

Structure of the Lie algebra of polynomial vector fields on the Riemann sphere with three punctures

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The Lie algebra of polynomial vector fields on the sphere with two punctures (also known as the Witt algebra or the centerless Virasoro algebra) can be generalized in different ways. In this paper the Lie algebra obtained by allowing poles at three points rather than only two is considered. Using some elementary representation theory of the symmetric group on three letters, a convenient basis for this Lie algebra is determined, such that the commutator of two basis elements is a linear combination of at most three other basis elements. The expressions for the basis elements show that this Lie algebra is related to modular forms for a level-2 congruence subgroup of $SL_2(\mathbf{Z})$. A conclusion is given with some suggestions for further investigation.

I. INTRODUCTION

In recent years much effort has been devoted to studying the representation theory of the Virasoro algebra, that is, the universal central extension of the Witt algebra, which is the Lie algebra of polynomial vector fields on the Riemann sphere with two punctures. (Feigin and Fuchs¹ provide the most complete survey to date of this subject.) Numerous authors have sought for natural generalizations of the Virasoro algebra, for example, superconformal algebras,^{2,3} infinitely generated Lie algebras related to Casimir operators for simple Lie algebras,^{4,5} and Krichever–Novikov algebras.^{6,7} The last-named algebras generalize the Witt algebra by considering Lie algebras of holomorphic vector fields on Riemann surfaces of positive genus with two punctures. One may also consider a Riemann surface of arbitrary genus with more than two punctures.⁸ A basic problem in this theory is to find a basis for the Lie algebra such that there is a uniform bound on the number of terms arising in any commutator of basis elements. In this paper we solve this problem for the case of the sphere with three points removed by using some elementary representation theory of the symmetric group S_3 permuting the three points.

II. THE LIE ALGEBRA \mathcal{S}

Let Σ denote the Riemann sphere. We remove three points from Σ and choose a coordinate λ on Σ such that one of the punctures has coordinate $\lambda = \infty$. Then we can identify Σ with the twice-punctured complex plane $\mathbf{C} \setminus \{\alpha, \beta\}$. By the ring of polynomial functions on Σ we mean the commutative ring

$$P_{\alpha, \beta} = \mathbf{C}[\lambda, (\lambda - \alpha)^{-1}, (\lambda - \beta)^{-1}].$$

It suffices to consider the case $\alpha = 0, \beta = 1$ since an isomorphism between $P_{0,1}$ and $P_{\alpha, \beta}$ is given by $\lambda \mapsto (\lambda - \alpha)/(\beta - \alpha)$.

We now write Σ' for the noncompact genus-zero Riemann surface $\mathbf{C} \setminus \{0, 1\}$, and P for $P_{0,1}$. By the Lie algebra of polynomial vector fields on Σ' we mean the derivation algebra $\mathcal{S} = \text{Der } P$. One easily checks that the set $\{\lambda^{m+1}\partial_\lambda \mid m \in \mathbf{Z}\} \cup \{(1-\lambda)^{-m}\partial_\lambda \mid m \geq 1\}$ is a basis for \mathcal{S} (writing $\partial_\lambda = d/d\lambda$). This basis has a convenient property,

that it extends the basis $\{\lambda^{m+1}\partial_\lambda \mid m \in \mathbf{Z}\}$ of the Witt algebra, and a very inconvenient property, that the commutator of $\lambda^{m+1}\partial_\lambda$ with $(1-\lambda)^{-n}\partial_\lambda$ can have an arbitrarily large number of terms.⁸ In the next section we show how to find a basis such that commutators of basis elements are linear combinations of at most three basis elements.

III. THE ACTION OF THE SYMMETRIC GROUP S_3 ON $\Sigma, P,$ AND \mathcal{S}

The group S_3 is generated by a transposition and a three-cycle. We think of the transposition as interchanging 0 and ∞ on Σ , and write $\sigma = (0\infty) = 1/\lambda$; similarly we write $\tau = (01\infty) = 1/(1-\lambda)$. In this way S_3 acts as a group of automorphisms of Σ and Σ' . These automorphisms can be directly interpreted as automorphisms of P , and then as automorphisms of \mathcal{S} [using the chain rule $\partial_{f(\lambda)} = (1/f'(\lambda))\partial_\lambda$]. In this way both P and \mathcal{S} become S_3 -modules. In Table I we list the elements of S_3 with the corresponding permutations, and their action on λ and ∂_λ .

The idea now is to find elements of P and \mathcal{S} that are basis elements for irreducible S_3 -submodules of P and \mathcal{S} . Recall that S_3 has three inequivalent irreducible finite-dimensional representations:⁹ The one-dimensional trivial module M_0 with $\sigma = \tau = 1$; the one-dimensional signature module M_1 with $\sigma = -1, \tau = 1$; and the two-dimensional module M_2 with

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix},$$

TABLE I. Action of S_3 on $\Sigma, P,$ and \mathcal{S} .

Element	Permutation	Action on λ	Action on ∂_λ
1	identity	λ	∂_λ
τ	(01∞)	$1/(1-\lambda)$	$(1-\lambda)^2\partial_\lambda$
τ^2	$(0\infty 1)$	$(\lambda-1)/\lambda$	$\lambda^2\partial_\lambda$
σ	(0∞)	$1/\lambda$	$-\lambda^2\partial_\lambda$
$\tau\sigma$	(01)	$1-\lambda$	$-\partial_\lambda$
$\tau^2\sigma$	(1∞)	$\lambda/(\lambda-1)$	$-(1-\lambda)^2\partial_\lambda$

where ω is a primitive cube root of 1. To find a basis of M_1 in P one computes

$$\begin{aligned} \rho &:= \frac{1}{3} (1 + \tau + \tau^2 - \sigma - \tau\sigma - \tau^2\sigma) \cdot \lambda \\ &= \frac{1}{3} \frac{(\lambda - 2)(\lambda - \frac{1}{2})(\lambda + 1)}{\lambda(\lambda - 1)}. \end{aligned}$$

Thus the even (odd) powers of ρ span S_3 -submodules of P isomorphic to M_0 (M_1). This turns out to be all the information we need about P as an S_3 -module. However, we also need some information about \mathcal{S} as an S_3 -module. Notice first that the elements $\partial_\lambda, \lambda\partial_\lambda, \lambda^2\partial_\lambda$ span a Lie subalgebra of \mathcal{S} [isomorphic to $\mathfrak{sl}_2(\mathbb{C})$] in which every vector field is holomorphic on Σ . One easily checks that this subalgebra is S_3 -invariant; in fact it contains one copy each of $M_1 = \mathbb{C}m'_0$ and $M_2 = \mathbb{C}p'_0 \oplus \mathbb{C}q'_0$, where

$$m'_0 := (\lambda^2 - \lambda + 1)\partial_\lambda,$$

$$p'_0 := (\omega\lambda^2 + 2\lambda + \omega^2)\partial_\lambda, \quad q'_0 := (\omega^2\lambda^2 + 2\lambda + \omega)\partial_\lambda.$$

IV. COMMUTATION RELATIONS FOR THE NEW BASIS

One now defines $m'_a := \rho^a m'_0$, $p'_a := \rho^a p'_0$, and $q'_a := \rho^a q'_0$, where $a \geq 0$ in each case. These vector fields form a basis of \mathcal{S} ; this is a simple corollary of the Riemann-Roch theorem, as follows. For meromorphic functions on Σ , the theorem says that the dimension of the space of functions with divisor $\text{div } f \geq -D$ for $D = \sum n_i [p_i]$ is $\deg D + 1 = \sum n_i + 1$, as long as $\deg D \geq 0$. For vector fields $f\partial_\lambda$, since ∂_λ has a double zero at ∞ (if $\lambda = 1/\mu$ then $\partial_\lambda = -\mu^2\partial_\mu$), the dimension is $\deg D + 3$. Thus there are three holomorphic vector fields; another three with at most simple poles at 0, 1, ∞ ; another three with at most double poles; etc. Now $m'_0, p'_0, q'_0, \dots, m'_a, p'_a, q'_a$ is a list of $3a + 3$ vector fields with poles of order at most a at 0, 1, ∞ . So it suffices to show linear independence, but that is straightforward.

It is now a simple calculation to obtain the commutation relations. To avoid irrelevant scalars in the relations we first define $m_a := \gamma^{a+1} m'_a$, $p_a := \gamma^{a+1} p'_a$ and $q_a := \gamma^{a+1} q'_a$, where $\gamma := -2i/3\sqrt{3}$. We then have the following theorem.

Theorem: The vector fields m_a, p_a and q_a for $a \geq 0$ form a basis of the Lie algebra \mathcal{S} . The commutation relations are as follows:

$$\begin{aligned} [m_a, m_b] &= (b-a)(m_{a+b+1} - m_{a+b-1}), \\ [m_a, p_b] &= 2p_{a+b} + (b-a)(p_{a+b+1} - p_{a+b-1}), \\ [m_a, q_b] &= -2q_{a+b} + (b-a) \\ &\quad \times (q_{a+b+1} - q_{a+b-1}), \\ [p_a, p_b] &= (b-a)(q_{a+b+1} - 2q_{a+b} + q_{a+b-1}), \\ [q_a, q_b] &= (b-a)(p_{a+b+1} + 2p_{a+b} + p_{a+b-1}), \\ [p_a, q_b] &= -4m_{a+b} + (b-a) \\ &\quad \times (m_{a+b+1} - m_{a+b-1}). \end{aligned}$$

V. RELATION WITH MODULAR FUNCTIONS OF LEVEL 2

The reason for using the notation λ for the coordinate on Σ is as follows. Let $\Gamma(2)$ denote the usual congruence subgroup¹⁰ of level 2 of $\text{SL}_2(\mathbb{Z})$, and let $X(2) := \Gamma(2) \backslash \mathcal{H}$ denote the corresponding modular curve, where \mathcal{H} is the upper half-plane. Let z be the coordinate on \mathcal{H} , and hence on $X(2)$. The field of modular functions for $\Gamma(2)$ [equivalently, the function field of $X(2)$] has form $\mathbb{C}(\lambda)$ where $\lambda = \lambda(z)$ is the classical λ function. On a fundamental domain for $\Gamma(2)$, $\lambda(z)$ takes every complex value exactly once, omitting only 0, 1 and ∞ ; thus $\lambda: X(2) \rightarrow \Sigma'$ is an isomorphism. The classical modular function J for the full modular group $\text{SL}_2(\mathbb{Z})$ is related to λ by the formula

$$J = \frac{4}{27} \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2} = \frac{4}{27} \rho^2 + 1.$$

VI. DIRECTIONS FOR FURTHER WORK

In order to generalize the Virasoro algebra to this situation, one needs to determine the universal central extension \mathcal{R} of the Lie algebra \mathcal{S} by calculating $H^2(\mathcal{S}, \mathbb{C})$. The resulting cocycle(s) might have a simple expression in terms of the basis introduced in this paper. It would also be interesting to find formulas for the standard Witt algebra basis elements $l_m = \lambda^{m+1}\partial_\lambda$ in terms of m_a, p_a, q_a . Preliminary computer calculations suggest (unfortunately) that one needs arbitrarily large linear combinations of the m_a, p_a, q_a to do this.

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