THE FUNDAMENTAL INVARIANTS OF $3 \times 3 \times 3$ ARRAYS

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Abstract. We use computer algebra to determine explicitly the three fundamental invariants of a $3 \times 3 \times 3$ array over $\mathbb{C}$ as polynomials in the 27 variables x_{ijk} for $1 \le i, j, k \le 3$. By the work of Vinberg, it is known that these invariants have degrees 6, 9 and 12; they freely generate the algebra of invariants for $\mathfrak{sl}_3(\mathbb{C})^3$ acting irreducibly on the tensor cube $(\mathbb{C}^3)^{\otimes 3}$ of the natural representation of $\mathfrak{sl}_3(\mathbb{C})$. We find expressions for these invariants in terms of the orbits of the symmetry group $(S_3 \times S_3 \times S_3) \rtimes S_3$ acting on monomials.

1. Introduction

We consider $3 \times 3 \times 3$ arrays X over the complex numbers:

$$X = (x_{ijk}), \qquad x_{ijk} \in \mathbb{C}, \qquad 1 \le i, j, k \le 3.$$

Following Bremner et al. [2, §6], we use a computational approach, based on the representation theory of the special linear Lie algebra $\mathfrak{sl}_3(\mathbb{C})$, to determine the invariant polynomials as elements in the nullspace of a large integer matrix.

Vinberg [12] generalized the notion of Weyl group from Lie groups to θ-groups, by which is meant the group of fixed points of an automorphism of finite order of a complex semisimple Lie group. He showed that these generalized Weyl groups are generated by complex reflections, and obtained the corollary that the algebra of invariants is a polynomial algebra. A general computational framework for studying θ-groups has been developed recently by de Graaf [4].

Vinberg embedded the direct sum of three copies of $\mathfrak{sl}_3(\mathbb{C})$ into an exceptional Lie algebra of type E_6, and deduced that the algebra of invariants for the natural representation of $\mathfrak{sl}_3(\mathbb{C})^3$ is freely generated by polynomials in degrees 6, 9 and 12. Nurmiev [11] obtained an implicit description of these generators in terms of convolutions of volume forms. An approach using classical invariant theory has been given by Briand et al. [3]. For an application of these invariants to quantum information theory, see Duff and Ferrara [5].

In this paper we obtain explicit expressions for the fundamental invariants in degrees 6, 9 and 12 for the algebra of invariants of the semisimple Lie algebra $\mathfrak{sl}_3(\mathbb{C})^3$ acting irreducibly on its natural representation, the tensor cube $(\mathbb{C}^3)^{\otimes 3}$ of the natural representation of $\mathfrak{sl}_3(\mathbb{C})$. We express these invariants in terms of the orbits of the symmetry group $(S_3 \times S_3 \times S_3) \rtimes S_3$ acting on monomials in the entries of a $3 \times 3 \times 3$ array. For related work on the hyperdeterminant of a $2 \times 2 \times 2 \times 2$ array, see Huggins et al. [7].

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2. Preliminaries

Unless otherwise stated, all vector spaces are over $\mathbb{C}$.

2.1. **Polynomial functions on $3 \times 3 \times 3$ arrays.** Let e_1, e_2, e_3 be the standard basis of the vector space $\mathbb{C}^3$. We consider the tensor product

$$\mathbb{C}^{333} = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3.$$

Every element of $\mathbb{C}^{333}$ is a finite sum of elements of the form $u \otimes v \otimes w$ where $u, v, w \in \mathbb{C}^3$. A basis for $\mathbb{C}^{333}$ consists of the 27 simple tensors

$$e_{ijk} = e_i \otimes e_j \otimes e_k, \qquad 1 \leq i, j, k \leq 3.$$

We may therefore identify the array $X = (x_{ijk})$ with the element

$$X = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} x_{ijk} e_{ijk}, \qquad x_{ijk} \in \mathbb{C}.$$

We introduce variables x_{ijk} for $1 \leq i, j, k \leq 3$ corresponding to the entries of X; this notation is ambiguous but should not be confusing. Strictly speaking, x_{ijk} is a coordinate function on $\mathbb{C}^{333}$ and is therefore a dual basis vector $e_i^* \otimes e_j^* \otimes e_k^*$, but this distinction will not be important for us. We consider the polynomial algebra P in these indeterminates over $\mathbb{C}$:

$$P = \mathbb{C}[\, x_{ijk} \mid 1 \leq i, j, k \leq 3 \,].$$

A basis of P consists of the monomials

$$M(E) = \prod_{i=1}^{3} \prod_{j=1}^{3} \prod_{k=1}^{3} x_{ijk}^{e_{ijk}} = x_{111}^{e_{111}} \cdots x_{ijk}^{e_{ijk}} \cdots x_{333}^{e_{333}},$$

where $E = (e_{ijk})$ is an exponent array of non-negative integers. The homogeneous subspace P_d of degree d has a basis consisting of monomials $M(E)$ for which

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} e_{ijk} = d.$$

A linear operator on $\mathbb{C}^3$ can be identified with a 3×3 matrix $A = (a_{ij})$ where $a_{ij} \in \mathbb{C}$ for $1 \leq i, j \leq 3$. The action of a triple of invertible operators (A, B, C) on a variable x_{ijk} corresponds to the action on the standard basis vectors in $\mathbb{C}^3$:

$$(A, B, C) \cdot x_{ijk} = \sum_{p=1}^{3} \sum_{q=1}^{3} \sum_{r=1}^{3} a_{pi} b_{qj} c_{rk} x_{pqr}.$$

This action extends to polynomials as follows:

$$(A, B, C) \cdot f\big(x_{111}, \dots, x_{ijk}, \dots, x_{333}\big) =$$
$$f\big((A, B, C) \cdot x_{111}, \dots, (A, B, C) \cdot x_{ijk}, \dots, (A, B, C) \cdot x_{333}\big).$$

Definition 2.1. The polynomial $f \in P$ is **invariant** if

$$(A, B, C) \cdot f = f \quad \text{whenever} \quad \det(A) = \det(B) = \det(C) = 1.$$

2.2. **Representations of Lie algebras.** The $n \times n$ complex matrices of determinant 1, with the usual operation of matrix multiplication, form the special linear group $SL_n(\mathbb{C})$. Finite-dimensional representations of $SL_n(\mathbb{C})$ can be studied in terms of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, which consists of all $n \times n$ complex matrices of trace 0; the bilinear product is the Lie bracket $[A,B] = AB - BA$. The standard basis of $\mathfrak{sl}_n(\mathbb{C})$ consists of

- the matrix units $U_{i,j}$ for $i \neq j$ with (i,j) entry 1 and other entries 0,
- the diagonal matrices $H_i = U_{i,i} - U_{i+1,i+1}$ for $i = 1, 2, \dots, n{-}1$.

The simple root vectors are the matrix units $T_i = U_{i,i+1}$ for $i = 1, 2, \dots, n{-}1$. The natural representation of $\mathfrak{sl}_n(\mathbb{C})$ is its irreducible action on $\mathbb{C}^n$.

Lemma 2.2. *In the natural representation of* $\mathfrak{sl}_n(\mathbb{C})$ *we have*

$$H_i \cdot e_j = \begin{cases} e_j & \textit{if } j = i \\ -e_j & \textit{if } j = i{+}1 \\ 0 & \textit{otherwise}, \end{cases} \qquad T_i \cdot e_j = \begin{cases} e_{j-1} & \textit{if } j = i{+}1 \\ 0 & \textit{otherwise.} \end{cases}$$

We are concerned exclusively with the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$; the diagonal matrices and the simple root vectors in the natural representation are

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; T_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; T_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Lemma 2.3. *The brackets of* H_i *and* T_j *are*

$$[H_1, T_1] = 2T_1, \quad [H_1, T_2] = -T_2, \quad [H_2, T_1] = -T_1, \quad [H_2, T_2] = 2T_2.$$

We consider the action of the semisimple Lie algebra

$$\mathfrak{sl}_{333}(\mathbb{C}) = \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}),$$

on its irreducible representation $\mathbb{C}^{333}$, the tensor product of the natural representations of its simple summands. For $\ell = 1, 2, 3$ we write $H_1^{(\ell)}$, $H_2^{(\ell)}$, $T_1^{(\ell)}$, $T_2^{(\ell)}$ for the corresponding elements in the ℓ-th summand $\mathfrak{sl}_3(\mathbb{C})$. Combining the preceding equations we obtain the action of these elements on the indeterminates x_{ijk}.

Lemma 2.4. *For* $m = 1, 2$ *we have*

$$H_m^{(1)} \cdot x_{ijk} = \begin{cases} x_{ijk} & \textit{if } i = m \\ -x_{ijk} & \textit{if } i = m{+}1 \\ 0 & \textit{otherwise} \end{cases} \qquad T_m^{(1)} \cdot x_{ijk} = \begin{cases} x_{i-1,j,k} & \textit{if } i = m{+}1 \\ 0 & \textit{otherwise} \end{cases}$$

$$H_m^{(2)} \cdot x_{ijk} = \begin{cases} x_{ijk} & \textit{if } j = m \\ -x_{ijk} & \textit{if } j = m{+}1 \\ 0 & \textit{otherwise} \end{cases} \qquad T_m^{(2)} \cdot x_{ijk} = \begin{cases} x_{i,j-1,k} & \textit{if } j = m{+}1 \\ 0 & \textit{otherwise} \end{cases}$$

$$H_m^{(3)} \cdot x_{ijk} = \begin{cases} x_{ijk} & \textit{if } k = m \\ -x_{ijk} & \textit{if } k = m{+}1 \\ 0 & \textit{otherwise} \end{cases} \qquad T_m^{(3)} \cdot x_{ijk} = \begin{cases} x_{i,j,k-1} & \textit{if } k = m{+}1 \\ 0 & \textit{otherwise} \end{cases}$$

The action of a Lie algebra L on a tensor product $V \otimes W$ of representations is given by the derivation rule:

$$A \cdot (v \otimes w) = (A \cdot v) \otimes w + v \otimes (A \cdot w), \qquad A \in L, \, v \in V, \, w \in W.$$

We identify the d-th symmetric power S^dV of the p-dimensional representation V with the space of homogeneous polynomials of degree d on a basis $v_1, \dots, v_p$ of V. It follows by induction on d that the action of L on S^dV is given by

$$A \cdot (v_1^{e_1} v_2^{e_2} \cdots v_p^{e_p}) = \sum_{i=1}^{p} v_1^{e_1} \cdots (A \cdot v_i^{e_i}) \cdots v_p^{e_p}$$
$$= \sum_{i=1}^{p} v_1^{e_1} \cdots \big(e_i v_i^{e_i - 1} (A \cdot v_i) \big) \cdots v_p^{e_p} = \sum_{i=1}^{p} e_i \, v_1^{e_1} \cdots v_i^{e_i - 1} \cdots v_p^{e_p} (A \cdot v_i).$$

We apply this to $L = \mathfrak{sl}_{333}(\mathbb{C})$ and $V = \mathbb{C}^{333}$.

Lemma 2.5. *For $m = 1, 2$ we have*

$$H_m^{(1)} \cdot \big(x_{111}^{e_{111}} \cdots x_{333}^{e_{333}} \big) = \sum_{j=1}^{3} \sum_{k=1}^{3} (e_{m,j,k} - e_{m+1,j,k}) \, x_{111}^{e_{111}} \cdots x_{333}^{e_{333}},$$
$$H_m^{(2)} \cdot \big(x_{111}^{e_{111}} \cdots x_{333}^{e_{333}} \big) = \sum_{i=1}^{3} \sum_{k=1}^{3} (e_{i,m,k} - e_{i,m+1,k}) \, x_{111}^{e_{111}} \cdots x_{333}^{e_{333}},$$
$$H_m^{(3)} \cdot \big(x_{111}^{e_{111}} \cdots x_{333}^{e_{333}} \big) = \sum_{i=1}^{3} \sum_{j=1}^{3} (e_{i,j,m} - e_{i,j,m+1}) \, x_{111}^{e_{111}} \cdots x_{333}^{e_{333}}.$$

Lemma 2.6. *For $m = 1, 2$ we have*

$$T_m^{(1)} \cdot \big(x_{111}^{e_{111}} \cdots x_{333}^{e_{333}} \big) = \sum_{j=1}^{3} \sum_{k=1}^{3} e_{m+1,j,k} \, x_{111}^{e_{111}} \cdots x_{m,j,k}^{e_{m,j,k}+1} \cdots x_{m+1,j,k}^{e_{m+1,j,k}-1} \cdots x_{333}^{e_{333}},$$
$$T_m^{(2)} \cdot \big(x_{111}^{e_{111}} \cdots x_{333}^{e_{333}} \big) = \sum_{i=1}^{3} \sum_{k=1}^{3} e_{i,m+1,k} \, x_{111}^{e_{111}} \cdots x_{i,m,k}^{e_{i,m,k}+1} \cdots x_{i,m+1,k}^{e_{i,m+1,k}-1} \cdots x_{333}^{e_{333}},$$
$$T_m^{(3)} \cdot \big(x_{111}^{e_{111}} \cdots x_{333}^{e_{333}} \big) = \sum_{i=1}^{3} \sum_{j=1}^{3} e_{i,j,m+1} \, x_{111}^{e_{111}} \cdots x_{i,j,m}^{e_{i,j,m}+1} \cdots x_{i,j,m+1}^{e_{i,j,m+1}-1} \cdots x_{333}^{e_{333}}.$$

Definition 2.7. The eigenvalues of the basis monomial $x_{111}^{e_{111}} \cdots x_{333}^{e_{333}}$ with respect to the elements $H_1^{(1)}$, $H_2^{(1)}$, $H_1^{(2)}$, $H_2^{(2)}$, $H_1^{(3)}$, $H_2^{(3)}$ will be denoted by

$$\omega_{11} = \sum_{j=1}^{3} \sum_{k=1}^{3} (e_{1jk} - e_{2jk}), \qquad \omega_{12} = \sum_{j=1}^{3} \sum_{k=1}^{3} (e_{2jk} - e_{3jk}),$$
$$\omega_{21} = \sum_{i=1}^{3} \sum_{k=1}^{3} (e_{i1k} - e_{i2k}), \qquad \omega_{22} = \sum_{i=1}^{3} \sum_{k=1}^{3} (e_{i2k} - e_{i3k}),$$
$$\omega_{31} = \sum_{i=1}^{3} \sum_{j=1}^{3} (e_{ij1} - e_{ij2}), \qquad \omega_{32} = \sum_{i=1}^{3} \sum_{j=1}^{3} (e_{ij2} - e_{ij3}).$$

Definition 2.8. The **weight** of a monomial $M = M(E)$ is the ordered 6-tuple

$$\Omega(E) = (\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}, \omega_{31}, \omega_{32}).$$

A **weight space** is a subspace of P spanned by the monomials of a given degree d and a given weight Ω, and will be denoted $W(\, d \mid \Omega \,)$. A monomial $M = M(E)$ has **weight zero** if $\Omega(E) = O$; that is, $\omega_{\ell m} = 0$ for $\ell = 1, 2, 3$ and $m = 1, 2$. We call $W(\,d \mid O\,)$ the **zero weight space** in degree d. A monomial $M = M(E)$ has **higher weight** if $\Omega(E) = \Omega_{\ell m}$ for some $\ell = 1, 2, 3$ and $m = 1, 2$ where

$$\begin{array}{lll} \Omega_{1,1} = (2,-1,0,0,0,0), & \Omega_{2,1} = (0,0,2,-1,0,0), & \Omega_{3,1} = (0,0,0,0,2,-1), \\ \Omega_{1,2} = (-1,2,0,0,0,0), & \Omega_{2,2} = (0,0,-1,2,0,0), & \Omega_{3,2} = (0,0,0,0,-1,2). \end{array}$$

We call $W(\,d \mid \Omega_{\ell m}\,)$ for $\ell = 1, 2, 3$ and $m = 1, 2$ the **higher weight spaces** in degree d.

Definition 2.9. By Lemma 2.3 the actions of $T_m^{(\ell)}$ induce these linear maps:

$$\begin{array}{ll} T_1^{(1)} \colon W(\,d \mid O\,) \longrightarrow W(\,d \mid \Omega_{1,1}\,), & T_2^{(1)} \colon W(\,d \mid O\,) \longrightarrow W(\,d \mid \Omega_{1,2}\,), \\ T_1^{(2)} \colon W(\,d \mid O\,) \longrightarrow W(\,d \mid \Omega_{2,1}\,), & T_2^{(2)} \colon W(\,d \mid O\,) \longrightarrow W(\,d \mid \Omega_{2,2}\,), \\ T_1^{(3)} \colon W(\,d \mid O\,) \longrightarrow W(\,d \mid \Omega_{3,1}\,), & T_2^{(3)} \colon W(\,d \mid O\,) \longrightarrow W(\,d \mid \Omega_{3,2}\,). \end{array}$$

Definition 2.10. We form the direct sum of these linear maps:

$$\Lambda_d = (T_1^{(1)}, \dots, T_2^{(3)}) \colon W(\,d \mid O\,) \longrightarrow \bigoplus_{\ell=1}^{3} \bigoplus_{m=1}^{2} W(\,d \mid \Omega_{\ell m}\,).$$

Lemma 2.11. *The invariant polynomials in degree d for the action of the Lie algebra $\mathfrak{sl}_{333}(\mathbb{C})$ on the space $\mathbb{C}^{333}$ of $3 \times 3 \times 3$ arrays $X = (x_{ijk})$ are the (nonzero) elements of the kernel of the linear map Λ_d.*

Theorem 2.12. *The algebra of invariants for $\mathfrak{sl}_{333}(\mathbb{C})$ is freely generated by three fundamental invariants in degrees 6, 9 and 12.*

Proof. Vinberg [12], especially item 2 in the table on page 491. □

Corollary 2.13. *For $d = 6$ and $d = 9$, the nullspace of Λ_d has dimension 1, but for $d = 12$, the nullspace of Λ_d has dimension 2.*

2.3. **Combinatorics of monomials.** Let $E = (e_{ijk})$ be a $3 \times 3 \times 3$ array with non-negative integer entries.

Definition 2.14. The **flattening** of an exponent array E is the ordered list obtained by applying lexicographical order to the subscripts:

$$\mathrm{flat}(E) = [e_{111}, e_{112}, e_{113}, e_{121}, e_{122}, e_{123}, \dots, e_{331}, e_{332}, e_{333}].$$

The **total order** on exponent arrays is defined to be the lex order on flattenings: that is, $E < E'$ for $\mathrm{flat}(E) = [f_1, f_2, \dots, f_{27}]$ and $\mathrm{flat}(E') = [f'_1, f'_2, \dots, f'_{27}]$ if and only if $f_i < f'_i$ where i is the least index with $f_i \neq f'_i$. The **matrix form** of E is:

$$\left[\begin{array}{ccc|ccc|ccc} e_{111} & e_{121} & e_{131} & e_{112} & e_{122} & e_{132} & e_{113} & e_{123} & e_{133} \\ e_{211} & e_{221} & e_{231} & e_{212} & e_{222} & e_{232} & e_{213} & e_{223} & e_{233} \\ e_{311} & e_{321} & e_{331} & e_{212} & e_{222} & e_{232} & e_{213} & e_{223} & e_{233} \end{array} \right].$$

The third index distinguishes the three 3×3 matrices, the frontal slices of E.

Definition 2.15. A **slice** of E is a 3×3 matrix obtained by fixing one subscript; there are three parallel slices in each of the three directions. We call E an **equal parallel slice (EPS) array** if all parallel slices have the same entry sum.

Lemma 2.16. *A basis for the zero weight space in degree d consists of the monomials of weight d whose exponent array are EPS arrays.*

Proof. This follows immediately from Lemma 2.5. □

- set `weightzeromonomials` $\leftarrow [\,]$ *(empty list)*
- for f_1 from 0 to $d/3$ do for f_2 from 0 to $d/3 - f_1$ do ...
  for f_8 from 0 to $d/3 - (f_1 + \cdots + f_7)$ do:
    - · set $f_9 \leftarrow d/3 - (f_1 + \cdots + f_8)$
    - · for f_{10} from 0 to $d/3$ do for f_{11} from 0 to $d/3 - f_{10}$ do ...
      for f_{17} from 0 to $d/3 - (f_{10} + \cdots + f_{16})$ do:
        - · set $f_{18} \leftarrow d/3 - (f_{10} + \cdots + f_{17})$
        - · for f_{19} from 0 to $d/3$ do for f_{20} from 0 to $d/3 - f_{19}$ do ...
          for f_{26} from 0 to $d/3 - (f_{19} + \cdots + f_{25})$ do:
            - · set $f_{27} \leftarrow d/3 - (f_{19} + \cdots + f_{26})$
            - · set $e \leftarrow$ `unflatten`(f)
            - · if $\sum_{j,k} e_{1jk} = \sum_{j,k} e_{2jk} = \sum_{j,k} e_{3jk}$ and
              $\sum_{i,k} e_{i1k} = \sum_{i,k} e_{i2k} = \sum_{i,k} e_{i3k}$ and
              $\sum_{i,j} e_{ij1} = \sum_{i,j} e_{ij2} = \sum_{i,j} e_{ij3}$ then
              append $[f_1, \dots, f_{27}]$ to `weightzeromonomials`
- return `weightzeromonomials`

Table 1. Pseudocode to generate weight zero monomials of degree d

- set $\ell \leftarrow 1$
- for $m = 1, 2$ do
    - · set `higherweightmonomials`$[\ell, m] \leftarrow [\,]$ *(empty list)*
    - · for $x \in$ `weightzeromonomials` do
        - · set $e \leftarrow$ `unflatten`(x)
        - · for $j = 1, 2, 3$ do for $k = 1, 2, 3$ do
            - · if $e_{m+1,j,k} > 0$ then
              set $f \leftarrow e$
              set $f_{m+1,j,k} \leftarrow f_{m+1,j,k} - 1$
              set $f_{m,j,k} \leftarrow f_{m,j,k} + 1$
              append `flatten`(f) to `higherweightmonomials`$[\ell, m]$
- return `higherweightmonomials`$[\ell, 1]$, `higherweightmonomials`$[\ell, 2]$

Table 2. Pseudocode to generate higher weight monomials ($\ell = 1$)

Corollary 2.17. *In degree d, there are no monomials of weight zero, and hence no invariants, unless d is a multiple of 3.*

To generate the exponent arrays for weight zero monomials of degree d, we use nested do loops together with the condition that the horizontal parallel slices have entry sum $d/3$. See Table 1, where the procedure `unflatten` takes a list representing a flattened array and returns the array.

To generate the higher weight monomials, we first use Lemma 2.6 to generate the monomials for weights Ω_{1m} ($m = 1, 2$). We then use symmetry to obtain the monomials for weights $\Omega_{\ell m}$ ($\ell = 2, 3,\ m = 1, 2$): if E is an exponent array for a monomial of weight Ω_{1m} then E' defined by $e'_{ijk} = e_{jik}$ is an exponent array for a monomial of weight Ω_{2m}, and similarly for monomials of weight Ω_{3m}. See Table 2, where the procedure `flatten` takes an array and returns its flattened form.

- `smallgrouporbit`(f)
  - · set $e \leftarrow$ `unflatten`(f)
  - · set `orbit` $\leftarrow \{\,\}$
  - · for $p \in S_3$ do for $q \in S_3$ do for $r \in S_3$ do:
    - · for $i = 1, 2, 3$ do for $j = 1, 2, 3$ do for $k = 1, 2, 3$ do:
      set $f_{i,j,k} \leftarrow e_{p(i),q(j),r(k)}$
    - · set `orbit` $\leftarrow$ `orbit` $\cup \{$`flatten`$(f)\}$
  - · return `orbit`

- `largegrouporbit`(f)
  - · set $e \leftarrow$ `unflatten`(f)
  - · set `orbit` $\leftarrow \{\,\}$
  - · for $s \in S_3$ do:
    - · for $i = 1, 2, 3$ do for $j = 1, 2, 3$ do for $k = 1, 2, 3$ do:
      set $t \leftarrow (i, j, k)$
      set $f_{i,j,k} \leftarrow e_{t_{s(1)},t_{s(2)},t_{s(3)}}$
    - · `orbit` $\leftarrow$ `orbit` $\cup$ `smallgrouporbit`(`flatten`(f))
  - · return `orbit`

TABLE 3. Pseudocode to generate the orbits of a weight zero monomial

Definition 2.18. The **symmetry group** of the set of exponent arrays E for weight zero monomials $X(E)$ is the semidirect product

$$G = (S_3 \times S_3 \times S_3) \rtimes S_3.$$

Each factor in the normal subgroup $S_3 \times S_3 \times S_3$ acts by permutation of the parallel slices in the corresponding direction; the last copy of S_3 acts by permuting the directions. More precisely, the group elements (α, β, γ) and δ act on $E = (e_{ijk})$ by:

$$\big((\alpha, \beta, \gamma) \cdot E\big)_{i,j,k} = e_{\alpha(i),\beta(j),\gamma(k)}, \qquad \big(\delta \cdot E\big)_{i,j,k} = e_{i^\delta,j^\delta,k^\delta}.$$

Definition 2.19. For a weight zero monomial $M = M(E)$ we define the **orbit** under the action of the symmetry group G to be the set

$$\mathcal{O}(M) = \{\, M(g \cdot E) \mid g \in G \,\}.$$

The **symmetric orbit sum** for a monomial $M = M(E)$ is the sum of the monomials in the orbit, each occurring with coefficient 1:

$$\mathcal{O}^+(M) = \frac{|\mathcal{O}(M)|}{|G|} \sum_{g \in G} M(g \cdot E) = \sum_{Y \in \mathcal{O}(M)} Y.$$

The **alternating orbit sum** for a monomial $M = M(E)$ is

$$\mathcal{O}^-(M) = \frac{|\mathcal{O}(M)|}{|G|} \sum_{g \in G} \epsilon(g)\, M(g \cdot E),$$

where $\epsilon(g)$ is the product of the signs of the components of $g = (\alpha, \beta, \gamma, \delta) \in G$. For some monomials, the alternating orbit sum will be zero.

To generate the orbit of a weight zero monomial, we use the algorithms in Table 3: `smallgrouporbit` applies the permutations of the slices, and `largegrouporbit` applies the permutations of the directions.

- set $B \leftarrow$ Matrix(1944, 1152) modulo 101
- set $\ell \leftarrow 1$
- set $c \leftarrow 0$
- for $x \in$ `weightzeromonomials` do
  - · set $c \leftarrow c + 1$
  - · for $m = 1, 2$ do
    - · set $e \leftarrow$ `unflatten`(x)
    - · for $j = 1, 2, 3$ do for $k = 1, 2, 3$ do
      - · if $e_{m+1,j,k} > 0$ then
        set $f \leftarrow e$
        set $f_{m+1,j,k} \leftarrow f_{m+1,j,k} - 1$
        set $f_{m,j,k} \leftarrow f_{m,j,k} + 1$
        set $g \leftarrow$ `flatten`(f)
        set $r \leftarrow$ `monomialindex`(g, ℓ, m):
        set $B_{1152+r,c} \leftarrow (B_{1152+r,c} + e_{m+1,j,k})$ modulo 101
- compute the row canonical form of B

TABLE 4. Pseudocode to fill the matrix ($d = 6$, $\ell = 1$)

3. Degree 6

In this section we find an explicit form of the invariant polynomial in degree 6.

Lemma 3.1. *In degree 6, there are 1152 monomials of weight zero, and 792 monomials of each higher weight $\Omega_{\ell m}$ for $\ell = 1, 2, 3$ and $m = 1, 2$.*

Proof. We generate the monomials using an implementation in Maple 15 of the algorithms in Tables 1 and 2 with $d = 6$. □

Lemma 3.2. *The nullspace of the matrix representing the linear map L_6 has dimension 1 using arithmetic modulo $p = 101$. Using symmetric representatives, the canonical basis vector of the nullspace has coefficients $\{-10, -4, -2, 1, 2, 4, 8\}$. If we interpret these coefficients as integers then the corresponding polynomial is an invariant for the action of $\mathfrak{sl}_{333}(\mathbb{C})$.*

Proof. We use the `LinearAlgebra[Modular]` package in Maple 15 to create a matrix B of size 1944×1152 with an upper block of size 1152×1152 and a lower block of size 792×1152. For $\ell = 1, 2, 3$ and $m = 1, 2$ we apply the formulas of Lemma 2.6 to store the matrix representing the linear map $T_m^{(\ell)}$ in the lower block of B and then compute the row canonical form of B. See Table 4 for the case $\ell = 1$, where `monomialindex` is a binary search for a higher weight monomial of given weight in the lexicographically ordered list of all higher weight monomials of that weight. At the end of this computation, the matrix B has rank 1151. From the row canonical form of B, we extract a basis vector for the nullspace. We then perform another computation using rational arithmetic to verify that the corresponding polynomial is indeed an invariant when the modular coefficients are regarded as integers. □

Lemma 3.3. *The coefficients of the canonical basis vector for the nullspace of L_6 are constant on the orbits for the action of the symmetry group on the weight zero monomials. The coefficients, the orbit sizes, and the matrix forms of the minimal representatives of each orbit, are displayed in Table 5.*

I_6	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
-10	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&1&0 \\ 0&0&1&0&0&0&1&0&0 \\ 0&1&0&1&0&0&0&0&0 \end{array}\right]$	36	1
-4	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&0&1 \\ 0&1&0&0&0&0&1&0&0 \\ 1&0&0&0&1&0&0&0&0 \end{array}\right]$	324	2
-2	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&0&0&0&2 \\ 0&1&0&0&1&0&0&0&0 \\ 1&0&0&1&0&0&0&0&0 \end{array}\right]$	162	3
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&0&0&0&2 \\ 0&0&0&0&2&0&0&0&0 \\ 2&0&0&0&0&0&0&0&0 \end{array}\right]$	36	4
2	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&0&1 \\ 0&1&0&0&0&0&0&1&0 \\ 1&0&0&1&0&0&0&0&0 \end{array}\right]$	108	5
2	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&0&1 \\ 0&0&0&0&1&0&1&0&0 \\ 1&1&0&0&0&0&0&0&0 \end{array}\right]$	324	6
4	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&0&0&0&2 \\ 0&1&0&1&0&0&0&0&0 \\ 1&0&0&0&1&0&0&0&0 \end{array}\right]$	54	7
8	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&1&0 \\ 0&0&1&0&0&0&1&0&0 \\ 1&0&0&0&1&0&0&0&0 \end{array}\right]$	108	8

TABLE 5. The basis invariant degree 6

Proof. We implemented the algorithms of Table 3 in Maple 15. □

Theorem 3.4. *Every invariant polynomial in degree 6 for the* $3 \times 3 \times 3$ *array* $X = (x_{ijk})$ *is a scalar multiple of*

$$\begin{aligned} &- 10\, \mathcal{O}^+\big(x_{123}x_{132}x_{213}x_{231}x_{312}x_{321} \big) - 4\, \mathcal{O}^+\big(x_{132}x_{133}x_{213}x_{221}x_{311}x_{322} \big) \\ &- 2\, \mathcal{O}^+\big(x_{133}^2x_{221}x_{222}x_{311}x_{312} \big) + \mathcal{O}^+\big(x_{133}^2x_{222}^2x_{311}^2 \big) \\ &+ 2\, \mathcal{O}^+\big(x_{132}x_{133}x_{221}x_{223}x_{311}x_{312} \big) + 2\, \mathcal{O}^+\big(x_{132}x_{133}x_{213}x_{222}x_{311}x_{321} \big) \\ &+ 4\, \mathcal{O}^+\big(x_{133}^2x_{212}x_{221}x_{311}x_{322} \big) + 8\, \mathcal{O}^+\big(x_{123}x_{132}x_{213}x_{231}x_{311}x_{322} \big). \end{aligned}$$

Proof. This is a restatement of the results in Table 5. □

4. DEGREE 9

In this section we find an explicit form of the invariant polynomial in degree 9.

Lemma 4.1. *In degree 9, there are 22620 monomials of weight zero, and 17802 monomials of each higher weight* $\Omega_{\ell m}$ *for* $\ell = 1, 2, 3$ *and* $m = 1, 2$.

Proof. Similar to the proof of Lemma 3.1. □

Lemma 4.2. *The nullspace of the matrix representing the linear map* L_9 *has dimension 1 using arithmetic modulo* $p = 101$. *Using symmetric representatives, the canonical basis vector of the nullspace has coefficients* $\{-1, 0, 1\}$*; coefficients* ± 1 *each occur 4608 times and coefficient 0 occurs 13404 times. If we interpret these coefficients as integers then the corresponding polynomial is an invariant for the action of* $\mathfrak{sl}_{333}(\mathbb{C})$.

Proof. Similar to the proof of Lemma 3.2. □

I_9	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&1&1 \\ 0&1&0&0&0&1&1&0&0 \\ 2&0&0&0&1&0&0&0&0 \end{array}\right]$	648	1
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&1&1 \\ 0&1&1&0&0&0&1&0&0 \\ 0&1&0&2&0&0&0&0&0 \end{array}\right]$	648	2
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&1&1 \\ 0&1&0&0&1&0&1&0&0 \\ 1&0&1&1&0&0&0&0&0 \end{array}\right]$	1296	3
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&1&1 \\ 0&2&0&0&0&0&1&0&0 \\ 1&0&0&1&0&1&0&0&0 \end{array}\right]$	648	4
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&1&1 \\ 0&0&1&1&0&0&0&1&0 \\ 1&1&0&1&0&0&0&0&0 \end{array}\right]$	648	5
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&1&1 \\ 0&0&1&1&1&0&0&0&0 \\ 2&0&0&0&0&0&0&1&0 \end{array}\right]$	1296	6
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&1&1 \\ 0&1&0&1&0&1&0&0&0 \\ 1&1&0&0&0&0&1&0&0 \end{array}\right]$	648	7
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&1&1 \\ 0&1&1&1&0&0&0&0&0 \\ 1&0&0&0&1&0&1&0&0 \end{array}\right]$	1296	8
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&1&1 \\ 0&1&0&1&0&0&1&0&0 \\ 0&1&1&1&0&0&0&0&0 \end{array}\right]$	216	9
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&1&1 \\ 0&0&1&2&0&0&0&0&0 \\ 0&2&0&0&0&0&1&0&0 \end{array}\right]$	648	10
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&1&0&1&1 \\ 1&0&0&1&0&1&0&0&0 \\ 1&1&0&0&0&0&0&1&0 \end{array}\right]$	72	11
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&2&0&1&0 \\ 0&1&1&0&0&0&1&0&0 \\ 1&0&0&1&0&0&0&1&0 \end{array}\right]$	432	12
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&2&0&1&0 \\ 0&1&0&0&1&0&1&0&0 \\ 2&0&0&0&0&0&0&0&1 \end{array}\right]$	648	13
1	$\left[\begin{array}{ccc\|ccc\|ccc} 0&0&0&0&0&2&0&1&0 \\ 0&0&1&0&0&0&2&0&0 \\ 0&2&0&1&0&0&0&0&0 \end{array}\right]$	72	14

TABLE 6. The basis invariant in degree 9

Lemma 4.3. *The canonical basis vector for the nullspace of L_9 is a linear combination of alternating orbit sums for the action of the symmetry group on the weight zero monomials. The coefficients, the orbit sizes, and the matrix forms of the minimal representatives of each orbit, are displayed in Table 6.*

Proof. Similar to the proof of Lemma 3.3. (There are another 30 orbits for the action of the symmetry group but they all occur with coefficient 0.) □

Theorem 4.4. *Every invariant polynomial in degree 9 for the $3 \times 3 \times 3$ array $X = (x_{ijk})$ is a scalar multiple of*

$$
\begin{aligned}
&\mathcal{O}^-\big(x_{123}x_{132}x_{133}x_{213}x_{221}x_{232}x_{311}^2x_{322}\big) \\
+&\mathcal{O}^-\big(x_{123}x_{132}x_{133}x_{213}x_{221}x_{231}x_{312}^2x_{321}\big) \\
+&\mathcal{O}^-\big(x_{123}x_{132}x_{133}x_{213}x_{221}x_{222}x_{311}x_{312}x_{331}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}^{-}\big(\,x_{123}x_{132}x_{133}x_{213}x_{221}^2x_{311}x_{312}x_{332}\big)\\
&+\mathcal{O}^{-}\big(\,x_{123}x_{132}x_{133}x_{212}x_{223}x_{231}x_{311}x_{312}x_{321}\big)\\
&+\mathcal{O}^{-}\big(\,x_{123}x_{132}x_{133}x_{212}x_{222}x_{231}x_{311}^2x_{323}\big)\\
&+\mathcal{O}^{-}\big(\,x_{123}x_{132}x_{133}x_{212}x_{221}x_{232}x_{311}x_{313}x_{321}\big)\\
&+\mathcal{O}^{-}\big(\,x_{123}x_{132}x_{133}x_{212}x_{221}x_{231}x_{311}x_{313}x_{322}\big)\\
&+\mathcal{O}^{-}\big(\,x_{123}x_{132}x_{133}x_{212}x_{213}x_{221}x_{312}x_{321}x_{331}\big)\\
&+\mathcal{O}^{-}\big(\,x_{123}x_{132}x_{133}x_{212}^2x_{231}x_{313}x_{321}^2\big)\\
&+\mathcal{O}^{-}\big(\,x_{123}x_{132}x_{133}x_{211}x_{212}x_{232}x_{311}x_{321}x_{323}\big)\\
&+\mathcal{O}^{-}\big(\,x_{123}x_{132}^2x_{213}x_{221}x_{231}x_{311}x_{312}x_{323}\big)\\
&+\mathcal{O}^{-}\big(\,x_{123}x_{132}^2x_{213}x_{221}x_{222}x_{311}^2x_{333}\big)\\
&+\mathcal{O}^{-}\big(\,x_{123}x_{132}^2x_{213}^2x_{231}x_{312}x_{321}^2\big).
\end{aligned}
$$

Proof. This is a restatement of the results in Table 6. □

5. Degree 12

In this section we find explicit forms of two linearly independent invariant polynomials in degree 12, and verify that the simpler of these two invariants can be taken as the generator in degree 12.

Lemma 5.1. *In degree 12, there are 302274 monomials of weight zero, and 254961 monomials of each higher weight $\Omega_{\ell m}$ for $\ell = 1, 2, 3$ and $m = 1, 2$.*

Even using modular arithmetic, it is impossible to process efficiently a matrix with 302274 columns and $302274 + 254961 = 557235$ rows. Therefore we look for invariants that are linear combinations of the symmetric orbit sums.

Lemma 5.2. *There are 359 orbits of weight zero monomials in degree 12 for the group $(S_3 \times S_3 \times S_3) \rtimes S_3$. The orbit sizes, and the minimal representative of each orbit in the total order of Definition 2.14, are given in columns 3–4 of Tables 7–11.*

It follows that we must consider the nullspace of a matrix with 359 columns; each column corresponds to one of the symmetric orbit sums.

Lemma 5.3. *The nullspace of the matrix representing the restriction of the linear map L_{12} to the span of the symmetric orbit sums has dimension 2 modulo $p = 101$.*

Proof. We use modular arithmetic with $p = 101$ on a matrix M with 359 columns and $359 + 254961 = 255320$ rows consisting of an upper block of size 359×359 and a lower block of size 254961×359. For each $\ell = 1, 2, 3$ and $m = 1, 2$ we consider the restriction of the linear map $T_m^{(\ell)}$ to the subspace of $W(\,12 \mid O\,)$ spanned by the symmetric orbit sums, store the matrix representing this restriction in the lower block of M, and compute the row canonical form of M. After the first iteration ($\ell = m = 1$) the rank is 357, and does not increase for the remaining five iterations. The dimension of the nullspace is therefore $359 - 357 = 2$. □

Lemma 5.3 agrees with the results quoted in Corollary 2.13 to Vinberg's Theorem 2.12: a basis of the space of invariants in degree 12 consists of I_6^2 together with a new generator I_{12}. We must confirm this result with further computations in characteristic 0.

Lemma 5.4. *Let A be a matrix with entries in the ring $\mathbb{Z}$ of integers, let r_0 be its rank over the field $\mathbb{Q}$ of rational numbers, and let r_p be its rank over the field $\mathbb{F}_p$ with p elements. Then $r_p \leq r_0$ for every prime p. Hence the dimension of the nullspace of A over $\mathbb{Q}$ is no larger than the dimension of the nullspace over $\mathbb{F}_p$.*

Proof. The Smith normal form $S^{(0)} = (s^0_{ij})$ over $\mathbb{Q}$ satisfies $s^0_{ii} = 1$ for $1 \leq i \leq r_0$, other entries 0. Hence the Smith normal form $T = (t_{ij})$ over $\mathbb{Z}$ satisfies $t_{ii} \in \mathbb{Z}$, $t_{ii} > 0$ for $1 \leq i \leq r_0$, other entries 0. If $k \geq 0$ denotes the number of t_{ii} which are divisible by p, then the Smith normal form $S^{(p)} = (s^p_{ij})$ over $\mathbb{F}_p$ satisfies $s^p_{ii} = 1$ for $1 \leq i \leq r_0 - k$, other entries 0. Hence $r_p = r_0 - k$. □

Theorem 5.5. *The polynomials I_{12} and I'_{12} given in columns 1–2 of Tables 7–11 form a reduced basis for the lattice of invariant polynomials in degree 12 with integer coefficients.*

Proof. For each $\ell = 1, 2, 3$ and $m = 1, 2$ we totally order the set of 254961 higher weight monomials according to Definition 2.14. We partition each of these sets into 399 blocks each containing 639 consecutive monomials. For each $j = 1, 2, \ldots 359$ we compute the action of $T^{(\ell)}_m$ on the symmetric orbit sum corresponding to column j, and separate the terms of the image corresponding to the different blocks.

We create an integer matrix M with 359 columns and $359 + 639 = 998$ rows, with an upper block of size 359×359 and a lower block of size 639×359, initialized to zero. For each $\ell = 1, 2, 3$ and $m = 1, 2$ and $j = 1, 2, \ldots, 359$ and $k = 1, 2, \ldots, 399$ we consider the terms in block k obtained from symmetric orbit sum j under the action of $T^{(\ell)}_m$, store the integer coefficients of these higher weight monomials in the lower block of M, and compute the Hermite normal form (HNF) of M.

We obtain the same behavior as in the modular case (Lemma 5.3): after the first iteration ($\ell = m = 1$) the rank is 357, and does not increase for the remaining five iterations. At the end of this computation, we obtain an integer matrix of size 357×359, which we also call M, whose integer nullspace consists of the coefficient vectors of the invariant polynomials of degree 12.

To find a lattice basis for the integer nullspace of M, we compute the HNF of the transpose M^t, obtaining an integer matrix H of size 359×357 and an invertible integer matrix U of size 359×359 for which $UM^t = H$. The last two rows of U form a lattice basis for the integer nullspace of M.

We then perform lattice basis reduction on the last two rows of U to find a short basis of the integer nullspace of M. Since the nullspace lattice is 2-dimensional, the output consists of a shortest nonzero lattice vector and a shortest lattice vector which is not a multiple of the first vector. The components of these reduced nullspace basis vectors I_{12} and I'_{12} are given in columns 1–2 of Tables 7–11.

Finally, we perform an independent check on these two polynomials to confirm that they are annihilated by the action of $T^{(\ell)}_m$ for $\ell = 1, 2, 3$ and $m = 1, 2$. □

For an introduction to lattice basis reduction, with applications to computing the HNF of an integer matrix, see Bremner [1].

Lemma 5.6. *We have $I'_{12} = I_6^2 + 21\, I_{12}$. Hence I_{12} is the fundamental invariant in degree 12; it involves 235 of the 359 orbits, with a total of 209061 monomials.*

Proof. We computed the expression for I_6^2 as a linear combination of the basis invariants in degree 12, obtaining $I_6^2 = I'_{12} - 21 I_{12}$. This computation can be performed very efficiently in Maple 15 using the algorithms of Monagan and Pearce [9, 10] which are based on the work of Johnson [8]. □

We can now state our main result.

Theorem 5.7. *The three fundamental invariants for* $3 \times 3 \times 3$ *arrays are:*

- *the polynomial* I_6 *of Table 5, which is a linear combination of the 8 symmetric orbits in degree 6, and has altogether 1152 terms;*
- *the polynomial* I_9 *of Table 6, which is the sum of 14 alternating orbits in degree 9, and has altogether 9216 terms;*
- *the polynomial* I_{12} *of Tables 7–11, which is a linear combination of 235 symmetric orbits in degree 12, and has altogether 209061 terms.*

Corollary 3.9 of Gelfand et al. [6] implies that the hyperdeterminant of a $3 \times 3 \times 3$ array has degree 36, and hence has the form

$$a\, I_6^6 + b\, I_6^4 I_{12} + c\, I_6^3 I_9^2 + d\, I_6^2 I_{12}^2 + e\, I_6 I_9^2 I_{12} + f\, I_9^4 + g\, I_{12}^3,$$

for some $a, b, c, d, e, f, g \in \mathbb{C}$. It is an open problem to determine these coefficients.

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I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
0	1	000000004000040000400000000	36	1
0	−4	000000004000130000310000000	324	2
0	6	000000004000220000220000000	162	3
0	4	000000004010120000210100000	324	4
0	8	000000004010120000300010000	216	5
0	−16	000000004010210000210010000	324	6
0	16	000000004020200000200020000	54	7
0	56	000000004110110000110110000	27	8
0	12	000000013000121000310000000	324	9
0	4	000000013000130000301000000	216	10
0	−12	000000013000211000220000000	648	11
0	4	000000013001030000300100000	648	12
0	−4	000000013001120000210100000	1296	13
0	−8	000000013001120000300010000	1296	14
0	16	000000013001210000210010000	648	15
0	12	000000013010021000300100000	648	16
0	−8	000000013010111000210100000	1296	17
0	−16	000000013010111000300010000	648	18
0	−4	000000013010120000201100000	1296	19
0	16	000000013010201000210010000	648	20
0	−12	000000013011020000200200000	1296	21
0	−8	000000013011110000110200000	648	22
0	32	000000013011110000200110000	1296	23
0	−32	000000013011200000200020000	648	24
0	−4	000000013020101000110200000	1296	25
0	16	000000013020101000200110000	1296	26
0	16	000000013020110000200101000	1296	27
0	−24	000000013021100000100210000	648	28
0	−8	000000013030100000100201000	648	29
0	32	000000013100111000120100000	648	30
0	−56	000000013101110000110110000	648	31
0	6	000000022000202000220000000	108	32
0	24	000000022000211000211000000	162	33
0	8	000000022001111000210100000	1296	34
0	16	000000022001111000300010000	648	35
0	4	000000022001120000201100000	1296	36
0	8	000000022001120000300001000	648	37
0	−16	000000022001201000210010000	648	38
0	−16	000000022001210000210001000	324	39
0	6	000000022002020000200200000	324	40
0	4	000000022002110000110200000	648	41
0	−16	000000022002110000200110000	1296	42
0	16	000000022002200000200020000	324	43
0	24	000000022011011000200200000	162	44
0	8	000000022011101000110200000	648	45
0	−32	000000022011101000200110000	1296	46
0	64	000000022011200000200011000	162	47
0	24	000000022012100000100210000	648	48
0	−16	000000022100102000120100000	324	49
0	−64	000000022100111000111100000	162	50
0	56	000000022101101000110110000	324	51
0	56	000000022101110000110101000	162	52
0	−12	000000112000121000211000000	108	53
0	28	000000112001021000210100000	1296	54
0	−4	000000112001021000300010000	1296	55
0	−8	000000112001030000300001000	324	56
0	−24	000000112001111000120100000	1296	57
0	8	000000112001120000210001000	324	58
0	−16	000000112002020000110200000	1296	59
0	4	000000112002020000200110000	648	60
0	56	000000112002110000110110000	324	61
0	8	000000112010012000210100000	1296	62
0	16	000000112010012000300010000	648	63
0	−32	000000112010021000201100000	648	64
0	−32	000000112010102000210010000	648	65
0	32	000000112010111000111100000	324	66
0	16	000000112010111000201010000	648	67
0	−24	000000112011011000110200000	1296	68
0	8	000000112011020000101200000	648	69
0	28	000000112011020000200101000	648	70
0	−8	000000112011101000110110000	648	71
0	48	000000112011101000200020000	648	72

TABLE 7. The two basis invariants in degree 12: page 1

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
0	72	0 0 0 0 0 0 1 1 2 0 1 1 1 1 0 0 0 0 1 1 0 1 0 1 0 0 0	324	73
0	−64	0 0 0 0 0 0 1 1 2 0 1 1 1 1 0 0 0 0 2 0 0 0 1 1 0 0 0	648	74
0	−12	0 0 0 0 0 1 0 1 2 0 0 1 0 2 0 1 0 0 2 1 0 1 0 0 0 0 0	324	75
0	16	0 0 0 0 0 1 0 1 2 0 0 1 0 2 0 1 0 0 3 0 0 0 1 0 0 0 0	648	76
1	13	0 0 0 0 0 1 0 1 2 0 0 1 1 1 0 0 1 0 2 1 0 1 0 0 0 0 0	648	77
−1	−5	0 0 0 0 0 1 0 1 2 0 0 1 1 1 0 0 1 0 3 0 0 0 1 0 0 0 0	1296	78
−1	19	0 0 0 0 0 1 0 1 2 0 0 1 1 1 0 1 0 0 1 2 0 1 0 0 0 0 0	1296	79
1	−27	0 0 0 0 0 1 0 1 2 0 0 1 1 1 0 1 0 0 2 1 0 0 1 0 0 0 0	1296	80
−1	11	0 0 0 0 0 1 0 1 2 0 0 1 1 2 0 0 0 0 2 1 0 0 0 0 1 0 0	1296	81
1	13	0 0 0 0 0 1 0 1 2 0 0 1 1 2 0 0 0 0 3 0 0 0 0 0 0 1 0	1296	82
−1	−9	0 0 0 0 0 1 0 1 2 0 0 1 2 0 0 0 1 0 1 2 0 1 0 0 0 0 0	1296	83
1	5	0 0 0 0 0 1 0 1 2 0 0 1 2 0 0 0 1 0 2 1 0 0 1 0 0 0 0	1296	84
1	−7	0 0 0 0 0 1 0 1 2 0 0 1 2 0 0 1 0 0 0 3 0 1 0 0 0 0 0	648	85
2	2	0 0 0 0 0 1 0 1 2 0 0 1 2 1 0 0 0 0 1 2 0 0 0 0 1 0 0	1296	86
−2	−26	0 0 0 0 0 1 0 1 2 0 0 1 2 1 0 0 0 0 2 1 0 0 0 0 0 1 0	1296	87
−1	−5	0 0 0 0 0 1 0 1 2 0 0 1 3 0 0 0 0 0 0 3 0 0 0 0 1 0 0	648	88
1	13	0 0 0 0 0 1 0 1 2 0 0 1 3 0 0 0 0 0 1 2 0 0 0 0 0 1 0	648	89
−2	−18	0 0 0 0 0 1 0 1 2 0 1 0 1 0 1 0 1 0 2 1 0 1 0 0 0 0 0	324	90
2	26	0 0 0 0 0 1 0 1 2 0 1 0 1 0 1 0 1 0 3 0 0 0 1 0 0 0 0	648	91
1	13	0 0 0 0 0 1 0 1 2 0 1 0 1 0 1 1 0 0 1 2 0 1 0 0 0 0 0	1296	92
−1	−21	0 0 0 0 0 1 0 1 2 0 1 0 1 0 1 1 0 0 2 1 0 0 1 0 0 0 0	1296	93
0	16	0 0 0 0 0 1 0 1 2 0 1 0 1 1 0 0 0 1 2 1 0 1 0 0 0 0 0	648	94
0	32	0 0 0 0 0 1 0 1 2 0 1 0 1 1 0 0 0 1 3 0 0 0 1 0 0 0 0	216	95
1	13	0 0 0 0 0 1 0 1 2 0 1 0 1 1 0 0 1 0 2 0 1 1 0 0 0 0 0	1296	96
−1	−5	0 0 0 0 0 1 0 1 2 0 1 0 1 1 0 0 1 0 3 0 0 0 0 1 0 0 0	1296	97
0	−8	0 0 0 0 0 1 0 1 2 0 1 0 1 1 0 1 0 0 1 1 1 1 0 0 0 0 0	1296	98
1	−27	0 0 0 0 0 1 0 1 2 0 1 0 1 1 0 1 0 0 2 0 1 0 1 0 0 0 0	1296	99
−1	−21	0 0 0 0 0 1 0 1 2 0 1 0 1 1 0 1 0 0 2 1 0 0 0 1 0 0 0	1296	100
0	−12	0 0 0 0 0 1 0 1 2 0 1 0 1 2 0 0 0 0 2 0 1 0 0 0 1 0 0	216	101
0	16	0 0 0 0 0 1 0 1 2 0 1 0 1 2 0 0 0 0 3 0 0 0 0 0 0 0 1	648	102
0	8	0 0 0 0 0 1 0 1 2 0 1 0 2 0 0 0 0 1 1 2 0 1 0 0 0 0 0	1296	103
0	−32	0 0 0 0 0 1 0 1 2 0 1 0 2 0 0 0 0 1 2 1 0 0 1 0 0 0 0	648	104
1	13	0 0 0 0 0 1 0 1 2 0 1 0 2 0 0 0 1 0 1 1 1 1 0 0 0 0 0	1296	105
−2	−26	0 0 0 0 0 1 0 1 2 0 1 0 2 0 0 0 1 0 2 0 1 0 1 0 0 0 0	1296	106
1	5	0 0 0 0 0 1 0 1 2 0 1 0 2 0 0 0 1 0 2 1 0 0 0 1 0 0 0	1296	107
−1	−29	0 0 0 0 0 1 0 1 2 0 1 0 2 0 0 1 0 0 0 2 1 1 0 0 0 0 0	216	108
1	53	0 0 0 0 0 1 0 1 2 0 1 0 2 0 0 1 0 0 1 1 1 0 1 0 0 0 0	1296	109
0	8	0 0 0 0 0 1 0 1 2 0 1 0 2 0 0 1 0 0 1 2 0 0 0 1 0 0 0	1296	110
−1	−9	0 0 0 0 0 1 0 1 2 0 1 0 2 0 1 0 0 0 1 2 0 0 0 0 1 0 0	1296	111
1	5	0 0 0 0 0 1 0 1 2 0 1 0 2 0 1 0 0 0 2 1 0 0 0 0 0 1 0	1296	112
−1	19	0 0 0 0 0 1 0 1 2 0 1 0 2 1 0 0 0 0 1 1 1 0 0 0 1 0 0	1296	113
1	5	0 0 0 0 0 1 0 1 2 0 1 0 2 1 0 0 0 0 2 0 1 0 0 0 0 1 0	1296	114
0	−32	0 0 0 0 0 1 0 1 2 0 1 0 2 1 0 0 0 0 2 1 0 0 0 0 0 0 1	648	115
−1	−5	0 0 0 0 0 1 0 1 2 0 1 0 3 0 0 0 0 0 1 1 1 0 0 0 0 1 0	1296	116
−2	14	0 0 0 0 0 1 0 1 2 0 1 1 1 0 0 1 0 0 0 2 0 2 0 0 0 0 0	648	117
4	−60	0 0 0 0 0 1 0 1 2 0 1 1 1 0 0 1 0 0 1 1 0 1 1 0 0 0 0	648	118
−2	54	0 0 0 0 0 1 0 1 2 0 1 1 1 0 0 1 0 0 2 0 0 0 2 0 0 0 0	648	119
2	22	0 0 0 0 0 1 0 1 2 0 1 1 2 0 0 0 0 0 0 2 0 1 0 0 1 0 0	648	120
−3	−15	0 0 0 0 0 1 0 1 2 0 1 1 2 0 0 0 0 0 1 1 0 0 1 0 1 0 0	1296	121
−1	−21	0 0 0 0 0 1 0 1 2 0 1 1 2 0 0 0 0 0 1 1 0 1 0 0 0 1 0	1296	122
2	42	0 0 0 0 0 1 0 1 2 0 1 1 2 0 0 0 0 0 2 0 0 0 1 0 0 1 0	1296	123
2	−38	0 0 0 0 0 1 0 1 2 0 2 0 2 0 0 0 0 0 1 0 1 0 1 0 1 0 0	648	124
1	5	0 0 0 0 0 1 0 1 2 0 2 0 2 0 0 0 0 0 1 0 1 1 0 0 0 1 0	1296	125
−2	−10	0 0 0 0 0 1 0 1 2 0 2 0 2 0 0 0 0 0 2 0 0 0 0 1 0 1 0	648	126
0	64	0 0 0 0 0 1 0 1 2 0 2 0 2 0 0 0 0 0 2 0 0 0 1 0 0 0 1	216	127
−1	−21	0 0 0 0 0 1 0 1 2 1 0 0 0 1 1 1 0 0 1 2 0 1 0 0 0 0 0	1296	128
1	13	0 0 0 0 0 1 0 1 2 1 0 0 0 1 1 1 0 0 2 1 0 0 1 0 0 0 0	1296	129
0	48	0 0 0 0 0 1 0 1 2 1 0 0 0 2 0 0 0 1 2 1 0 1 0 0 0 0 0	108	130
1	−27	0 0 0 0 0 1 0 1 2 1 0 0 0 2 0 1 0 0 1 1 1 1 0 0 0 0 0	1296	131
−1	−9	0 0 0 0 0 1 0 1 2 1 0 0 0 2 0 1 0 0 2 1 0 0 0 1 0 0 0	648	132
2	10	0 0 0 0 0 1 0 1 2 1 0 0 1 0 1 0 1 0 1 2 0 1 0 0 0 0 0	1296	133
−2	−18	0 0 0 0 0 1 0 1 2 1 0 0 1 0 1 0 1 0 2 1 0 0 1 0 0 0 0	648	134
0	−64	0 0 0 0 0 1 0 1 2 1 0 0 1 1 0 0 0 1 1 2 0 1 0 0 0 0 0	648	135
−2	−42	0 0 0 0 0 1 0 1 2 1 0 0 1 1 0 0 1 0 1 1 1 1 0 0 0 0 0	1296	136
−2	−2	0 0 0 0 0 1 0 1 2 1 0 0 1 1 0 1 0 0 1 1 1 0 1 0 0 0 0	648	137
1	13	0 0 0 0 0 1 0 1 2 1 0 0 1 1 0 1 0 0 1 2 0 0 0 1 0 0 0	1296	138
−1	−21	0 0 0 0 0 1 0 1 2 1 0 0 1 1 1 0 0 0 1 2 0 0 0 0 1 0 0	1296	139
1	13	0 0 0 0 0 1 0 1 2 1 0 0 1 1 1 0 0 0 2 1 0 0 0 0 0 1 0	648	140
1	−27	0 0 0 0 0 1 0 1 2 1 0 0 1 2 0 0 0 0 1 1 1 0 0 0 1 0 0	1296	141
−1	−9	0 0 0 0 0 1 0 1 2 1 0 0 2 0 0 0 1 0 1 2 0 0 0 1 0 0 0	648	142
−1	−9	0 0 0 0 0 1 0 1 2 1 0 0 2 0 1 0 0 0 1 2 0 0 0 0 0 1 0	216	143
−2	78	0 0 0 0 0 1 0 1 2 1 0 1 0 1 0 1 0 0 1 1 0 1 1 0 0 0 0	648	144

TABLE 8. The two basis invariants in degree 12: page 2

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
0	−32	0 0 0 0 0 1 0 1 2 1 0 1 0 1 0 1 0 0 2 0 0 0 2 0 0 0 0	648	145
−2	−2	0 0 0 0 0 1 0 1 2 1 0 1 1 0 0 0 1 0 1 1 0 1 1 0 0 0 0	1296	146
1	5	0 0 0 0 0 1 0 1 2 1 0 1 1 0 0 0 1 0 2 0 0 0 2 0 0 0 0	1296	147
0	32	0 0 0 0 0 1 0 1 2 1 0 1 1 1 0 0 0 0 1 1 0 0 1 0 1 0 0	648	148
4	60	0 0 0 0 0 1 0 1 2 1 0 1 1 1 0 0 0 0 1 1 0 1 0 0 0 1 0	1296	149
−1	−21	0 0 0 0 0 1 0 1 2 1 0 1 1 1 0 0 0 0 2 0 0 0 1 0 0 1 0	1296	150
0	112	0 0 0 0 0 1 0 1 2 1 1 0 1 0 0 0 0 1 1 1 0 1 1 0 0 0 0	216	151
−2	−2	0 0 0 0 0 1 0 1 2 1 1 0 1 0 0 0 1 0 1 1 0 1 0 1 0 0 0	648	152
−1	−21	0 0 0 0 0 1 0 1 2 1 1 0 1 0 0 0 1 0 2 0 0 0 1 1 0 0 0	1296	153
2	10	0 0 0 0 0 1 0 1 2 1 1 0 1 1 0 0 0 0 2 0 0 0 0 1 0 1 0	648	154
1	−11	0 0 0 0 0 1 0 2 1 0 0 1 0 1 1 1 0 0 3 0 0 0 1 0 0 0 0	1296	155
−1	−21	0 0 0 0 0 1 0 2 1 0 0 1 0 2 0 1 0 0 3 0 0 0 0 1 0 0 0	1296	156
0	−24	0 0 0 0 0 1 0 2 1 0 0 1 1 0 1 1 0 0 1 2 0 1 0 0 0 0 0	648	157
0	16	0 0 0 0 0 1 0 2 1 0 0 1 1 0 1 1 0 0 2 1 0 0 1 0 0 0 0	1296	158
1	−11	0 0 0 0 0 1 0 2 1 0 0 1 1 1 0 0 0 1 3 0 0 0 1 0 0 0 0	1296	159
2	−14	0 0 0 0 0 1 0 2 1 0 0 1 1 1 0 1 0 0 1 1 1 1 0 0 0 0 0	1296	160
−2	22	0 0 0 0 0 1 0 2 1 0 0 1 1 1 0 1 0 0 2 0 1 0 1 0 0 0 0	648	161
0	16	0 0 0 0 0 1 0 2 1 0 0 1 1 1 0 1 0 0 2 1 0 0 0 1 0 0 0	1296	162
−1	−21	0 0 0 0 0 1 0 2 1 0 0 1 1 2 0 0 0 0 3 0 0 0 0 0 0 0 1	1296	163
1	5	0 0 0 0 0 1 0 2 1 0 0 1 2 0 0 0 0 1 1 2 0 1 0 0 0 0 0	1296	164
−1	11	0 0 0 0 0 1 0 2 1 0 0 1 2 0 0 0 0 1 2 1 0 0 1 0 0 0 0	1296	165
1	−27	0 0 0 0 0 1 0 2 1 0 0 1 2 0 0 1 0 0 1 1 1 0 1 0 0 0 0	1296	166
1	5	0 0 0 0 0 1 0 2 1 0 0 1 2 0 0 1 0 0 1 2 0 0 0 1 0 0 0	1296	167
−1	11	0 0 0 0 0 1 0 2 1 0 0 1 2 0 1 0 0 0 1 2 0 0 0 0 1 0 0	1296	168
−3	−15	0 0 0 0 0 1 0 2 1 0 0 1 2 1 0 0 0 0 1 1 1 0 0 0 1 0 0	1296	169
2	42	0 0 0 0 0 1 0 2 1 0 0 1 2 1 0 0 0 0 2 1 0 0 0 0 0 0 1	648	170
2	10	0 0 0 0 0 1 0 2 1 0 0 1 3 0 0 0 0 0 0 2 1 0 0 0 1 0 0	648	171
1	−15	0 0 0 0 0 1 0 2 1 0 0 2 0 1 0 1 0 0 1 1 0 2 0 0 0 0 0	648	172
−1	11	0 0 0 0 0 1 0 2 1 0 0 2 0 1 0 1 0 0 2 0 0 1 1 0 0 0 0	1296	173
2	−2	0 0 0 0 0 1 0 2 1 0 0 2 1 0 0 1 0 0 0 2 0 2 0 0 0 0 0	648	174
−4	28	0 0 0 0 0 1 0 2 1 0 0 2 1 0 0 1 0 0 1 1 0 1 1 0 0 0 0	1296	175
2	−22	0 0 0 0 0 1 0 2 1 0 0 2 1 0 0 1 0 0 2 0 0 0 2 0 0 0 0	1296	176
0	16	0 0 0 0 0 1 0 2 1 0 0 2 1 1 0 0 0 0 1 1 0 1 0 0 1 0 0	1296	177
−1	11	0 0 0 0 0 1 0 2 1 0 0 2 1 1 0 0 0 0 2 0 0 0 1 0 1 0 0	1296	178
−2	−10	0 0 0 0 0 1 0 2 1 0 0 2 2 0 0 0 0 0 0 2 0 1 0 0 1 0 0	648	179
3	−1	0 0 0 0 0 1 0 2 1 0 0 2 2 0 0 0 0 0 1 1 0 0 1 0 1 0 0	1296	180
1	13	0 0 0 0 0 1 0 2 1 0 1 0 0 0 2 1 0 0 3 0 0 0 1 0 0 0 0	648	181
−1	−5	0 0 0 0 0 1 0 2 1 0 1 0 0 1 1 1 0 0 3 0 0 0 0 1 0 0 0	1296	182
−2	−42	0 0 0 0 0 1 0 2 1 0 1 0 1 0 1 0 0 1 3 0 0 0 1 0 0 0 0	648	183
−2	−2	0 0 0 0 0 1 0 2 1 0 1 0 1 0 1 1 0 0 1 1 1 1 0 0 0 0 0	1296	184
0	16	0 0 0 0 0 1 0 2 1 0 1 0 1 0 1 1 0 0 2 0 1 0 1 0 0 0 0	1296	185
2	10	0 0 0 0 0 1 0 2 1 0 1 0 1 0 1 1 0 0 2 1 0 0 0 1 0 0 0	1296	186
0	16	0 0 0 0 0 1 0 2 1 0 1 0 1 1 0 1 0 0 2 0 1 0 0 1 0 0 0	648	187
1	−11	0 0 0 0 0 1 0 2 1 0 1 0 1 1 1 0 0 0 3 0 0 0 0 0 0 0 1	1296	188
−1	−21	0 0 0 0 0 1 0 2 1 0 1 0 2 0 0 0 0 1 1 1 1 1 0 0 0 0 0	1296	189
2	42	0 0 0 0 0 1 0 2 1 0 1 0 2 0 0 0 0 1 2 0 1 0 1 0 0 0 0	648	190
−1	−29	0 0 0 0 0 1 0 2 1 0 1 0 2 0 0 1 0 0 1 0 2 0 1 0 0 0 0	1296	191
−1	−21	0 0 0 0 0 1 0 2 1 0 1 0 2 0 0 1 0 0 1 1 1 0 0 1 0 0 0	1296	192
3	7	0 0 0 0 0 1 0 2 1 0 1 0 2 0 1 0 0 0 1 1 1 0 0 0 1 0 0	648	193
1	−15	0 0 0 0 0 1 0 2 1 0 1 0 2 1 0 0 0 0 1 0 2 0 0 0 1 0 0	432	194
−2	2	0 0 0 0 0 1 0 2 1 0 1 0 3 0 0 0 0 0 0 1 2 0 0 0 1 0 0	648	195
3	7	0 0 0 0 0 1 0 2 1 0 1 1 0 0 1 1 0 0 1 1 0 2 0 0 0 0 0	648	196
−3	−15	0 0 0 0 0 1 0 2 1 0 1 1 0 0 1 1 0 0 2 0 0 1 1 0 0 0 0	1296	197
3	47	0 0 0 0 0 1 0 2 1 0 1 1 0 1 0 1 0 0 2 0 0 1 0 1 0 0 0	648	198
1	53	0 0 0 0 0 1 0 2 1 0 1 1 1 0 0 0 0 1 2 0 0 1 1 0 0 0 0	1296	199
0	−24	0 0 0 0 0 1 0 2 1 0 1 1 1 0 0 1 0 0 0 1 1 2 0 0 0 0 0	648	200
2	26	0 0 0 0 0 1 0 2 1 0 1 1 1 0 0 1 0 0 1 0 1 1 1 0 0 0 0	1296	201
−2	38	0 0 0 0 0 1 0 2 1 0 1 1 1 0 0 1 0 0 1 1 0 1 0 1 0 0 0	1296	202
0	−64	0 0 0 0 0 1 0 2 1 0 1 1 1 0 0 1 0 0 2 0 0 0 1 1 0 0 0	1296	203
−2	−2	0 0 0 0 0 1 0 2 1 0 1 1 1 0 1 0 0 0 1 1 0 1 0 0 1 0 0	1296	204
3	47	0 0 0 0 0 1 0 2 1 0 1 1 1 0 1 0 0 0 2 0 0 0 1 0 1 0 0	1296	205
2	−14	0 0 0 0 0 1 0 2 1 0 1 1 1 1 0 0 0 0 1 0 1 1 0 0 1 0 0	1296	206
−3	−15	0 0 0 0 0 1 0 2 1 0 1 1 1 1 0 0 0 0 2 0 0 0 0 1 1 0 0	1296	207
1	53	0 0 0 0 0 1 0 2 1 0 1 1 1 1 0 0 0 0 2 0 0 1 0 0 0 0 1	1296	208
−1	59	0 0 0 0 0 1 0 2 1 0 1 1 2 0 0 0 0 0 1 0 1 0 1 0 1 0 0	1296	209
1	−27	0 0 0 0 0 1 0 2 1 0 1 1 2 0 0 0 0 0 1 1 0 0 0 1 1 0 0	1296	210
−2	−106	0 0 0 0 0 1 0 2 1 0 1 1 2 0 0 0 0 0 2 0 0 0 1 0 0 0 1	648	211
−2	−58	0 0 0 0 0 1 0 2 1 0 1 2 1 0 0 0 0 0 1 0 0 1 1 0 1 0 0	1296	212
1	5	0 0 0 0 0 1 0 2 1 0 2 0 0 0 1 1 0 0 2 0 0 1 0 1 0 0 0	648	213
−1	11	0 0 0 0 0 1 0 2 1 0 2 0 1 0 0 0 0 1 2 0 0 1 0 1 0 0 0	648	214
4	4	0 0 0 0 0 1 0 2 1 0 2 0 1 0 0 1 0 0 1 0 1 1 0 1 0 0 0	1296	215
−2	−10	0 0 0 0 0 1 0 2 1 0 2 0 1 0 0 1 0 0 2 0 0 0 0 2 0 0 0	648	216

TABLE 9. The two basis invariants in degree 12: page 3

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
0	−24	0 0 0 0 0 1 0 2 1 0 2 0 1 0 1 0 0 0 1 0 1 1 0 0 1 0 0	1296	217
1	5	0 0 0 0 0 1 0 2 1 0 2 0 1 0 1 0 0 0 2 0 0 0 0 1 1 0 0	1296	218
−1	11	0 0 0 0 0 1 0 2 1 0 2 0 1 0 1 0 0 0 2 0 0 1 0 0 0 0 1	1296	219
−3	17	0 0 0 0 0 1 0 2 1 0 2 0 2 0 0 0 0 0 1 0 1 0 0 1 1 0 0	1296	220
2	−22	0 0 0 0 0 1 0 2 1 0 2 0 2 0 0 0 0 0 2 0 0 0 0 1 0 0 1	648	221
2	10	0 0 0 0 0 1 0 2 1 0 2 1 1 0 0 0 0 0 1 0 0 1 0 1 1 0 0	1296	222
0	32	0 0 0 0 0 1 0 2 1 1 0 0 0 1 1 1 0 0 1 1 1 1 0 0 0 0 0	1296	223
−2	22	0 0 0 0 0 1 0 2 1 1 0 0 1 0 1 0 0 1 1 2 0 1 0 0 0 0 0	324	224
0	−8	0 0 0 0 0 1 0 2 1 1 0 0 1 0 1 1 0 0 1 1 1 0 1 0 0 0 0	1296	225
−2	−18	0 0 0 0 0 1 0 2 1 1 0 0 1 0 1 1 0 0 1 2 0 0 0 1 0 0 0	648	226
2	106	0 0 0 0 0 1 0 2 1 1 0 0 1 1 0 0 0 1 1 1 1 1 0 0 0 0 0	648	227
0	−8	0 0 0 0 0 1 0 2 1 1 0 0 1 1 0 1 0 0 1 1 1 0 0 1 0 0 0	1296	228
0	32	0 0 0 0 0 1 0 2 1 1 0 0 1 1 1 0 0 0 1 1 1 0 0 0 1 0 0	1296	229
4	−20	0 0 0 0 0 1 0 2 1 1 0 1 0 0 1 1 0 0 1 1 0 1 1 0 0 0 0	1296	230
1	53	0 0 0 0 0 1 0 2 1 1 0 1 0 1 0 0 0 1 1 1 0 2 0 0 0 0 0	648	231
2	−54	0 0 0 0 0 1 0 2 1 1 0 1 0 1 0 1 0 0 1 0 1 1 1 0 0 0 0	1296	232
−2	−82	0 0 0 0 0 1 0 2 1 1 0 1 0 1 0 1 0 0 1 1 0 1 0 1 0 0 0	1296	233
0	32	0 0 0 0 0 1 0 2 1 1 0 1 0 1 1 0 0 0 1 1 0 1 0 0 1 0 0	1296	234
−2	22	0 0 0 0 0 1 0 2 1 1 0 1 0 2 0 0 0 0 1 0 1 1 0 0 1 0 0	1296	235
2	−54	0 0 0 0 0 1 0 2 1 1 0 1 1 0 0 0 0 1 1 1 0 1 1 0 0 0 0	1296	236
0	32	0 0 0 0 0 1 0 2 1 1 0 1 1 0 0 1 0 0 1 1 0 0 1 1 0 0 0	1296	237
−4	−36	0 0 0 0 0 1 0 2 1 1 0 1 1 0 1 0 0 0 1 1 0 0 1 0 1 0 0	1296	238
2	26	0 0 0 0 0 1 0 2 1 1 0 1 1 1 0 0 0 0 1 1 0 0 0 1 1 0 0	1296	239
−4	−116	0 0 0 0 0 1 0 2 1 1 0 1 1 1 0 0 0 0 1 1 0 1 0 0 0 0 1	648	240
−2	22	0 0 0 0 0 1 0 2 1 1 1 0 0 0 1 0 0 1 1 1 0 2 0 0 0 0 0	648	241
−2	−2	0 0 0 0 0 1 0 2 1 1 1 0 0 0 1 1 0 0 1 1 0 1 0 1 0 0 0	1296	242
2	10	0 0 0 0 0 1 0 2 1 1 1 0 0 0 2 0 0 0 1 1 0 1 0 0 1 0 0	648	243
1	21	0 0 0 0 0 1 0 3 0 0 0 1 0 1 1 1 0 0 3 0 0 0 0 1 0 0 0	648	244
0	24	0 0 0 0 0 1 0 3 0 0 0 1 1 0 1 1 0 0 1 1 1 1 0 0 0 0 0	648	245
−1	−5	0 0 0 0 0 1 0 3 0 0 0 1 1 0 1 1 0 0 2 1 0 0 0 1 0 0 0	1296	246
−2	−10	0 0 0 0 0 1 0 3 0 0 0 1 1 1 0 0 0 1 3 0 0 0 0 1 0 0 0	108	247
2	10	0 0 0 0 0 1 0 3 0 0 0 1 2 0 0 0 0 1 2 1 0 0 0 1 0 0 0	648	248
3	−17	0 0 0 0 0 1 0 3 0 0 0 2 1 0 0 1 0 0 1 1 0 1 0 1 0 0 0	432	249
−2	22	0 0 0 0 0 1 0 3 0 0 0 2 1 0 0 1 0 0 2 0 0 0 1 1 0 0 0	648	250
2	42	0 0 0 0 0 1 0 3 0 0 0 2 2 0 0 0 0 0 2 0 0 0 1 0 0 0 1	216	251
1	1	0 0 0 0 0 1 0 3 0 0 0 3 0 0 0 1 0 0 0 1 0 3 0 0 0 0 0	72	252
−4	−4	0 0 0 0 0 1 0 3 0 0 1 1 1 0 0 1 0 0 1 0 1 1 0 1 0 0 0	648	253
2	−22	0 0 0 0 0 1 0 3 0 1 0 0 1 0 1 0 0 1 1 1 1 1 0 0 0 0 0	648	254
2	18	0 0 0 0 0 1 0 3 0 1 0 0 1 0 1 1 0 0 1 1 1 0 0 1 0 0 0	216	255
2	58	0 0 0 0 0 1 0 3 0 1 0 1 0 1 0 1 0 0 1 0 1 1 0 1 0 0 0	648	256
−4	4	0 0 0 0 0 1 1 1 1 0 0 1 0 2 0 1 0 0 1 1 1 1 0 0 0 0 0	324	257
3	47	0 0 0 0 0 1 1 1 1 0 0 1 0 2 0 1 0 0 2 1 0 0 0 1 0 0 0	1296	258
0	24	0 0 0 0 0 1 1 1 1 0 0 1 1 1 0 0 1 0 1 1 1 1 0 0 0 0 0	648	259
−2	−42	0 0 0 0 0 1 1 1 1 0 0 1 1 1 0 0 1 0 2 1 0 0 0 1 0 0 0	1296	260
4	20	0 0 0 0 0 1 1 1 1 0 0 1 1 2 0 0 0 0 1 1 1 0 0 0 1 0 0	648	261
−1	−21	0 0 0 0 0 1 1 1 1 0 0 1 1 2 0 0 0 0 2 1 0 0 0 0 0 0 1	648	262
−3	17	0 0 0 0 0 1 1 1 1 0 0 2 0 1 0 1 0 0 0 2 0 2 0 0 0 0 0	648	263
4	−20	0 0 0 0 0 1 1 1 1 0 0 2 0 1 0 1 0 0 1 1 0 1 1 0 0 0 0	1296	264
−1	11	0 0 0 0 0 1 1 1 1 0 0 2 0 1 0 1 0 0 2 0 0 0 2 0 0 0 0	1296	265
0	16	0 0 0 0 0 1 1 1 1 0 0 2 0 2 0 0 0 0 1 1 0 1 0 0 1 0 0	648	266
4	4	0 0 0 0 0 1 1 1 1 0 0 2 1 1 0 0 0 0 0 2 0 1 0 0 1 0 0	1296	267
−4	−36	0 0 0 0 0 1 1 1 1 0 0 2 1 1 0 0 0 0 1 1 0 0 1 0 1 0 0	1296	268
0	32	0 0 0 0 0 1 1 1 1 0 1 0 0 1 1 1 0 0 2 0 1 0 1 0 0 0 0	1296	269
2	18	0 0 0 0 0 1 1 1 1 0 1 0 1 0 1 0 1 0 1 1 1 1 0 0 0 0 0	648	270
2	−22	0 0 0 0 0 1 1 1 1 0 1 0 1 0 1 1 0 0 1 1 1 0 1 0 0 0 0	648	271
−2	−42	0 0 0 0 0 1 1 1 1 0 1 0 1 1 0 0 0 1 2 0 1 0 1 0 0 0 0	648	272
−4	−84	0 0 0 0 0 1 1 1 1 0 1 0 1 1 1 0 0 0 1 1 1 0 0 0 1 0 0	216	273
−2	−42	0 0 0 0 0 1 1 1 1 0 1 0 2 0 0 0 0 1 1 1 1 0 1 0 0 0 0	324	274
−2	−2	0 0 0 0 0 1 1 1 1 0 1 0 2 0 0 0 1 0 1 1 1 0 0 1 0 0 0	324	275
−4	36	0 0 0 0 0 1 1 1 1 0 1 1 0 0 1 1 0 0 1 1 0 1 1 0 0 0 0	1296	276
2	58	0 0 0 0 0 1 1 1 1 0 1 1 0 1 0 1 0 0 1 0 1 1 1 0 0 0 0	648	277
−2	−50	0 0 0 0 0 1 1 1 1 0 1 1 0 1 0 1 0 0 1 1 0 1 0 1 0 0 0	1296	278
2	18	0 0 0 0 0 1 1 1 1 0 1 1 0 1 1 0 0 0 1 1 0 1 0 0 1 0 0	648	279
0	32	0 0 0 0 0 1 1 1 1 0 1 1 0 1 1 0 0 0 2 0 0 1 0 0 0 1 0	1296	280
−2	−10	0 0 0 0 0 1 1 1 1 0 1 1 1 0 0 0 0 1 1 1 0 1 1 0 0 0 0	1296	281
0	−64	0 0 0 0 0 1 1 1 1 0 1 1 1 0 0 0 0 1 2 0 0 0 2 0 0 0 0	1296	282
0	−56	0 0 0 0 0 1 1 1 1 0 1 1 1 0 0 0 1 0 1 1 0 1 0 1 0 0 0	1296	283
2	106	0 0 0 0 0 1 1 1 1 0 1 1 1 0 0 0 1 0 2 0 0 0 1 1 0 0 0	648	284
0	−48	0 0 0 0 0 1 1 1 1 0 1 1 1 0 0 1 0 0 1 0 1 0 2 0 0 0 0	1296	285
2	58	0 0 0 0 0 1 1 1 1 0 1 1 1 0 0 1 0 0 1 1 0 0 1 1 0 0 0	1296	286
6	46	0 0 0 0 0 1 1 1 1 0 1 1 1 0 1 0 0 0 1 1 0 1 0 0 0 1 0	648	287
−4	−36	0 0 0 0 0 1 1 1 1 0 1 1 1 0 1 0 0 0 2 0 0 0 1 0 0 1 0	1296	288

TABLE 10. The two basis invariants in degree 12: page 4

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
0	−96	0 0 0 0 0 1 1 1 1 0 1 1 1 1 0 0 0 0 1 1 0 1 0 0 0 0 1	648	289
2	106	0 0 0 0 0 1 1 1 1 0 1 1 1 1 0 0 0 0 2 0 0 0 1 0 0 0 1	1296	290
4	−108	0 0 0 0 0 1 1 1 1 1 1 0 0 0 1 0 0 1 1 1 0 1 1 0 0 0 0	324	291
2	10	0 0 0 0 0 1 1 2 0 0 0 1 1 1 0 0 0 1 2 1 0 0 0 1 0 0 0	324	292
−4	−20	0 0 0 0 0 1 1 2 0 0 0 1 2 0 0 0 0 1 1 2 0 0 0 1 0 0 0	324	293
1	21	0 0 0 0 0 1 1 2 0 0 0 2 0 2 0 0 0 0 2 0 0 1 0 0 0 0 1	1296	294
−1	−13	0 0 0 0 0 1 1 2 0 0 0 2 1 0 0 0 0 1 0 2 0 2 0 0 0 0 0	432	295
0	16	0 0 0 0 0 1 1 2 0 0 0 2 1 0 0 0 0 1 1 1 0 1 1 0 0 0 0	1296	296
4	84	0 0 0 0 0 1 1 2 0 0 0 2 1 1 0 0 0 0 1 1 0 1 0 0 0 0 1	648	297
−3	−63	0 0 0 0 0 1 1 2 0 0 0 2 1 1 0 0 0 0 2 0 0 0 1 0 0 0 1	1296	298
2	10	0 0 0 0 0 1 1 2 0 0 1 0 1 0 1 0 0 1 2 0 1 0 1 0 0 0 0	648	299
−2	−26	0 0 0 0 0 1 1 2 0 0 1 0 2 0 0 0 0 1 1 0 2 0 1 0 0 0 0	324	300
0	−48	0 0 0 0 0 1 1 2 0 0 1 1 1 0 0 0 0 1 1 0 1 1 1 0 0 0 0	1296	301
2	26	0 0 0 0 0 1 1 2 0 0 1 1 1 0 0 0 0 1 1 1 0 1 0 1 0 0 0	1296	302
−2	22	0 0 0 0 0 1 1 2 0 0 1 1 1 0 0 0 0 1 2 0 0 0 1 1 0 0 0	648	303
−4	−36	0 0 0 0 0 1 1 2 0 0 1 1 1 0 1 0 0 0 1 1 0 1 0 0 0 0 1	1296	304
1	53	0 0 0 0 0 1 1 2 0 0 1 1 1 0 1 0 0 0 2 0 0 0 1 0 0 0 1	1296	305
3	−33	0 0 0 0 0 1 1 2 0 0 1 1 1 1 0 0 0 0 2 0 0 0 0 1 0 0 1	1296	306
−2	−58	0 0 0 0 0 1 1 2 0 0 2 0 1 0 0 0 0 1 1 0 1 1 0 1 0 0 0	1296	307
4	4	0 0 0 0 0 1 1 2 0 0 2 0 1 0 1 0 0 0 1 0 1 1 0 0 0 0 1	648	308
−2	22	0 0 0 0 0 1 1 2 0 0 2 1 1 0 0 0 0 0 1 0 0 1 0 1 0 0 1	324	309
2	26	0 0 0 0 0 1 1 2 0 1 0 1 0 1 0 0 0 1 1 0 1 1 1 0 0 0 0	1296	310
0	−48	0 0 0 0 0 1 1 2 0 1 0 1 0 1 0 0 0 1 1 1 0 1 0 1 0 0 0	648	311
−4	44	0 0 0 0 0 1 1 2 0 1 0 1 1 1 0 0 0 0 1 1 0 0 0 1 0 0 1	648	312
−4	18	0 0 0 0 0 2 0 2 0 0 0 2 0 0 0 2 0 0 0 2 0 2 0 0 0 0 0	36	313
6	−34	0 0 0 0 0 2 0 2 0 0 0 2 0 0 0 2 0 0 1 1 0 1 1 0 0 0 0	324	314
−2	22	0 0 0 0 0 2 0 2 0 0 0 2 0 0 0 2 0 0 2 0 0 0 2 0 0 0 0	108	315
−5	23	0 0 0 0 0 2 0 2 0 0 0 2 1 0 0 1 0 0 1 1 0 0 1 0 1 0 0	648	316
−12	92	0 0 0 0 0 2 0 2 0 0 1 1 1 0 0 1 0 0 0 1 1 1 0 0 1 0 0	162	317
4	−76	0 0 0 0 0 2 0 2 0 0 1 1 1 0 0 1 0 0 1 0 1 0 1 0 1 0 0	648	318
0	128	0 0 0 0 0 2 0 2 0 0 1 1 1 0 0 1 0 0 2 0 0 0 1 0 0 0 1	324	319
−8	40	0 0 0 0 0 2 0 2 0 1 0 1 0 0 0 1 0 1 1 1 0 1 1 0 0 0 0	324	320
−4	108	0 0 0 0 0 2 0 2 0 1 0 1 0 1 0 1 0 0 1 0 1 0 1 0 1 0 0	324	321
4	164	0 0 0 0 0 2 0 2 0 1 0 1 0 1 0 1 0 0 1 1 0 1 0 0 0 0 1	324	322
2	10	0 0 0 0 0 2 0 2 0 1 0 1 1 0 0 0 0 1 1 1 0 0 1 0 1 0 0	648	323
8	−72	0 0 0 0 0 2 1 1 0 0 1 1 0 0 0 1 0 1 1 1 0 1 1 0 0 0 0	1296	324
−6	50	0 0 0 0 0 2 1 1 0 0 1 1 0 1 0 1 0 0 1 0 1 1 0 0 0 1 0	648	325
−2	38	0 0 0 0 0 2 1 1 0 0 1 1 0 1 0 1 0 0 1 1 0 1 0 0 0 0 1	1296	326
5	41	0 0 0 0 0 2 1 1 0 0 1 1 0 1 0 1 0 0 2 0 0 0 1 0 0 0 1	648	327
0	112	0 0 0 0 0 2 1 1 0 0 1 1 1 0 0 0 0 1 1 1 0 0 1 0 1 0 0	1296	328
−6	−30	0 0 0 0 0 2 1 1 0 0 1 1 1 0 0 0 0 1 1 1 0 1 0 0 0 1 0	432	329
−2	158	0 0 0 0 0 2 1 1 0 0 1 1 1 0 0 0 1 0 1 0 1 0 1 0 1 0 0	324	330
−1	−181	0 0 0 0 0 2 1 1 0 0 1 1 1 0 0 0 1 0 2 0 0 0 1 0 0 0 1	648	331
−4	−116	0 0 0 0 0 2 1 1 0 0 1 1 1 0 0 1 0 0 1 1 0 0 1 0 0 0 1	1296	332
2	−54	0 0 0 0 0 2 1 1 0 0 1 1 1 1 0 0 0 0 1 1 0 0 0 0 1 0 1	648	333
−6	34	0 0 0 0 0 2 1 1 0 0 1 1 1 1 0 0 0 0 2 0 0 0 0 0 0 1 1	324	334
4	20	0 0 0 0 0 2 1 1 0 1 1 0 0 1 0 0 0 1 1 1 0 1 0 0 0 0 1	324	335
24	−88	0 0 0 0 1 1 0 1 1 0 1 1 1 0 0 1 0 0 0 1 1 1 0 0 1 0 0	27	336
−8	8	0 0 0 0 1 1 0 1 1 0 1 1 1 0 0 1 0 0 1 0 1 0 1 0 1 0 0	648	337
10	−6	0 0 0 0 1 1 0 1 1 1 0 1 0 0 0 1 0 1 1 1 0 1 1 0 0 0 0	108	338
−8	48	0 0 0 0 1 1 0 1 1 1 0 1 0 0 0 1 1 0 1 0 1 1 1 0 0 0 0	324	339
−2	70	0 0 0 0 1 1 0 1 1 1 0 1 0 0 0 1 1 0 1 1 0 1 0 1 0 0 0	648	340
4	172	0 0 0 0 1 1 0 1 1 1 0 1 0 1 0 1 0 0 1 0 1 1 0 0 0 1 0	324	341
−2	−130	0 0 0 0 1 1 0 1 1 1 0 1 0 1 0 1 0 0 1 1 0 1 0 0 0 0 1	324	342
−12	60	0 0 0 0 1 1 1 0 1 0 1 1 0 0 0 1 1 0 1 0 1 1 1 0 0 0 0	108	343
−4	76	0 0 0 0 1 1 1 0 1 0 1 1 0 0 0 1 1 0 1 1 0 1 0 1 0 0 0	324	344
2	−22	0 0 0 0 1 1 1 0 1 0 1 1 0 0 1 1 0 0 1 1 0 1 0 0 0 1 0	1296	345
4	−108	0 0 0 0 1 1 1 0 1 0 1 1 0 1 0 1 0 0 1 0 1 1 0 0 0 1 0	648	346
0	−136	0 0 0 0 1 1 1 0 1 0 1 1 1 0 0 0 0 1 1 1 0 0 1 0 1 0 0	216	347
6	46	0 0 0 0 1 1 1 0 1 0 1 1 1 0 0 0 1 0 1 0 1 1 0 0 0 1 0	648	348
4	92	0 0 0 0 1 1 1 0 1 0 1 1 1 0 0 0 1 0 1 1 0 0 0 1 1 0 0	648	349
0	−16	0 0 0 0 1 1 1 0 1 0 1 1 1 0 0 0 1 0 1 1 0 1 0 0 0 0 1	1296	350
8	88	0 0 1 0 0 1 1 1 0 0 0 1 0 2 0 1 0 0 1 1 0 1 0 0 0 0 1	216	351
−4	−164	0 0 1 0 0 1 1 1 0 0 0 1 1 1 0 0 1 0 1 1 0 1 0 0 0 0 1	648	352
4	180	0 0 1 0 0 1 1 1 0 0 1 0 1 0 1 0 1 0 2 0 0 0 1 0 0 0 1	324	353
0	144	0 0 1 0 0 1 1 1 0 0 1 0 1 0 1 1 0 0 1 1 0 0 1 0 0 0 1	648	354
−4	236	0 0 1 0 0 1 1 1 0 0 1 0 1 1 0 0 0 1 1 1 0 1 0 0 0 0 1	108	355
0	−128	0 0 1 0 0 1 1 1 0 0 1 0 1 1 0 0 0 1 2 0 0 0 1 0 0 0 1	216	356
−2	−82	0 0 1 0 1 0 1 0 1 0 1 0 1 1 0 0 0 1 2 0 0 0 0 1 0 1 0	324	357
6	6	0 0 1 0 1 0 1 0 1 1 0 0 0 1 1 0 1 0 1 1 0 1 0 0 0 0 1	72	358
2	222	0 0 1 0 1 0 2 0 0 0 2 0 1 0 0 0 0 1 1 0 0 0 0 2 0 1 0	36	359

TABLE 11. The two basis invariants in degree 12: page 5