=0, \tag{4.4.1}$$

which holds in Ω_R^1/dR. Now take $i=k-4$ and rearrange to obtain

$$(k+\tfrac{1}{2})r_k(b)=2(k-1)br_{k-1}(b)-(k-\tfrac{5}{2})r_{k-2}(b),$$

for $r=p$ or q. Comparing this with Eq. (4.3) we find the given values of the parameters. The initial conditions are immediate from the definition of $p_k(b)$ and $q_k(b)$. □

Since t^3-2bt^2+t is a symmetric polynomial, we can define an automorphism σ of R by $\sigma(t)=t^{-1}$, $\sigma(u)=t^{-2}u$. On Ω_R^1/dR, σ acts as the negative of the identity. To see this, observe that $\sigma(\overline{t^{-1}\,dt}) = \overline{td(t^{-1})} = \overline{t(-t^{-2})dt} = -\overline{t^{-1}\,dt}$. The calculations for $\overline{t^{-1}u\,dt}$ and $\overline{t^{-2}u\,dt}$ are similar. [For the former we need the equation $\overline{t^{-3}u\,dt} = \overline{t^{-1}u\,dt}$, which follows from (4.4.1) with $i=-3$.] Applying the automorphism σ we find that $p_k=p_{-k}$ and $q_k=q_{-k}$.

Proposition 4.5: (The odd case). For $i,j\in\mathbb{Z}$ we have

$$t^{i-1}ud(t^j)\equiv j(p_{|i+j|}(b)t^{-1}u\,dt+q_{|i+j|}(b)t^{-2}u\,dt) \quad (\text{mod } dR).$$

Proof: Apply the definition of p_k and q_k to the relation $t^{i-1}ud(t^j)=jt^{i+j-2}u\,dt$. □

We can now give explicit commutation relations for $\hat{\mathcal{G}}$. We write $\omega_0 = \overline{t^{-1}\,dt}$, $\omega_+ = \overline{t^{-1}u\,dt}$ and $\omega_- = \overline{t^{-2}u\,dt}$.

Theorem 4.6: *The elliptic affine Lie algebra $\hat{\mathcal{G}}$ has a $\mathbb{Z}/2$-grading in which*

$$\hat{\mathcal{G}}^0=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}\omega_0, \quad \hat{\mathcal{G}}^1=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]u\oplus\mathbb{C}\omega_+\oplus\mathbb{C}\omega_-.$$

A spanning subset of $\hat{\mathcal{G}}$ consists of the elements $x\otimes t^i$ and $x\otimes t^{j-1}u$ where $x\in\mathfrak{g}, i,j\in\mathbb{Z}$, together with the central elements ω_0, ω_+ and ω_-. The even subalgebra $\hat{\mathcal{G}}^0$ is an untwisted affine Kac–Moody Lie algebra with commutation relations

$$[x\otimes t^i,y\otimes t^j]=[xy]\otimes t^{i+j}+\delta_{i+j,0}(x,y)j\omega_0.$$

The commutator of two elements of $\hat{\mathcal{G}}^1$ lies in $\hat{\mathcal{G}}^0$:

$$[x\otimes t^{i-1}u,y\otimes t^{j-1}u]=[xy]\otimes(t^{i+j-1}-2bt^{i+j}+t^{i+j+1})+(x,y)\begin{cases}-2jb\omega_0, & \text{for } i+j=0\\ \tfrac{1}{2}(j-i)\omega_0, & \text{for } |i+j|=1\\ 0, & \text{for } |i+j|\geqslant 2.\end{cases}$$

The odd subspace $\hat{\mathcal{G}}^1$ is a $\hat{\mathcal{G}}^0$-module with relations

$$[x\otimes t^{i-1}u,y\otimes t^j]=[xy]\otimes t^{i+j-1}u+(x,y)j(p_{|i+j|}(b)\omega_++q_{|i+j|}(b)\omega_-).$$

Proof: Since the $\mathbb{Z}/2$-grading on R induces a $\mathbb{Z}/2$-grading on both $\mathcal{G}$ and Ω_R^1/dR, the Lie algebra $\hat{\mathcal{G}}$ also has a $\mathbb{Z}/2$-grading. The other assertions are clear. □

Note that $\hat{\mathcal{G}}$ is not a 1-quasigraded Lie algebra; to get a 1-quasigraded Lie algebra we must factor out the central ideal spanned by the odd central elements ω_+ and ω_-. The reason for this is that the cocycle connected to ω_0 is local with respect to the quasigrading on $\mathcal{G}$, but the cocycles connected to ω_- and ω_+ are not. (The construction of Ω_R^1/dR respects the grading on the subring

$\mathbb{C}[t,t^{-1}]$ of R but not the 1-quasigrading of R.) Thus it makes sense to set deg $\omega_0=0$, but there is no choice of values for deg ω_+ and deg ω_- consistent with the quasigrading of $\mathscr{G}$; locality of the cocycles is necessary[26] for the cocycles to extend to $\hat{\mathscr{G}}$.

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