The partially alternating ternary sum in an associative dialgebra

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Abstract. The alternating ternary sum in an associative algebra,

$$abc - acb - bac + bca + cab - cba,$$

gives rise to the partially alternating ternary sum in an associative dialgebra with products $\dashv$ and $\vdash$ by making the argument a the center of each term:

$$a \dashv b \dashv c - a \dashv c \dashv b - b \vdash a \dashv c + c \vdash a \dashv b + b \vdash c \vdash a - c \vdash b \vdash a.$$

We use computer algebra to determine the polynomial identities in degree ≤ 9 satisfied by this new trilinear operation. In degrees 3 and 5 we obtain

$$[a, b, c] + [a, c, b] \equiv 0, \quad [a, [b, c, d], e] + [a, [c, b, d], e] \equiv 0;$$

these identities define a new variety of partially alternating ternary algebras. We show that there is a 49-dimensional space of multilinear identities in degree 7, and we find equivalent nonlinear identities. We use the representation theory of the symmetric group to show that there are no new identities in degree 9.

1. Introduction

The Lie bracket $[a, b] = ab - ba$ gives rise to two distinct ternary operations: the Lie triple product $[[a, b], c] = abc - bac - cab + cba$, and the alternating ternary sum (ATS) $[a, b, c] = abc - acb - bac + bca + cab - cab$, also called the ternary commutator [3] or 'ternutator' [12]. Apart from the obvious skew-symmetry in degree 3, the simplest non-trivial identities for the ATS have degree 7 and were found by Bremner [3] using computer algebra. Subsequent work of Bremner and Hentzel showed that there are no new identities for the ATS in degree 9. (An identity is 'new' if it is not a consequence of identities in lower degree.) The ATS is closely related to the $n = 3$ case of n-Lie algebras introduced by Filippov [13].

The ATS appears in physics in the context of a novel formulation of quantum mechanics developed by Nambu [21]. He introduced a multilinear n-bracket (now called the Nambu n-bracket) which becomes the ATS for $n = 3$. This theory has been developed by Takhtajan [23], Gautheron [14], Curtright and Zachos [10], and Ataguema et al. [1], among many others. Recently, Curtright et al. [9] have generalized the identity of Bremner [3] to all odd n. However, the most important recent developments in physics related to n-ary algebras are the works of Bagger and

Lambert [2] and Gustavsson [15], which aim at a world-volume theory of multiple M2-branes. For a very recent comprehensive survey of this entire area, from both the physical and mathematical points of view, see de Azcárraga and Izquierdo [11].

Motivated by the importance of the ATS in theoretical physics, as well as the recent works of Bremner and Peresi [4, 7] on polynomial identities satisfied by the quasi-Jordan product in an associative dialgebra, the present paper will focus on the partially alternating ternary sum in an associative dialgebra. We use computational linear algebra and the representation theory of the symmetric group to determine the polynomial identities in degree ≤ 9 satisfied by this new trilinear operation.

2. Preliminaries on dialgebras

Unless otherwise stated, the base field F is the field $\mathbb{Q}$ of rational numbers.

2.1. Dialgebras and Leibniz algebras

Dialgebras were introduced by Loday [18, 19, 20] to provide a natural setting for Leibniz algebras, a 'noncommutative' generalization of Lie algebras.

Definition 2.1. (Loday [18].) A *Leibniz algebra* is a vector space L together with a bilinear map $L \times L \to L$, denoted $(a, b) \mapsto [a, b]$ and called the *Leibniz bracket*, satisfying the *Leibniz identity* which says that right multiplications are derivations:

$$[[a, b], c] \equiv [[a, c], b] + [a, [b, c]].$$

If $[a, a] \equiv 0$ then the Leibniz identity is the Jacobi identity and L is a Lie algebra.

Every associative algebra becomes a Lie algebra if the associative product is replaced by the Lie bracket. Loday introduced the notion of dialgebra which gives, by a similar procedure, a Leibniz algebra: one replaces ab and ba by two distinct operations, so that the resulting bracket is not necessarily skew-symmetric.

Definition 2.2. (Loday [19].) An *(associative) dialgebra* is a vector space A together with two bilinear maps $A \times A \to A$, denoted $a \dashv b$ and $a \vdash b$ and called the *left* and *right* products, satisfying the following identities:

$$(a \dashv b) \dashv c \equiv a \dashv (b \dashv c), \quad (a \vdash b) \vdash c \equiv a \vdash (b \vdash c), \quad (a \vdash b) \dashv c \equiv a \vdash (b \dashv c),$$
$$(a \dashv b) \vdash c \equiv (a \vdash b) \vdash c, \quad a \dashv (b \dashv c) \equiv a \dashv (b \vdash c).$$

Both products are associative, they satisfy *inner associativity* (with the operation symbols pointing inwards), and the *bar properties* (on the bar side, the operation symbols are interchangeable). The *dicommutator* is defined by $[a, b] = a \dashv b - b \vdash a$.

Every dialgebra becomes a Leibniz algebra if the associative products are replaced by the dicommutator.

2.2. Free dialgebras

Definition 2.3. (Loday [20].) A *(dialgebra) monomial* on a set X is a product $x = a_1 a_2 \cdots a_n$ where $a_1, \ldots, a_n \in X$ with some placement of parentheses and some choice of operations. The *center* of x is defined inductively: if $n = 1$ then $c(x) = x$; if $n \geq 2$ then $x = y \dashv z$ or $x = y \vdash z$ and we set $c(y \dashv z) = c(y)$ or $c(y \vdash z) = c(z)$.

A monomial is determined by the order of its factors and the position of its center.

Lemma 2.4. (Loday [20].) *If $x = a_1 a_2 \cdots a_n$ is a monomial with $c(x) = a_i$ then*

$$x = (a_1 \vdash \cdots \vdash a_{i-1}) \vdash a_i \dashv (a_{i+1} \dashv \cdots \dashv a_n).$$

Definition 2.5. The right side of the last equation is the *normal form* of x and is abbreviated by the *hat notation* $a_1 \cdots a_{i-1} \widehat{a}_i a_{i+1} \cdots a_n$.

Lemma 2.6. (Loday [20].) *The set of monomials $a_1 \cdots a_{i-1} \widehat{a}_i a_{i+1} \cdots a_n$ in normal form with $1 \le i \le n$ and $a_1, \dots, a_n \in X$ forms a basis of the free dialgebra on X.*

2.3. Identities for algebras and identities for dialgebras

Kolesnikov [17] and Pozhidaev [22] recently introduced an algorithm for passing from identities of algebras to identities of dialgebras; we do not assume associativity. Let I be a multilinear algebra identity in the variables $a_1, \dots, a_n$. For each $i = 1, \dots, n$ we convert I into a multilinear dialgebra identity by making a_i the center of each monomial. In this way, one algebra identity in degree n produces n dialgebra identities. For example, associativity $(ab)c - a(bc)$ gives rise to $(\widehat{a}b)c - \widehat{a}(bc)$, $(a\widehat{b})c - a(\widehat{b}c)$, $(ab)\widehat{c} - a(b\widehat{c})$: associativity of the left product, inner associativity, and associativity of the right product.

The same algorithm can be used to convert a multilinear algebra operation into a set of multilinear dialgebra operations. For example, the Lie bracket $ab - ba$ gives rise to the left and right Leibniz products $a \dashv b - b \vdash a$ and $a \vdash b - b \dashv a$. We use this method to obtain ternary dialgebra operations from ternary algebra operations.

3. Degree 3

The Kolesnikov-Pozhidaev algorithm produces the dialgebra version of the ATS.

Definition 3.1. The *partially alternating ternary sum* (*PATS*) is this trilinear operation in an associative dialgebra:

$$[a, b, c] = \widehat{a}bc - \widehat{a}cb - b\widehat{a}c + c\widehat{a}b + bc\widehat{a} - cb\widehat{a}.$$

It is easy to see that the PATS is skew-symmetric in its second and third arguments; Proposition 4.3 shows that it becomes completely alternating in the second and third arguments of a nested monomial. The PATS is obtained from the ATS by making a the center of each monomial; the skew-symmetry of the ATS implies that making b or c the center gives equivalent operations.

Definition 3.2. A *ternary algebra* is a vector space T together with a trilinear map $T \times T \times T \to T$ denoted by (a, b, c). In a ternary algebra we define the polynomial $P(a, b, c) = (a, b, c) + (a, c, b)$.

Proposition 3.3. *Every multilinear polynomial identity in degree ≤ 3 satisfied by the PATS is a consequence of $P(a, b, c) \equiv 0$.*

Proof. Consider the general trilinear identity of degree 3:

$$x_1[a, b, c] + x_2[a, c, b] + x_3[b, a, c] + x_4[b, c, a] + x_5[c, a, b] + x_6[c, b, a] = 0.$$

Each of the 6 ternary monomials expands using the PATS into a linear combination of 6 dialgebra monomials. Altogether we obtain 18 dialgebra monomials:

$$\widehat{a}bc, \widehat{a}cb, \widehat{b}ac, \widehat{b}ca, \widehat{c}ab, \widehat{c}ba,\ a\widehat{b}c, a\widehat{c}b, b\widehat{a}c, b\widehat{c}a, c\widehat{a}b, c\widehat{b}a,\ ab\widehat{c}, ac\widehat{b}, ba\widehat{c}, bc\widehat{a}, ca\widehat{b}, cb\widehat{a}.$$

Table 1. Transpose of the expansion matrix E in degree 3.

$$
\begin{bmatrix}
1 & -1 & . & . & . & . & . & . & -1 & . & 1 & . & . & . & . & 1 & . & -1 \\
-1 & 1 & . & . & . & . & . & . & 1 & . & -1 & . & . & . & . & -1 & . & 1 \\
. & . & 1 & -1 & . & . & -1 & . & . & . & . & 1 & . & 1 & . & . & -1 & . \\
. & . & -1 & 1 & . & . & 1 & . & . & . & . & -1 & . & -1 & . & . & 1 & . \\
. & . & . & . & 1 & -1 & . & -1 & . & 1 & . & . & 1 & . & -1 & . & . & . \\
. & . & . & . & -1 & 1 & . & 1 & . & -1 & . & . & -1 & . & 1 & . & . & .
\end{bmatrix}
$$

Let E be the 18×6 expansion matrix whose (i,j) entry is the coefficient of the i-th dialgebra monomial in the expansion of the j-th ternary monomial (Table 1). The coefficient vectors of the identities in degree 3 satisfied by the PATS are the vectors in the nullspace of E. We compute the row canonical form of E and find that the canonical integral basis of the nullspace consists of three permutations of $P(a,b,c)$, namely $[a,b,c]+[a,c,b]$, $[b,a,c]+[b,c,a]$ and $[c,a,b]+[c,b,a]$. □

4. Degree 5

For a ternary operation, there are three association types in degree 5: $((a,b,c),d,e)$, $(a,(b,c,d),e)$ and $(a,b,(c,d,e))$.

Lemma 4.1. *If a ternary operation satisfies $P(a,b,c) \equiv 0$, then every multilinear monomial in degree 5 equals one of the following 90 monomials:*

$$((a^\sigma, b^\sigma, c^\sigma), d^\sigma, e^\sigma) \text{ for } b^\sigma < c^\sigma,\ d^\sigma < e^\sigma, \quad (a^\sigma, (b^\sigma, c^\sigma, d^\sigma), e^\sigma) \text{ for } c^\sigma < d^\sigma.$$

Here σ is a permutation of $\{a,b,c,d,e\}$ and $<$ denotes alphabetical precedence.

Proof. Since $P(a,b,c) \equiv 0$ implies $(a,b,(c,d,e)) = -(a,(c,d,e),b)$, we can ignore the third association type. Applying $P(a,b,c) \equiv 0$ to the first and second types gives

$$((a,b,c),d,e) \equiv -((a,c,b),d,e) \equiv -((a,b,c),e,d) \equiv ((a,c,b),e,d),$$
$$(a,(b,c,d),e) \equiv -(a,(b,d,c),e).$$

Hence, there are $5!/4 = 30$ monomials in the first type and $5!/2 = 60$ in the second. □

Definition 4.2. In a ternary algebra, we define the polynomial $Q(a,b,c,d,e) = (a,(b,c,d),e) + (a,(c,b,d),e)$.

Proposition 4.3. *Every multilinear polynomial identity in degree ≤ 5 satisfied by the PATS is a consequence of $P(a,b,c) \equiv 0$ and $Q(a,b,c,d,e) \equiv 0$.*

Proof. We order the multilinear ternary monomials of Lemma 4.1 first by association type and then by lex order of the permutation. Each ternary monomial expands into a sum of 36 dialgebra monomials. For the first two association types we have:

$$
\begin{aligned}
&[[a,b,c],d,e] = \\
&\widehat{a}bcde - \widehat{a}bced - \widehat{a}cbde + \widehat{a}cbed - b\widehat{a}cde + b\widehat{a}ced + c\widehat{a}bde - c\widehat{a}bed - d\widehat{a}bce \\
&\ + d\widehat{a}cbe + e\widehat{a}bcd - e\widehat{a}cbd + bc\widehat{a}de - bc\widehat{a}ed - cb\widehat{a}de + cb\widehat{a}ed + db\widehat{a}ce - dc\widehat{a}be \\
&\ + de\widehat{a}bc - de\widehat{a}cb - eb\widehat{a}cd + ec\widehat{a}bd - ed\widehat{a}bc + ed\widehat{a}cb - dbc\widehat{a}e + dcb\widehat{a}e - deb\widehat{a}c \\
&\ + dec\widehat{a}b + ebc\widehat{a}d - ecb\widehat{a}d + edb\widehat{a}c - edc\widehat{a}b + debc\widehat{a} - decb\widehat{a} - edbc\widehat{a} + edcb\widehat{a}, \\
&[a,[b,c,d],e] =
\end{aligned}
$$

$$
\begin{aligned}
&\widehat{a}bcde - \widehat{a}bdce - \widehat{a}cbde + \widehat{a}cdbe + \widehat{a}dbce - \widehat{a}dcbe - \widehat{a}ebcd + \widehat{a}ebdc + \widehat{a}ecbd \\
&- \widehat{a}ecdb - \widehat{a}edbc + \widehat{a}edcb + e\widehat{a}bcd - e\widehat{a}bdc - e\widehat{a}cbd + e\widehat{a}cdb + e\widehat{a}dbc - e\widehat{a}dcb \\
&- bcd\widehat{a}e + bdc\widehat{a}e + cbd\widehat{a}e - cdb\widehat{a}e - dbc\widehat{a}e + dcb\widehat{a}e + bcde\widehat{a} - bdce\widehat{a} - cbde\widehat{a} \\
&+ cdbe\widehat{a} + dbce\widehat{a} - dcbe\widehat{a} - ebcd\widehat{a} + ebdc\widehat{a} + ecbd\widehat{a} - ecdb\widehat{a} - edbc\widehat{a} + edcb\widehat{a}.
\end{aligned}
$$

Altogether there are $5 \cdot 5! = 600$ dialgebra monomials; we order them first by the position of the center and then lexicographically. We construct the expansion matrix E of size 600×90 whose (i,j) entry is the coefficient of the i-th dialgebra monomial in the expansion of the j-th ternary monomial. We use the `Maple` package `LinearAlgebra` to manipulate this matrix: `Rank` returns the value 50 and so the nullspace has dimension 40; `ReducedRowEchelonForm` computes the row canonical form; `DeleteRow` removes the 550 zero rows at the bottom of the matrix. The result is a 50×90 matrix with the leading 1s of its rows in columns 1–33, 36, 43–45, 48, 55–57, 60, 67–69, 72, 79–81, 84, and so the free variables correspond to columns 34, 35, 37–42, 46, 47, 49–54, 58, 59, 61–66, 70, 71, 73–78, 82, 83, 85–90. For each free variable, we set that variable to 1 and the other free variables to 0, and solve for the leading variables. We obtain a basis of the nullspace consisting of the following identities, 20 of each form:

$$
\left.
\begin{array}{l}
[a^\sigma, [b^\sigma, c^\sigma, d^\sigma], e^\sigma] + [a^\sigma, [c^\sigma, b^\sigma, d^\sigma], e^\sigma] \equiv 0 \\
[a^\sigma, [b^\sigma, c^\sigma, d^\sigma], e^\sigma] - [a^\sigma, [d^\sigma, b^\sigma, c^\sigma], e^\sigma] \equiv 0
\end{array}
\right\}
\text{ for } b^\sigma < c^\sigma < d^\sigma.
$$

Every identity of the first form is equivalent to $Q(a,b,c,d,e) \equiv 0$. We have

$$
[a^\sigma, [b^\sigma, c^\sigma, d^\sigma], e^\sigma] \overset{P}{\equiv} -[a^\sigma, [b^\sigma, d^\sigma, c^\sigma], e^\sigma] \overset{Q}{\equiv} [a^\sigma, [d^\sigma, b^\sigma, c^\sigma], e^\sigma],
$$

and so every identity of the second form follows from P and Q. □

Corollary 4.4. *If a ternary algebra satisfies $P(a,b,c) \equiv 0$ and $Q(a,b,c,d,e) \equiv 0$, then every monomial $(a,(b,c,d),e)$ is an alternating function of b,c,d and every monomial $(a,b,(c,d,e))$ is an alternating function of c,d,e.*

5. Partially alternating ternary algebras

The PATS makes any dialgebra into a partially alternating ternary algebra as follows.

Definition 5.1. A *completely alternating ternary algebra (CATA)* is one with a *completely alternating product*: $(a^\sigma, b^\sigma, c^\sigma) = \epsilon(\sigma)(a,b,c)$ for all permutations σ where ϵ is the sign. A *partially alternating ternary algebra (PATA)* is one with a *partially alternating product*: the product satisfies $P(a,b,c) \equiv 0$ and $Q(a,b,c,d,e) \equiv 0$.

Lemma 5.2. *Let $(\cdot,\cdot,\cdot)$ be a partially alternating ternary product and let $s = a_1 \cdots a_n$ be a monomial with some placement of (ternary) parentheses. Then the second and third arguments of any submonomial (x,y,z) of s are completely alternating.*

Proof. We proceed by induction on n. For $n \leq 3$ the claim is obvious. Set $n > 3$ and assume that the result holds for any monomial of degree $< n$. We have the factorization $s = (t,u,v)$; then the second or third argument of a submonomial of s is either (i) u or v, or (ii) the second or third argument of a submonomial of t, u or v. In case (ii), the claim follows from the inductive hypothesis. In case (i), it suffices to prove the claim for the second argument u, since $P \equiv 0$ implies $s = -(t,v,u)$ and so the claim also holds for v. To show the claim for u, it is enough to note that $P \equiv 0$ (respectively $Q \equiv 0$) implies that u alternates in its second and third arguments

Table 2. Association types for ternary products.

a	(a,b,c)	$((a,b,c),d,e)$	$(((a,b,c),d,e),f,g)$	$((((a,b,c),d,e),f,g),h,i)$
			$((a,b,c),(d,e,f),g)$	$(((a,b,c),(d,e,f),g),h,i)$
				$(((a,b,c),d,e),(f,g,h),i)$
completely alternating types				$((a,b,c),(d,e,f),(g,h,i))$
a	(a,b,c)	$((a,b,c),d,e)$	$(((a,b,c),d,e),f,g)$	$((((a,b,c),d,e),f,g),h,i)$
		$(a,(b,c,d),e)$	$((a,(b,c,d),e),f,g)$	$(((a,(b,c,d),e),f,g),h,i)$
			$((a,b,c),(d,e,f),g)$	$(((a,b,c),(d,e,f),g),h,i)$
			$(a,((b,c,d),e,f),g)$	$((a,((b,c,d),e,f),g),h,i)$
			$(a,(b,c,d),(e,f,g))$	$((a,(b,c,d),(e,f,g)),h,i)$
				$(((a,b,c),d,e),(f,g,h),i)$
				$((a,(b,c,d),e),(f,g,h),i)$
				$((a,b,c),((d,e,f),g,h),i)$
				$((a,b,c),(d,e,f),(g,h,i))$
				$(a,(((b,c,d),e,f),g,h),i)$
				$(a,((b,c,d),(e,f,g),h),i)$
partially alternating types				$(a,((b,c,d),e,f),(g,h,i))$

(respectively its first and second arguments). Since these two transpositions generate the symmetric group, u is completely alternating. □

A ternary operation has monomials only in odd degrees. We generate the association types for a partially alternating ternary product inductively by degree; we simultaneously generate the completely alternating (CA) types and the partially alternating (PA) types (Table 2). Suppose that we have already generated ordered lists of the CA and PA types in degrees $< n$. To generate the CA and PA association types of degree n, we consider all partitions of $n = i + j + k$ into three odd parts:

(i) If $i > j > k$ then for all CA types t, u, v of degrees i, j, k respectively, we include (t, u, v) as a new CA type in degree n.

(ii) If $i = j > k$ then for all CA types t, u, v of degrees i, i, k where t precedes u in the types of degree i, we include (t, u, v) as a CA type in degree n.

(iii) If $i > j = k$ then for all CA types t, u, v of degrees i, j, j where u precedes v in the types of degree j, we include (t, u, v) as a CA type in degree n.

(iv) If $i = j = k$ then for all CA types t, u, v of degree i where t precedes u and u precedes v, we include (t, u, v) as a CA type in degree n.

(v) If $j > k$ then for all PA types t of degree i and all CA types u, v of degrees j, k respectively, we include (t, u, v) as a new PA type in degree n.

(vi) If $j = k$ then for all PA types t of degree i and all CA types u, v of degree j where u precedes v, we include (t, u, v) as a PA type in degree n.

To enumerate the multilinear monomials in a completely or partially alternating association type in degree n, we need to determine the number d of skew-symmetries of the association type; the number of monomials is then $n!/d$. To compute d we use the recursive procedure of Figure 1; before calling the procedure we set $d \leftarrow 1$.

6. Degree 7

From now on we usually omit the commas in all ternary monomials.

Figure 1. Recursive procedure `countsymmetry`(x, `flag`).

Input:
A completely or partially alternating ternary association type x;
a Boolean variable `flag` which is `true` for CA and `false` for PA.

Procedure:
If $\deg(x) > 1$ then write $x = (x_1, x_2, x_3)$:
  If `flag` = `true` then
    `countsymmetry`(x_1, `true`):
    `countsymmetry`(x_2, `true`):
    `countsymmetry`(x_3, `true`):
    If all three of x_1, x_2, x_3 have the same degree and association type then
      set $d \leftarrow 6d$.
    If only two of x_1, x_2, x_3 have the same degree and association type then
      set $d \leftarrow 2d$.
  else
    `countsymmetry`(x_1, `false`):
    `countsymmetry`(x_2, `true`):
    `countsymmetry`(x_3, `true`):
    If x_2, x_3 have the same degree and association type then set $d \leftarrow 2d$.

Lemma 6.1. *In a partially alternating ternary algebra, every multilinear monomial in degree 7 equals one of the following 1960 monomials:*

(1)	$(((a^\sigma b^\sigma c^\sigma) d^\sigma e^\sigma) f^\sigma g^\sigma)$	$(b^\sigma < c^\sigma, d^\sigma < e^\sigma, f^\sigma < g^\sigma)$
(2)	$((a^\sigma (b^\sigma c^\sigma d^\sigma) e^\sigma) f^\sigma g^\sigma)$	$(b^\sigma < c^\sigma < d^\sigma, f^\sigma < g^\sigma)$
(3)	$((a^\sigma b^\sigma c^\sigma)(d^\sigma e^\sigma f^\sigma) g^\sigma)$	$(b^\sigma < c^\sigma, d^\sigma < e^\sigma < f^\sigma)$
(4)	$(a^\sigma ((b^\sigma c^\sigma d^\sigma) e^\sigma f^\sigma) g^\sigma)$	$(b^\sigma < c^\sigma < d^\sigma, e^\sigma < f^\sigma)$
(5)	$(a^\sigma (b^\sigma c^\sigma d^\sigma)(e^\sigma f^\sigma g^\sigma))$	$(b^\sigma < c^\sigma < d^\sigma, e^\sigma < f^\sigma < g^\sigma, b^\sigma < e^\sigma)$

Here σ is a permutation of $\{a, b, c, d, e, f, g\}$ and $<$ is alphabetical precedence.

Proof. Degree 7 has 12 ternary association types, but in a PATA these reduce to 5:

$$
\begin{array}{ll}
(((abc)de)fg) = \text{type (1)}, & ((a(bcd)e)fg) = \text{type (2)}, \\
((ab(cde))fg) = -((a(cde)b)fg), & ((abc)(def)g) = \text{type (3)}, \\
((abc)d(efg)) = -((abc)(efg)d), & (a((bcd)ef)g) = \text{type (4)}, \\
(a(b(cde)f)g) = -(a((cde)bf)g), & (a(bc(def))g) = (a((def)bc)g), \\
(a(bcd)(efg)) = \text{type (5)}. & (ab((cde)fg)) = -(a((cde)fg)b), \\
(ab(c(def)g)) = (a((def)cg)b), & (ab(cd(efg))) = -(a((efg)cd)b).
\end{array}
$$

To enumerate the multilinear monomials in each type, we count the skew-symmetries:

$(((abc)de)fg)$	alternates in b, c and d, e and f, g	$7!/8 = 630$
$((a(bcd)e)fg)$	alternates in b, c, d and f, g	$7!/12 = 420$
$((abc)(def)g)$	alternates in b, c and d, e, f	$7!/12 = 420$
$(a((bcd)ef)g)$	alternates in b, c, d and e, f	$7!/12 = 420$
$(a(bcd)(efg))$	alternates in b, c, d and e, f, g and bcd, efg	$7!/72 = 70$

We order these monomials first by association type and then lexicographically. □

Definition 6.2. In a PATA we consider the following polynomials of degree 7:

$$R(a,b,c,d,e,f,g) =$$
$$\frac{1}{12}\sum_{\sigma\in S_6}\epsilon(\sigma)\,((a(b^\sigma c^\sigma d^\sigma)e^\sigma)f^\sigma g^\sigma) - \frac{1}{12}\sum_{\sigma\in S_6}\epsilon(\sigma)\,((ab^\sigma c^\sigma)(d^\sigma e^\sigma f^\sigma)g^\sigma),$$
$$S(a,b,c,d,e,f,g) =$$
$$\frac{1}{4}\sum_{\sigma\in S_5}\epsilon(\sigma)\,(((ab^\sigma c^\sigma)d^\sigma g)e^\sigma f^\sigma) - \frac{1}{6}\sum_{\sigma\in S_5}\epsilon(\sigma)\,((a(b^\sigma c^\sigma d^\sigma)e^\sigma)f^\sigma g)$$
$$+\frac{1}{4}\sum_{\sigma\in S_5}\epsilon(\sigma)\,((a(b^\sigma c^\sigma g)d^\sigma)e^\sigma f^\sigma) + \frac{1}{12}\sum_{\sigma\in S_5}\epsilon(\sigma)\,((ab^\sigma c^\sigma)(d^\sigma e^\sigma f^\sigma)g)$$
$$-\frac{1}{6}\sum_{\sigma\in S_5}\epsilon(\sigma)\,(a((b^\sigma c^\sigma d^\sigma)e^\sigma g)f^\sigma) - \frac{1}{12}\sum_{\sigma\in S_5}\epsilon(\sigma)\,(a(bc^\sigma d^\sigma)(e^\sigma f^\sigma g^\sigma)).$$

Some of the permutations σ in R and S produce monomials which are not 'straightened': P and Q need to be applied to convert such a monomial into ($\pm$) an equivalent monomial in which the permutation σ' satisfies $\sigma' < \sigma$ in lex order. This produces repetitions; to cancel the resulting coefficients, we use appropriate fractions. (This 'straightening' algorithm is described in detail in Subsection 8.2.) After this process, each monomial has coefficient ± 1, and each identity has 120 terms.

Theorem 6.3. *Every multilinear polynomial identity of degree ≤ 7 satisfied by the PATS is a consequence of $P \equiv 0$, $Q \equiv 0$, $R \equiv 0$ and $S \equiv 0$.*

Proof. There are $7 \cdot 7! = 35280$ dialgebra monomials in degree 7; this and Lemma 6.1 show that the expansion matrix E has size 35280×1960. It is not practical to use rational arithmetic with such a large matrix so we use `LinearAlgebra[Modular]`. The vector space of multilinear ternary polynomials is a representation of the symmetric group which acts by permuting the variables. The group algebra FS_n is semisimple over a field F of characteristic 0 or $p > n$, and so for degree 7 any $p > 7$ will give the correct ranks. The procedure `RowReduce` computes the row canonical form of E and its rank 1911. The 49-dimensional nullspace consists of the multilinear polynomial identities in degree 7 for the PATS which are not consequences of $P \equiv 0$ and $Q \equiv 0$. The procedure `Basis` computes the canonical basis of the nullspace, which we sort by increasing number of terms in the corresponding identities:

The first 7 identities: 120 terms; coefficients ± 1; association types 2, 3.
The next 28: 120 terms; coefficients ± 1; association types 1, 2, 3, 4.
The next 7: 120 terms; coefficients ± 1; association types 1, 2, 3, 4, 5.
The last 7: 180 terms; coefficients ± 1, ± 2; association types 1, 2, 3, 4.

Further computations verify that identities 1 and 36 (R and S) generate the nullspace: every identity is a linear combination of permutations of R and S. Moreover, R (respectively, S) generates a subspace of dimension 7 (respectively, 42). Hence the nullspace is the direct sum of these two subspaces, and so R and S are independent. □

7. Representation theory of the symmetric group

To study nonlinear identities, and identities of higher degree, we use the representation theory of the symmetric group S_n; our main reference is James and Kerber [16].

Table 3. Representation matrix X_λ in partition λ.

$$
\left[
\begin{array}{ccccc|ccccc}
\rho_\lambda(E_1^1) & \rho_\lambda(E_2^1) & \cdots & \rho_\lambda(E_{n-1}^1) & \rho_\lambda(E_n^1) & -I_d & O & \cdots & O & O \\
\rho_\lambda(E_1^2) & \rho_\lambda(E_2^2) & \cdots & \rho_\lambda(E_{n-1}^2) & \rho_\lambda(E_n^2) & O & -I_d & \cdots & O & O \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\rho_\lambda(E_1^{t-1}) & \rho_\lambda(E_2^{t-1}) & \cdots & \rho_\lambda(E_{n-1}^{t-1}) & \rho_\lambda(E_n^{t-1}) & O & O & \cdots & -I_d & O \\
\rho_\lambda(E_1^t) & \rho_\lambda(E_2^t) & \cdots & \rho_\lambda(E_{n-1}^t) & \rho_\lambda(E_n^t) & O & O & \cdots & O & -I_d
\end{array}
\right]
$$

Definition 7.1. A *partition* of n is a tuple $\lambda = (n_1, \dots, n_\ell)$ with $n = n_1 + \cdots + n_\ell$ and $n_1 \geq \cdots \geq n_\ell \geq 1$. The *frame* $[\lambda]$ consists of n boxes in ℓ left-justified rows with n_i boxes in row i. A *tableau* is a bijection between $\{1, \dots, n\}$ and the boxes of $[\lambda]$. In a *standard* tableau the numbers increase from left to right and from top to bottom.

The irreducible representations of S_n correspond to the partitions of n. The dimension d_λ associated to λ is the number of standard tableaux with frame $[\lambda]$. If $F = \mathbb{Q}$ or $F = \mathbb{F}_p$ for $p > n$ then the group algebra FS_n decomposes into an orthogonal direct sum of two-sided ideals isomorphic to simple matrix algebras:

$$
FS_n \approx \bigoplus_\lambda M_{d_\lambda}(F).
$$

Given a permutation π and a partition λ, we need to compute the projection of π onto $M_{d_\lambda}(F)$: the matrix for π in representation λ. A simple algorithm for this was found by Clifton [8]. We fix an ordering of the standard tableaux $T_1, \dots, T_d$ ($d = d_\lambda$) and construct a matrix R^λ_π as follows: Apply π to T_j, obtaining a (possibly non-standard) tableau πT_j. If there exist two numbers in the same column of T_i and the same row of πT_j, then $(R^\lambda_\pi)_{ij} = 0$. Otherwise, $(R^\lambda_\pi)_{ij}$ is the sign of the permutation of T_i which leaves the columns invariant as sets and moves the numbers to the correct rows of πT_j. The matrix R^λ_{id} for the identity permutation may not be the identity matrix but is invertible. An algorithm for computing R^λ_π is given by Bremner and Peresi [7].

Lemma 7.2. (Clifton [8].) *The matrix representing π in partition λ equals $(R^\lambda_{\mathrm{id}})^{-1} R^\lambda_\pi$.*

Any polynomial identity (not necessarily multilinear or even homogeneous) of degree $\leq n$ over a field F of characteristic 0 or $p > n$ is equivalent to a finite set of multilinear identities [24]. We consider a multilinear identity $I(x_1, \dots, x_n)$ of degree n and collect the terms with the same association type: $I = I_1 + \cdots + I_t$. The monomials in each I_k differ only by a permutation of $x_1, \dots, x_n$; hence I_k is an element of FS_n and I is an element of $(FS_n)^t$. If $U \subseteq (FS_n)^t$ is the span of all permutations of I, then U is a representation of S_n, and so U is the direct sum of components corresponding to the irreducible representations of S_n. This breaks down a large computational problem into smaller pieces. We fix a partition λ with associated dimension $d = d_\lambda$. To determine the λ-component of U, we construct a $d \times dt$ matrix M_λ consisting of t blocks of size $d \times d$; in block j we put the representation matrix for the terms of I_j.

Definition 7.3. The rank of M_λ is the *rank of identity I in partition λ*.

We modify this procedure to determine the nullspace of the expansion matrix E for the PATS. In degree n, let $t = t_n$ be the number of association types for a partially alternating ternary product. Consider the monomial in ternary association type i with the identity permutation of the variables, and let E^i be its expansion using the PATS. We have $E^i = E^i_1 + \cdots + E^i_n$ where E^i_j contains the dialgebra monomials with center in position j. We construct a $td \times (n{+}t)d$ matrix X_λ with t rows and $n{+}t$ columns

Table 4. Skew-symmetries of ternary association types in degree 7.

$(((abc)de)fg) + (((acb)de)fg) \equiv 0$	$(((abc)de)fg) + (((abc)ed)fg) \equiv 0$
$(((abc)de)fg) + (((abc)de)gf) \equiv 0$	$((a(bcd)e)fg) + ((a(cbd)e)fg) \equiv 0$
$((a(bcd)e)fg) + ((a(bdc)e)fg) \equiv 0$	$((a(bcd)e)fg) + ((a(bcd)e)gf) \equiv 0$
$((abc)(def)g) + ((acb)(def)g) \equiv 0$	$((abc)(def)g) + ((abc)(edf)g) \equiv 0$
$((abc)(def)g) + ((abc)(dfe)g) \equiv 0$	$(a((bcd)ef)g) + (a((cbd)ef)g) \equiv 0$
$(a((bcd)ef)g) + (a((bdc)ef)g) \equiv 0$	$(a((bcd)ef)g) + (a((bcd)fe)g) \equiv 0$
$(a(bcd)(efg)) + (a(cbd)(efg)) \equiv 0$	$(a(bcd)(efg)) + (a(bdc)(efg)) \equiv 0$
$(a(bcd)(efg)) + (a(efg)(bcd)) \equiv 0$	

of $d \times d$ blocks (Table 3). In the right side, in block $(i, n+i)$ for $1 \le i \le t$, we put $-I_d$ (identity matrix); the other blocks of the right side are zero. In the left side, in block (i,j) for $1 \le i \le t$ and $1 \le j \le n$, we put $\rho_\lambda(E^i_j)$, the representation matrix of E^i_j. The matrix X_λ is the representation matrix for the components in partition λ of the expansions of the ternary association types in degree n. We compute the row canonical form of X_λ and distinguish the upper (respectively lower) part containing the rows with leading ones in the left (respectively right) side. The rows of the lower right part represent polynomial identities satisfied by the PATS as a result of dependence relations among the dialgebra expansions of the ternary association types.

Definition 7.4. The number of (nonzero) rows in the lower right block of the row canonical form of X_λ is the *rank of identities satisfied by the PATS in partition* λ.

8. Degree 7: nonlinear identities

In this section we find nonlinear identities for the PATS in degree 7 which are shorter than the 120-term multilinear identities R and S of Definition 6.2.

Definition 8.1. A polynomial identity is *nonlinear* if it is homogeneous of degree n and there is a partition $(n_1, \dots, n_\ell)$ of n with some $n_i \ge 2$ (equivalently $\ell < n$) such that the variables in each monomial are a permutation of

$$\overbrace{a_1, \dots, a_1}^{n_1}, \overbrace{a_2, \dots, a_2}^{n_2}, \dots, \overbrace{a_\ell, \dots, a_\ell}^{n_\ell}.$$

The representation theory of the symmetric group tells us which partitions to use in our search for shorter nonlinear identities.

8.1. Application of representation theory

In Section 6, we found the inequivalent multilinear monomials in each ternary association type. That method was based on monomials and used the skew-symmetries implied by $P \equiv 0$ and $Q \equiv 0$ to reduce the number of monomials in each type. In contrast, the representation theory is based on association types and expresses the skew-symmetries as multilinear polynomial identities.

Lemma 8.2. *In a partially alternating ternary algebra, every skew-symmetry of the 5 association types in degree 7 is a consequence of the 15 identities in Table 4.*

Proof. This follows by applying the identities $P \equiv 0$ and $Q \equiv 0$. □

Table 5. Matrix ranks of each representation in degree 7 for the PATS.

	partition	dimension	symrank	exprank	newrank
1	7	1	5	5	0
2	61	6	30	30	0
3	52	14	70	70	0
4	511	15	75	75	0
5	43	14	69	69	0
6	421	35	170	170	0
7	4111	20	96	96	0
8	331	21	99	99	0
9	322	21	96	96	0
10	3211	35	156	156	0
11	31111	15	63	64	1
12	2221	14	56	56	0
13	22111	14	52	53	1
14	211111	6	17	20	3
15	1111111	1	0	2	2

Let λ be a partition of $n = 7$ with associated irreducible representation of dimension $d = d_\lambda$. There are 15 skew-symmetry identities and 5 association types, requiring a $15d \times 5d$ matrix M_λ. In each skew-symmetry, the first term has the identity permutation with representation matrix I_d, and the second term has a permutation π of order 2 with representation matrix $(R^\lambda_{\mathrm{id}})^{-1} R^\lambda_\pi$ from Lemma 7.2. The $d \times d$ block in position (i, j) contains the sum of these matrices, where i and j are the index number and the association type of the skew-symmetry. The rank of M_λ is 'symrank' in Table 5. For each λ, we construct the representation matrix X_λ (Table 3) and compute the rank of its lower right part (Definition 7.4); this is 'exprank' in Table 5. Column 'newrank' is the difference between 'symrank' and 'exprank': this is the rank of the new identities in degree 7 for partition λ; that is, the identities which are not trivial consequences of the skew-symmetries of the association types. We check the results by summing, over all representations, the product of 'newrank' and 'dimension': $1 \cdot 15 + 1 \cdot 14 + 3 \cdot 6 + 2 \cdot 1 = 49$. This is the dimension of the nullspace of the expansion matrix from Section 6; column 'newrank' gives the decomposition of the nullspace into irreducible representations.

There are four representations where 'newrank' is positive: 11, 13, 14, 15. This suggests that a slight modification of the techniques of Section 6 will produce nonlinear identities in these partitions: identities in which the variables in each term are a permutation of $\{a, a, a, b, c, d, e\}$, $\{a, a, b, b, c, d, e\}$ or $\{a, a, b, c, d, e, f\}$. (We omit representation 15 since it corresponds to the multilinear case.)

8.2. Straightening algorithm

We need an algorithm to convert a monomial (multilinear or nonlinear) to its 'straightened' form with respect to $P \equiv 0$ and $Q \equiv 0$. We apply P and Q to convert a monomial with a given permutation of the variables into $(\pm)$ a monomial with a different permutation which lexicographically precedes the original permutation. We use the recursiveprocedures `completestraighten` (`CS`) and `partialstraighten` (`PS`); for both the input is a monomial x. If the straightened form of x is 0, then both return 0. If $\deg(x) = 1$ then both return x. If $\deg(x) > 1$ then we write $x = (x_1, x_2, x_3)$ and proceed as follows:

- CS recursively computes $\texttt{CS}(x_1)$, $\texttt{CS}(x_2)$, $\texttt{CS}(x_3)$. If any of them is 0, then CS returns 0. If two or more are equal, then CS returns 0. Otherwise, CS puts them in the correct order using `strictlyprecedes` (see below).
- PS recursively computes $\texttt{PS}(x_1)$, $\texttt{CS}(x_2)$, $\texttt{CS}(x_3)$. If any of them is 0, then PS returns 0. If $\texttt{CS}(x_2) = \texttt{CS}(x_3)$, then PS returns 0. Otherwise, PS puts $\texttt{CS}(x_2)$ and $\texttt{CS}(x_3)$ in the correct order using `strictlyprecedes`.

Procedure `strictlyprecedes` compares monomials x and y. If $\deg(x) \neq \deg(y)$ then it returns `true` if $\deg(x) < \deg(y)$, `false` if $\deg(x) > \deg(y)$. If $\deg(x) = \deg(y)$ then:

- If both x and y have degree 1, it uses the total order on the generators.
- If both have degree > 1 then $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ and it finds the least i with $x_i \neq y_i$ and recursively calls $\texttt{strictlyprecedes}(x_i, y_i)$.

In other words, first compare the degrees; if the degrees are equal then compare the association types; and if the types are equal then compare the permutations.

Theorem 8.3. *There is one identity for partition 31111; it has 60 terms:*

$$\begin{aligned} I^{(1)}_{31111} = {} & \frac{1}{4} \sum_{\sigma \in S_5} \epsilon(\sigma)\, [[[aa^\sigma b^\sigma] a c^\sigma] d^\sigma e^\sigma] - \frac{1}{12} \sum_{\sigma \in S_5} \epsilon(\sigma)\, [[a[a^\sigma b^\sigma c^\sigma]a] d^\sigma e^\sigma] \\ & - \frac{1}{12} \sum_{\sigma \in S_5} \epsilon(\sigma)\, [[aa^\sigma b^\sigma][c^\sigma d^\sigma e^\sigma]a] - \frac{1}{6} \sum_{\sigma \in S_5} \epsilon(\sigma)\, [a[[b^\sigma c^\sigma d^\sigma] a e^\sigma] a^\sigma]. \end{aligned}$$

Proof. We first generate all 840 permutations of a, a, a, b, c, d, e. For each association type, we apply the type to each permutation, find the straightened form of the resulting monomial, and retain only those monomials which equal their own straightened forms. We sort the remaining monomials in each type by lex order of the permutation. For partition 31111, the five association types contain respectively $60{+}34{+}34{+}34{+}3 = 165$ monomials. The expansion matrix E has 165 columns and $7 \cdot 840 = 5880$ rows. For $j = 1, \dots, 165$ we store in column j the PATS expansion of ternary monomial j. The rank is 164, and so the nullspace has dimension 1. Hence, up to a scalar multiple, there is exactly one identity for the PATS with these variables; this identity has 60 terms with coefficients ± 1 in association types 1–4. □

Theorem 8.4. *There are two identities for partition 22111; both have 60 terms:*

$$\begin{aligned} I^{(1)}_{22111} = {} & \frac{1}{4} \sum_{\sigma \in S_5} \epsilon(\sigma)\, [[[aa^\sigma b^\sigma] b c^\sigma] d^\sigma e^\sigma] - \frac{1}{12} \sum_{\sigma \in S_5} \epsilon(\sigma)\, [[a(a^\sigma b^\sigma c^\sigma]b] d^\sigma e^\sigma] \\ & - \frac{1}{12} \sum_{\sigma \in S_5} \epsilon(\sigma)\, [[aa^\sigma b^\sigma][c^\sigma d^\sigma e^\sigma]b] - \frac{1}{6} \sum_{\sigma \in S_5} \epsilon(\sigma)\, [a[[a^\sigma b^\sigma c^\sigma] b d^\sigma] e^\sigma], \end{aligned}$$

and $I^{(2)}_{22111}$ *which is obtained by interchanging* a *and* b.

Proof. Similar to the proof of Theorem 8.3. □

Theorem 8.5. *There are 12 identities for partition 211111; only 5 have 60 terms:*

$$\begin{aligned} I^{(1)}_{211111} = {} & \frac{1}{4} \sum_{\sigma \in S_5} \epsilon(\sigma)\, [[[ba^\sigma c^\sigma] a d^\sigma] e^\sigma f^\sigma] - \frac{1}{12} \sum_{\sigma \in S_5} \epsilon(\sigma)\, [[b[a^\sigma c^\sigma d^\sigma]a] e^\sigma f^\sigma] \\ & - \frac{1}{12} \sum_{\sigma \in S_5} \epsilon(\sigma)\, [[ba^\sigma c^\sigma][d^\sigma e^\sigma f^\sigma]a] - \frac{1}{6} \sum_{\sigma \in S_5} \epsilon(\sigma)\, [b[[a^\sigma c^\sigma d^\sigma] a e^\sigma] f^\sigma], \end{aligned}$$

and $I^{(2)}_{211111}, \dots, I^{(5)}_{211111}$ *obtained by interchanging* b *and* c, d, e, f *respectively.*

Proof. Similar to the proof of Theorem 8.3. □

9. Degree 9

In degree 9 there are 12 association types for a PATA (Table 2). We compute the following matrix ranks using the representation theory of the symmetric group:

- symrank: the rank of the skew-symmetry identities of the association types.
- symlifrank: the rank of the skew-symmetries combined with the consequences in degree 9 of the identities R and S in degree 7 from Definition 6.2; that is,

$$T((ahi), b, c, d, e, f, g),\ T(a, (bhi), c, d, e, f, g),\ \ldots,\ T(a, b, c, d, e, f, (ghi)),$$
$$(T(a, b, c, d, e, f, g), h, i),\ (h, T(a, b, c, d, e, f, g), i),\ (h, i, T(a, b, c, d, e, f, g)),$$

  where $T = R$ and $T = S$.
- exprank: the rank of the lower right part of the expansion matrix (Definition 7.4).

For every partition 'symlifrank' equals 'exprank': there are no new identities.

10. Conclusion

Trilinear operations in an associative algebra have recently been classified by Bremner and Peresi [6]: there are six isolated operations (the alternating, symmetric, and cyclic sums, the cyclic commutator, the weakly commutative and anticommutative operations), and four infinite families (the Lie, Jordan and anti-Jordan families, and a fourth family which seems unrelated to Lie and Jordan structures). The Kolesnikov-Pozhidaev algorithm can be applied to all these operations: we choose one of the three arguments and make it the center of each monomial. Unlike the alternating sum, which corresponds to the 1-dimensional sign representation of the symmetric group S_3, most of the other algebra operations will produce essentially different dialgebra operations for each choice of the central argument. The polynomial identities satisfied by these new dialgebra operations will define varieties of ternary algebras with great potential for applications in pure mathematics and theoretical physics.

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