

# LIE INVARIANTS IN TWO AND THREE VARIABLES

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ABSTRACT. We use computer algebra to determine the Lie invariants of degree  $\leq 12$  in the free Lie algebra on two generators corresponding to the natural representation of the simple 3-dimensional Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$ . We then consider the free Lie algebra on three generators, and compute the Lie invariants of degree  $\leq 7$  corresponding to the adjoint representation of  $\mathfrak{sl}_2(\mathbb{C})$ , and the Lie invariants of degree  $\leq 9$  corresponding to the natural representation of  $\mathfrak{sl}_3(\mathbb{C})$ . We represent the action of  $\mathfrak{sl}_2(\mathbb{C})$  and  $\mathfrak{sl}_3(\mathbb{C})$  on Lie polynomials by computing the coefficient matrix with respect to the basis of Hall words. We then use algorithms for linear algebra (row canonical form, Hermite normal form, lattice basis reduction) to compute a basis of the nullspace.

## 1. INTRODUCTION

The theory of Lie invariants (that is, elements of the free Lie algebra which are invariant under the action of some group of automorphisms) has been studied since the pioneering work of Magnus in 1940 and Wever in 1949. Magnus [15, page 147] found two Lie invariants, in degrees 2 and 6, for the action of the general linear group in two dimensions in its natural representation:

$$[a, b], \quad [[a, [a, b]], [b, [a, b]]].$$

Wever [21, 22] found a Lie invariant in degree 9 for the action of the general linear group in three dimensions in its natural representation:

$$[[[[[a, b], [a, c]], [[a, b], [b, c]], c] + [[[[[b, c], [b, a]], [[b, c], [c, a]], a] + [[[[[c, a], [c, b]], [[c, a], [a, b]], b].$$

He also showed that the space of invariants in degree 9 has dimension 4. Wever observed that the degree  $d$  of a Lie invariant for the action of the general linear group in  $q$  dimensions in its natural representation (that is, in the free Lie algebra on  $q$  generators) must be a multiple of  $q$ ; more precisely,  $d = mq$ , where  $m \geq 2$  unless  $q = 2$ . He found that there are no Lie invariants for  $q = 2$  generators in degree  $d = 4$ ; for  $q = 3$  generators in degree  $d = 6$ ; and for  $q = d \geq 3$ . He gave an explicit formula for the dimension of the space of invariants for  $q = 2$  generators and arbitrary degree  $d$ . In 1958, Burrow [6] showed that there is always a Lie invariant of degree  $d = mq$  for all  $m \geq 2$  and  $q \geq 2$ , except in the two cases noticed by Wever. In 1967, Burrow [7] extended Wever's formula to the case  $d = mq$  for all  $m \geq 2$  and  $q \geq 2$ . A brief survey of these developments may be found in Reutenauer [16, §8.6.2]. Lie invariants in the natural representation of the general linear group are important in group theory owing to the connection between free Lie rings and the

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lower central series of free groups; see Magnus et al. [15]. We emphasize that all of these references do not consider the Lie invariants corresponding to any other irreducible representations of the general linear group.

By the Shirshov-Witt theorem [17, 18, 24], we know that every subalgebra  $J$  of a free Lie algebra  $L$  is also free, and so the Lie subalgebra of invariants is a free Lie algebra. We are primarily interested in the primitive invariants; that is, a set of free generators for  $J$ . If we have computed the dimensions of the homogeneous subspaces of  $J$  in all degrees  $< n$ , then we can use the generalized Witt dimension formula of Kang and Kim [14] and Jurisich [13] to compute the dimension of the subspace of non-primitive invariants in degree  $n$ ; that is, the subspace generated by the invariants of lower degree. The rapid growth of the number of Lie invariants of degree  $mq$  as a function of  $m$  suggests that the free Lie subalgebra of Lie invariants in  $q$  generators may not be finitely generated. So it is most likely impossible to determine all the Lie invariants, or even all the primitive invariants, even in the simplest case of the natural representation of the general linear group in two dimensions.

In 1998, Bremner [4] used computer algebra to determine the Lie invariants (primitive and non-primitive) of degree  $\leq 10$  in this simplest case. In this paper we extend those explicit results to degree 12 (Section 3), and compute the dimensions in degree 14. We also consider Lie invariants in three variables in two cases:

- For the adjoint representation of the general linear group in two dimensions (Section 4), we obtain explicit invariants up to degree 7, and dimensions up to degree 12. This seems to be the first time that Lie invariants have been considered in a representation other than the natural representation.
- For the natural representation of the general linear group in three dimensions (Section 5), we obtain explicit results up to degree 9, and partial results in degree 12. This seems to be the first time that an explicit basis has been computed for the space of invariants in degree 9.

These are essentially the only cases where a simple Lie algebra has an irreducible 3-dimensional representation; the only other possibility is the dual of the natural representation of  $\mathfrak{sl}_3(\mathbb{C})$ , and the spaces of invariants for that case will be linearly isomorphic to the spaces of invariants for the natural representation.

Most of our results depend on calculations with the computer algebra system Maple, especially the packages `LinearAlgebra` and `LinearAlgebra[Modular]`. Our calculations depend on computing the row canonical form of a large integer matrix over the field of rational numbers, and extracting the canonical basis of the nullspace. However, in a few cases, we obtain better results by combining an algorithm for the Hermite normal form of an integer matrix with the LLL algorithm for lattice basis reduction. See Bremner and Peresi [5] for another application of these methods to problems on free nonassociative algebras. A comprehensive reference on modern computer algebra is von zur Gathen and Gerhard [20].

## 2. PRELIMINARIES

**2.1. The Hall basis of a free Lie algebra.** Our main reference for free Lie algebras is Reutenauer [16]; see also Bahturin [2]. The Hall basis [10] of a free Lie algebra is a subset of the free magma on an alphabet.

**Definition 1.** [16, §4] The *free magma*  $M(A)$  on a set of generators  $A$  is the set of all nonassociative words on  $A$ , and can be identified with the set of complete rooted binary trees with leaves labeled by elements of  $A$ . Each binary tree with at least 2 leaf nodes can be written uniquely as  $t = (t', t'')$  where  $t'$  and  $t''$  are its immediate left and right subtrees. The binary operation  $M(A) \times M(A) \rightarrow M(A)$  is the mapping  $(t', t'') \rightarrow t$ . There is a canonical map from  $M(A)$  onto  $A^*$ , the set of all associative words on  $A$ , defined by  $f(a) = a$  if  $a \in A$  and  $f(t) = f(t')f(t'')$  if  $t = (t', t'') \in M(A) \setminus A$ . The *degree* of  $t$ , denoted  $\deg(t)$ , is the length of  $f(t)$ .

**Definition 2.** [10] Let  $M(A)$  be the free magma on the set of generators  $A$ . A subset  $H \subset M(A)$  is called a *Hall basis* if the following conditions hold:

- (1)  $H$  has a total order  $<$ ;
- (2)  $A \subset H$ ;
- (3) if  $t = (t', t'') \in H \setminus A$  then  $t', t'' \in H$ ;
- (4) if  $t = (t', t'') \in H \setminus A$  then  $t' > t''$ ;
- (5) if  $t = (t', t'') \in H \setminus A$  then either  $t' \in A$  or  $(t')'' \leq t''$ .

The elements of the Hall basis are called *Hall words*.

The relation between trees and Hall words is given by interpreting each non-leaf node of a tree  $t$  corresponding to a given Hall word as the Lie bracket of the Hall words corresponding to its subtrees  $t'$  and  $t''$ . We follow Hall's original method of constructing the Hall basis inductively by degree.

**Definition 3.** Let  $A = \{a_1, \dots, a_q\}$  and define a total order on  $A$  by  $a_i < a_j$  if and only if  $i < j$ . We extend this total order inductively to  $M(A)$  by making  $<$  agree with  $<$  on  $A$ , and then for  $t, u \in M(A)$  we define  $t < u$  if and only if:

- (1) either  $\deg(t) < \deg(u)$ , or
- (2)  $\deg(t) = \deg(u) = 1$  and  $t < u$  in  $A$ , or
- (3)  $\deg(t) = \deg(u) \geq 2$  and  $t' < u'$ , or
- (4)  $\deg(t) = \deg(u) \geq 2$  and  $t' = u'$  and  $t'' < u''$ .

**Theorem 4. Hall's Theorem.** [10, Theorem 3.1] *The Hall words on  $A$  form a basis for the free Lie ring generated by  $A$ .*

The algorithm in the proof of Theorem 3.1 of [10] transforms any nonassociative word in  $M(A)$  into a linear combination of Hall words; see Figure 1. This process is crucial for us because it guarantees that we can always obtain a linear combination of Hall words after applying an element of  $\mathfrak{sl}_2(\mathbb{C})$  or  $\mathfrak{sl}_3(\mathbb{C})$  to a Hall word. We emphasize that a single call to this algorithm is not always sufficient; the terms  $((q, s), p)$  and  $((p, s), q)$  may not be Hall words. (For example, consider  $x = [[[[[b, a], a], a], b], a]$ .) The algorithm must be called recursively until every word is a Hall word.

**2.2. Irreducible representations of  $\mathfrak{sl}_2(\mathbb{C})$ .** We recall some basic information about the representation theory of Lie algebras; our reference is Humphreys [12]. The simple 3-dimensional Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$  has the following standard basis:

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

All Lie brackets in  $\mathfrak{sl}_2(\mathbb{C})$  follow from the standard commutation relations using bilinearity and anticommutativity:

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$

- Input:** An arbitrary element  $x$  in the free magma  $M(A)$ .  
**Output:** A linear combination of Hall words.
- (1) If  $\deg(x) = 1$  then set **result**  $\leftarrow x$ .
  - (2) If  $\deg(x) \geq 2$  then  $x = (y, z)$ :
    - (a) Set **result**  $\leftarrow 0$ .
    - (b) For each term  $au$  in  $\text{HallForm}(y)$  and each term  $bv$  in  $\text{HallForm}(z)$  do:
      - [*apply anticommutativity*]
      - (i) Set  $\epsilon \leftarrow 1$ .
      - (ii) If  $u = v$  then set **newterm**  $\leftarrow 0$ .
      - (iii) If  $u < v$  then set **newterm**  $\leftarrow (v, u)$  and  $\epsilon \leftarrow -1$ .
      - (iv) If  $u > v$  then set **newterm**  $\leftarrow (u, v)$ .
      - (v) If **newterm**  $\neq 0$  then **newterm**  $= (r, s)$  with  $r > s$ :
        - If  $\deg(r) = 1$  then **result**  $\leftarrow \text{result} + \epsilon ab(r, s)$
        - If  $\deg(r) > 1$  then  $r = (p, q)$ :
          - [*apply the Jacobi identity*]
          - If  $s \geq q$  then
            - result**  $\leftarrow \text{result} + \epsilon ab((p, q), s)$
          - else
            - result**  $\leftarrow \text{result} - \epsilon ab((q, s), p) + \epsilon ab((p, s), q)$ .
- (3) Return **result**.

FIGURE 1. Hall's recursive algorithm  $\text{HallForm}(x)$ 

Every finite-dimensional representation  $M$  of  $\mathfrak{sl}_2(\mathbb{C})$  is the direct sum of its weight spaces  $M_w$  for the action of  $h$ ; that is,  $M_w$  is the eigenspace for  $h$  with eigenvalue  $w$ . In this paper we are primarily concerned with the weight spaces  $M_0$  and  $M_2$  and the linear map  $X: M_0 \rightarrow M_2$  given by the action of  $x$ . For every  $n \in \mathbb{Z}$ ,  $n \geq 0$ , there is an irreducible representation  $V(n)$  of  $\mathfrak{sl}_2(\mathbb{C})$  with highest weight  $n$  and dimension  $n + 1$ ; the action of  $\mathfrak{sl}_2(\mathbb{C})$  with respect to the basis  $\{v_{n-2i} \mid i = 0, \dots, n\}$  of weight vectors in  $V(n)$  is as follows:

$$\begin{aligned} h.v_{n-2i} &= (n-2i)v_{n-2i}, \\ x.v_n &= 0, \quad x.v_{n-2i} = (n-i+1)v_{n-2i+2} \quad (i = 1, \dots, n), \\ y.v_{n-2i} &= (i+1)v_{n-2i-2} \quad (i = 0, \dots, n-1), \quad y.v_{-n} = 0. \end{aligned}$$

Any irreducible finite-dimensional representation of  $\mathfrak{sl}_2(\mathbb{C})$  is isomorphic to  $V(n)$  for some  $n$ . Any finite-dimensional representation of  $\mathfrak{sl}_2(\mathbb{C})$  is isomorphic to a direct sum of irreducible representations.

**Lemma 5.** *If  $M$  is a finite-dimensional representation of  $\mathfrak{sl}_2(\mathbb{C})$ , then the  $x$ -action map  $X: M_0 \rightarrow M_2$  is surjective.*

*Proof.* We first write  $M = V(n_1) \oplus \dots \oplus V(n_k)$  as the direct sum of irreducible representations. For any weight  $w \in \mathbb{Z}$  we have  $M_w = V(n_1)_w \oplus \dots \oplus V(n_k)_w$  and so it suffices to prove the claim for an irreducible representation  $M = V(n)$ . If  $n = 0$  or  $n$  is odd then  $M_2 = \{0\}$  and the claim is vacuous. If  $n \geq 2$  is even then let  $m \in M_2$  be arbitrary; thus  $m = cv_2$  for some  $c \in \mathbb{C}$ . By the action of  $\mathfrak{sl}_2(\mathbb{C})$  on  $V(n)$  we have

$$x.(y.v_2) = x.\left(\frac{n}{2}v_0\right) = \frac{n}{2}(x.v_0) = \frac{n}{2}\left(\frac{n}{2} + 1\right)v_2.$$

Since  $\frac{n}{2}(\frac{n}{2} + 1) \neq 0$ , this completes the proof.  $\square$

This result holds more generally for any simple (finite-dimensional) Lie algebra  $G$  and any finite-dimensional representation  $M$ . Consider a simple root vector  $x_i \in G$  and the  $x_i$ -action map  $X_i: M_{\mathbf{0}} \rightarrow M_{\mathbf{w}_i}$  where  $\mathbf{w}_i$  is the weight of  $x_i$ . There exist  $h_i, y_i \in G$  for which the span of  $x_i, h_i, y_i$  is a subalgebra isomorphic to  $\mathfrak{sl}_2(\mathbb{C})$ . We regard  $M$  as a representation of this subalgebra and apply Lemma 5 to conclude that  $X_i$  is surjective. We use this result in the cases  $G = \mathfrak{sl}_2(\mathbb{C})$  and  $G = \mathfrak{sl}_3(\mathbb{C})$ .

### 3. LIE INVARIANTS IN THE NATURAL REPRESENTATION OF $\mathfrak{sl}_2(\mathbb{C})$

We consider the free Lie algebra  $L$  generated by the ordered set  $A = \{a, b\}$  with  $a < b$ . The generators  $a$  and  $b$  are the Hall words of degree 1; they form a basis of the subspace  $L_1$ . We regard the 2-dimensional space  $L_1$  as the natural representation of  $\mathfrak{sl}_2(\mathbb{C})$  by identifying  $a$  and  $b$  with column vectors as follows:

$$a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The natural action of  $\mathfrak{sl}_2(\mathbb{C})$  on  $L_1$  by matrix-vector multiplication gives

$$x.a = 0, \quad x.b = a, \quad h.a = a, \quad h.b = -b, \quad y.a = b, \quad y.b = 0.$$

We define the action of  $\mathfrak{sl}_2(\mathbb{C})$  on the Hall basis of  $L_j$  inductively by degree. For degree 1, the Hall words are  $a$  and  $b$ , and we use the previous equations. If  $[t, u]$  is a Hall word of degree  $j \geq 2$ , and  $D$  is any element of  $\mathfrak{sl}_2(\mathbb{C})$ , then we use the derivation rule  $D.[t, u] = [D.t, u] + [t, D.u]$ . This action extends linearly to  $L_j$ .

**Definition 6.** By the *weight*  $w$  of the Hall word  $t$ , we mean its eigenvalue with respect to the action by  $h$ : that is,  $h.t = wt$ . The *weight space*  $L_j^w$  is the subspace of  $L_j$  spanned by the Hall words of weight  $w$ .

**Notation 7.** If  $t$  is any Hall word then  $\#a(t)$  and  $\#b(t)$  denote respectively the number of  $a$ 's and  $b$ 's occurring in  $t$ .

**Lemma 8.** *We have  $L_j^w = \text{span}\{t \mid \#a(t) - \#b(t) = w\}$ .*

*Proof.* By induction on  $j$  since the weight of  $a$  is 1 and the weight of  $b$  is  $-1$ .  $\square$

Our problem is to determine which linear combinations of the Hall words of degree  $j$  are invariant under the natural action of  $\mathfrak{sl}_2(\mathbb{C})$ . That is, we want to find the elements  $Z \in L_j$  which satisfy  $D.Z = 0$  for all  $D \in \mathfrak{sl}_2(\mathbb{C})$ .

**Lemma 9.** *The Lie invariants of degree  $j$  for  $\mathfrak{sl}_2(\mathbb{C})$  in the natural representation are the kernel of the linear  $x$ -action map  $X: L_j^0 \rightarrow L_j^2$  defined by  $X(Z) = x.Z$ .*

*Proof.* Since  $L_j$  is finite-dimensional and  $\mathfrak{sl}_2(\mathbb{C})$  is semisimple Lie algebra, Weyl's Theorem implies that  $L_j$  is isomorphic to a direct sum of simple highest weight modules. If  $Z \in L_j$  satisfies  $D.Z = 0$  for all  $D \in \mathfrak{sl}_2(\mathbb{C})$ , then clearly  $h.Z = 0$ , and so  $Z$  is in  $L_j^0$ , the weight space of weight 0. If  $D \neq h$  then it suffices to consider  $D = x$  and  $D = y$ . Since  $x.a = 0$  and  $x.b = a$ , it is easy to see that the action of  $x$  induces a linear map from  $L_j^0$  to  $L_j^2$ , and the action of  $y$  induces a linear map from  $L_j^0$  to  $L_j^{-2}$ . Thus we need to find all  $Z \in L_j^0$  such that  $x.Z = 0$  and  $y.Z = 0$ . But the equations  $h.Z = 0$  and  $x.Z = 0$  imply that  $Z$  spans a 1-dimensional simple  $\mathfrak{sl}_2(\mathbb{C})$ -submodule of  $L_j^0$  isomorphic to the highest weight module with highest weight 0,

and this implies that  $y.Z = 0$ . So it suffices to find the  $Z \in L_j$  such that  $h.Z = 0$  and  $x.Z = 0$ .  $\square$

Bremner [4] determined all the Lie invariants of degree  $\leq 10$  for the natural representation of  $\mathfrak{sl}_2(\mathbb{C})$ . By the results of Wever [21, 22], we know that these invariants can occur only in even degree. We now extend these computations to degree 12. We first review the results for degree  $\leq 10$ : we confirm and simplify previously known results to illustrate our computational methods.

**Lemma 10.** *Any primitive Lie invariant of degree  $\leq 10$  in the natural representation of  $\mathfrak{sl}_2(\mathbb{C})$  is a linear combination of the following six Lie polynomials:*

$$\begin{aligned} I_2 &= [b, a], & I_6 &= [[[ba]b][[ba]a]], & I_{10}^{(1)} &= [[[[[ba]b][ba]][[[ba]a][ba]]], \\ I_{10}^{(2)} &= [[[[[ba]a]a]b][[[ba]b][ba]]] - 2[[[[[ba]a]b]b][[[ba]a][ba]]] - 3[[[[[ba]a]b]b][[[ba]a]a]b]] \\ &\quad + [[[[[ba]b]b]b][[[ba]a]a]a]], \\ I_{10}^{(3)} &= [[[[[ba]a]a][ba]][[[ba]b]b]] - 2[[[[[ba]a]b][ba]][[[ba]a]b]] + [[[[[ba]b]b][ba]][[[ba]a]a]], \\ I_{10}^{(4)} &= [[[[[ba]b][[ba]a]][[[ba]a]b]] + [[[[[ba]a]a][[ba]b]][[ba]b]] - 2[[[[[ba]a]b][[ba]a]][[ba]b]] \\ &\quad + [[[[[ba]b]b][[ba]a]][[ba]a]]. \end{aligned}$$

*Proof.* For degree 2, there is only one Hall word, namely  $I_2 = [b, a]$ , and we easily verify that this is an invariant:

$$x.[b, a] = [x.b, a] + [b, x.a] = [a, a] + [b, 0] = 0.$$

For degree 4, there is only one Hall word of weight 0, namely  $[[[b, a], a], b]$ , and only one Hall word of weight 2, namely  $[[[b, a], a], a]$ . We calculate

$$\begin{aligned} x.[[[b, a], a], b] &= [[[x.b, a], a], b] + [[[b, x.a], a], b] + [[[b, a], x.a], b] + [[[b, a], a], x.b] \\ &= [[[a, a], a], b] + [[[b, 0], a], b] + [[[b, a], 0], b] + [[[b, a], a], a] \\ &= 0 + 0 + 0 + [[[b, a], a], a]. \end{aligned}$$

Thus  $x.[[[b, a], a], b] \neq 0$  and so there is no invariant in degree 4. (This also follows from the surjectivity of the  $x$ -action map; see Lemma 5.) For degree 6, we have three Hall words of weight 0, and two Hall words of weight 2:

$$[[[ba]b][[ba]a]], \quad [[[[ba]a]b][ba]], \quad [[[[[ba]a]a]b]b]; \quad [[[[ba]a]a][ba]], \quad [[[[[ba]a]a]a]b].$$

(From now on we omit the commas in Hall words.) We calculate

$$\begin{aligned} x.[[[ba]b][[ba]a]] &= 0, & x.[[[[ba]a]b][ba]] &= [[[[ba]a]a][ba]], \\ x.[[[[ba]a]a]b]b] &= [[[[[ba]a]a][ba]] + 2[[[[[ba]a]a]a]b]. \end{aligned}$$

Hence the kernel of the  $x$ -action has dimension 1 and basis  $I_6 = [[[[ba]b][[ba]a]]$ . (The invariants of degree  $\leq 6$  were first found by Magnus [15].) For degree 8, we have 8 Hall words of weight 0, and 7 of weight 2:

$$\begin{aligned} &[[[[[ba]b]b][[[ba]a]a]], \quad [[[[[ba]a]a][ba]][[ba]b]], \quad [[[[[ba]b][ba]][[ba]a]], \quad [[[[[ba]a]a]b][[ba]b]], \\ &[[[[[ba]a]b]b][[[ba]a]a]], \quad [[[[[[ba]a]b][ba]][[ba]a]], \quad [[[[[[[ba]a]a]b]b][ba]], \quad [[[[[[[ba]a]a]a]b]b]b]], \\ &[[[[[ba]a]b][[[ba]a]a]], \quad [[[[[ba]a]a][ba]][[ba]a]], \quad [[[[[[ba]a]a]a][[ba]b]], \quad [[[[[[ba]a]a]b][[ba]a]], \\ &[[[[[[ba]a]a][ba]][[ba]a]], \quad [[[[[[[ba]a]a]a]b][ba]], \quad [[[[[[[[ba]a]a]a]a]b]b]]. \end{aligned}$$

Using these ordered bases, we compute the matrix  $[X]$  representing the  $x$ -action map  $X: L_8^0 \rightarrow L_8^2$ :

$$[X] = \begin{bmatrix} 2 & . & . & . & . & . & . & . \\ . & 1 & 1 & . & 1 & . & . & . \\ . & . & . & 1 & . & . & . & 2 \\ . & . & . & 1 & 2 & . & . & . \\ . & . & . & . & . & 1 & 1 & . \\ . & . & . & . & . & . & 2 & 3 \\ . & . & . & . & . & . & . & 3 \end{bmatrix}$$

This matrix has rank 7, and so its nullspace has dimension 1; a basis for the nullspace consists of the invariant

$$[[[ba]a][ba]][[ba]b] - [[[ba]b][ba]][[ba]a].$$

But this is the Hall form of  $[I_2, I_6] = [[ba][[ba]b][[ba]a]]$ , the Lie bracket of the invariants of degree 2 and 6, so there are no primitive invariants in degree 8. For degree 10, there are 25 words of weight 0, and 20 words of weight 2. The  $x$ -action matrix  $[X]$  has size  $20 \times 25$ ; using Maple we find that its rank is 20, and so its nullspace has dimension 5. From the row canonical form of  $[X]$  we obtain the canonical basis of the space of invariants:

$$J_{10}^{(1)} = [[[[ba]b][ba]][[ba]a][ba]],$$

$$J_{10}^{(2)} = [[[[[ba]a]a]b][[[ba]b][ba]]] - 2[[[[ba]a]b]b][[[ba]a][ba]]] - 3[[[[ba]a]b]b][[[ba]a]a]b] + [[[[[ba]b]b]b][[[ba]a]a]a]],$$

$$J_{10}^{(3)} = [[[[[ba]a]a][ba]][[[ba]b]b]] - 2[[[[ba]a]b][ba]][[[ba]a]b]] + [[[[[ba]b]b][ba]][[[ba]a]a]],$$

$$J_{10}^{(4)} = [[[[ba]b][[ba]a]][[[ba]a]b]] + [[[[[ba]a]a][ba]b]][[ba]b]] - 2[[[[ba]a]b][[ba]a]][[ba]b]] + [[[[[ba]b]b][[ba]a]][[ba]a]],$$

$$J_{10}^{(5)} = -[[[[[ba]a][ba]][ba]][[ba]b]] + [[[[[ba]b][ba]][ba]][[ba]a]].$$

From the primitive invariants  $I_2$  and  $I_6$  in degrees 2 and 6 we can obtain only one invariant in degree 10, namely  $[I_2, [I_2, I_6]]$ . The Hall form of this invariant is

$$K_{10} = 2[[[[ba]b][ba]][[[ba]a][ba]]] - [[[[[ba]a][ba]][ba]][[ba]b]]] + [[[[[ba]b][ba]][ba]][[ba]a]]].$$

Hence there is a 4-dimensional space of primitive invariants in degree 10: we can choose any subspace, of the 5-dimensional space with basis  $\{J_{10}^{(1)}, \dots, J_{10}^{(5)}\}$ , which is complementary to the 1-dimensional span of  $K_{10}$ . In particular, we easily check that the elements  $\{K_{10}, J_{10}^{(1)}, \dots, J_{10}^{(4)}\}$  are linearly independent, and so we may take  $\{J_{10}^{(1)}, \dots, J_{10}^{(4)}\}$  as the canonical set of primitive invariants in degree 10: these are the invariants  $I_{10}^{(1)}, \dots, I_{10}^{(4)}$  in the statement of the Lemma. (The invariants of degree 10 were first found by Bremner [4] in a slightly more complicated form.)  $\square$

We now extend these results to degree 12.

**Theorem 11.** *Any linear combination of the following four Lie polynomials is a primitive Lie invariant of degree 12 in the natural representation of  $\mathfrak{sl}_2(\mathbb{C})$ :*

$$I_{12}^{(1)} = [[[[[ba]a][ba]][ba]][[[ba]b][ba]]] - [[[[[ba]b][ba]][ba]][[[ba]a][ba]]],$$

$$I_{12}^{(2)} = [[[[[[ba]a][ba]][ba]][ba]][[ba]b]] - [[[[[ba]b][ba]][ba]][ba]][[ba]a]],$$

$$\begin{aligned}
I_{12}^{(3)} &= [[[[[ba]a]a][ba]][ba]][[ba]b]b] - 2[[[[[ba]a]b][ba]][ba]][[ba]a]b] \\
&\quad + [[[[[ba]b]b][ba]][ba]][[ba]a]a], \\
I_{12}^{(4)} &= [[[[[ba]a]b][ba]][[ba]b][ba]a]] - [[[[[ba]a]a][ba]][[ba]b]][[ba]b]] \\
&\quad + 2[[[[[ba]a]b][ba]][[ba]a]][[ba]b] - [[[[[ba]b]b][ba]][[ba]a]][[ba]a]].
\end{aligned}$$

*Proof.* There are 335 Hall words in degree 12, with 75 of weight 0 and 66 of weight 2. We now use Maple for the following calculations:

- (1) compute the action of  $x$  on each of the Hall words  $w$  of weight 0;
- (2) compute the Hall form of each resulting word of weight 2;
- (3) collect and sort the Hall words of weight 2 appearing in  $x.w$ ;
- (4) construct the matrix  $[X]$  representing the linear map  $X: L_{12}^0 \rightarrow L_{12}^2$ ;
- (5) compute the row canonical form of the  $66 \times 75$  matrix  $[X]$ ;
- (6) extract the canonical basis of the nullspace of  $[X]$ ;
- (7) sort the canonical basis vectors by increasing Euclidean norm.

The rank of  $[X]$  is 66, and so the nullspace has dimension 9. The canonical basis of the subspace of invariants in degree 12 consists of the following elements:

$$\begin{aligned}
J_{12}^{(1)} &= -[[[[[ba]a][ba]][ba]][[ba]b][ba]] + [[[[[ba]b][ba]][ba]][[ba]a][ba]], \\
J_{12}^{(2)} &= -[[[[[ba]a][ba]][ba]][ba]][[ba]b] + [[[[[ba]b][ba]][ba]][ba]][[ba]a], \\
J_{12}^{(3)} &= [[[[[ba]a]a][[ba]b]][[ba]b][ba]] - [[[[[ba]a]b][[ba]a]][[ba]b][ba]] \\
&\quad - [[[[[ba]a]b][[ba]b]][[ba]a][ba]] + [[[[[ba]b]b][[ba]a]][[ba]a][ba]], \\
J_{12}^{(4)} &= [[[[[ba]a][ba]][[ba]a]][[ba]b]b] - [[[[[ba]a][ba]][[ba]b]][[ba]a]b] \\
&\quad - [[[[[ba]b][ba]][[ba]a]][[ba]a]b] + [[[[[ba]b][ba]][[ba]b]][[ba]a]a], \\
J_{12}^{(5)} &= [[[[[ba]a]a][ba]][ba]][[ba]b]b] - 2[[[[[ba]a]b][ba]][ba]][[ba]a]b] \\
&\quad + [[[[[ba]b]b][ba]][ba]][[ba]a]a], \\
J_{12}^{(6)} &= -[[[[[ba]a]b][ba]][[ba]b][[ba]a]] + [[[[[ba]a]a][ba]][[ba]b]][[ba]b] \\
&\quad - 2[[[[[ba]a]b][ba]][[ba]a]][[ba]b] + [[[[[ba]b]b][ba]][[ba]a]][[ba]a], \\
J_{12}^{(7)} &= [[[[[ba]a][ba]][[ba]a]][[ba]b]b] + [[[[[ba]a][ba]][[ba]b]][[ba]a]b] \\
&\quad - [[[[[ba]b][ba]][[ba]a]][[ba]a]b] - [[[[[ba]a]a][[ba]b]][[ba]b]b] \\
&\quad + [[[[[ba]a]a]b][[ba]a]][[ba]b]b] + 2[[[[[ba]a]a]b][[ba]b]][[ba]a]b] \\
&\quad - 2[[[[[ba]a]b]b][[ba]a]][[ba]a]b] - [[[[[ba]a]b]b][[ba]b]][[ba]a]a] \\
&\quad + [[[[[ba]b]b]b][[ba]a]][[ba]a]a], \\
J_{12}^{(8)} &= -[[[[[ba]a]a][[ba]a]][[ba]b]b] + [[[[[ba]a]a][[ba]b]][[ba]b][ba]] \\
&\quad + [[[[[ba]a]a][[ba]b]][[ba]a]b] + 2[[[[[ba]a]b][[ba]a]][[ba]a]b] \\
&\quad - 2[[[[[ba]a]b][[ba]b]][[ba]a][ba]] - 2[[[[[ba]a]b][[ba]b]][[ba]a]a] \\
&\quad - [[[[[ba]b]b][[ba]a]][[ba]a]b] + [[[[[ba]b]b][[ba]b]][[ba]a]a], \\
J_{12}^{(9)} &= 2[[[[[ba]a][ba]][ba]][[ba]a]b] - [[[[[ba]b][ba]][ba]][[ba]a]a] \\
&\quad - [[[[[ba]a]a][ba]][[ba]b]b] + [[[[[ba]a]a]b][ba]][[ba]b][ba]] \\
&\quad + 3[[[[[ba]a]a]b][ba]][[ba]a]b] - 2[[[[[ba]a]b]b][ba]][[ba]a][ba]]
\end{aligned}$$

$$- 3[[[[[ba]a]b]b][ba]][[[[ba]a]a]b]] + [[[[[ba]b]b]b][ba]][[[[ba]a]a]a]].$$

We compute the non-primitive invariants in degree 12 by taking the Lie bracket of the invariant  $I_2$  in degree 2 with the basis invariants  $J_{10}^{(1)}, \dots, J_{10}^{(5)}$  in degree 10, and reducing all the words to their Hall form:

$$\begin{aligned} [I_2, J_{10}^{(1)}] &\rightarrow K_{12}^{(1)} = [[[[[ba]a][ba]][ba]][[[[ba]b][ba]]] - [[[[[ba]b][ba]][ba]][[[[ba]a][ba]]], \\ [I_2, J_{10}^{(2)}] &\rightarrow K_{12}^{(2)} = -2[[[[[ba]a][ba]][ba]][[[[ba]a]b]b]] + [[[[[ba]b][ba]][ba]][[[[ba]a]a]b]] \\ &\quad + [[[[[ba]a]a]a][ba]][[[[ba]b]b]b]] - [[[[[ba]a]a]b][ba]][[[[ba]b][ba]]] \\ &\quad - 3[[[[[ba]a]a]b][ba]][[[[ba]a]b]b]] + 2[[[[[ba]a]b]b][ba]][[[[ba]a][ba]]] \\ &\quad + 3[[[[[ba]a]b]b][ba]][[[[ba]a]a]b]] - [[[[[ba]b]b]b][ba]][[[[ba]a]a]a], \\ [I_2, J_{10}^{(3)}] &\rightarrow K_{12}^{(3)} = -[[[[[ba]a]a][ba]][ba]][[[[ba]b]b]] + 2[[[[[ba]a]b][ba]][ba]][[[[ba]a]b]] \\ &\quad - [[[[[ba]b]b][ba]][ba]][[[[ba]a]a]], \\ [I_2, J_{10}^{(4)}] &\rightarrow K_{12}^{(4)} = [[[[[ba]a]b][ba]][[[[ba]b][ba]a]]] - 2[[[[[ba]a]a][ba]b]][[[[ba]b][ba]]] \\ &\quad + 2[[[[[ba]a]b][ba]a]][[[[ba]b][ba]]] + 2[[[[[ba]a]b][ba]b]][[[[ba]a][ba]]] \\ &\quad - 2[[[[[ba]b]b][ba]a]][[[[ba]a][ba]]] + [[[[[ba]a][ba]][ba]a]][[[[ba]b]b]] \\ &\quad - [[[[[ba]a][ba]][ba]b]][[[[ba]a]b]] - [[[[[ba]b][ba]][ba]a]][[[[ba]a]b]] \\ &\quad + [[[[[ba]b][ba]][ba]b]][[[[ba]a]a]] - [[[[[ba]a]a][ba]][ba]b]][[[[ba]b]]] \\ &\quad + 2[[[[[ba]a]b][ba]][ba]a]][[[[ba]b]]] - [[[[[ba]b]b][ba]][ba]a]][[[[ba]a]], \\ [I_2, J_{10}^{(5)}] &\rightarrow K_{12}^{(5)} = [[[[[ba]a][ba]][ba]][[[[ba]b][ba]]] - [[[[[ba]b][ba]][ba]][[[[ba]a][ba]]] \\ &\quad + [[[[[ba]a][ba]][ba]][ba]][[[[ba]b]]] - [[[[[ba]b][ba]][ba]][ba]][[[[ba]a]]. \end{aligned}$$

Hence there is a 4-dimensional space of primitive invariants in degree 12: we can choose any subspace, of the 9-dimensional space with basis  $\{J_{12}^{(1)}, \dots, J_{12}^{(9)}\}$ , which is complementary to the 5-dimensional span of  $\{K_{12}^{(1)}, \dots, K_{12}^{(5)}\}$ . In particular, we easily check that the elements  $\{K_{12}^{(1)}, \dots, K_{12}^{(5)}, J_{12}^{(1)}, J_{12}^{(2)}, J_{12}^{(5)}, J_{12}^{(6)}\}$  are linearly independent, and so we may take  $\{J_{12}^{(1)}, J_{12}^{(2)}, J_{12}^{(5)}, J_{12}^{(6)}\}$  as the canonical set of primitive invariants in degree 12. These (with sign changes) are the invariants  $I_{12}^{(1)}, \dots, I_{12}^{(4)}$  in the statement of the Theorem. This completes the proof.  $\square$

We extended these calculations to degree 14, and obtained a 33-dimensional space of invariants, with a 10-dimensional subspace of non-primitive invariants spanned by Lie brackets of invariants of lower degree. Hence there is a 23-dimensional space of primitive invariants in degree 14; see Hu [11] for details.

#### 4. LIE INVARIANTS IN THE ADJOINT REPRESENTATION OF $\mathfrak{sl}_2(\mathbb{C})$

In this section, we consider for the first time the Lie invariants for a simple finite-dimensional Lie algebra in a representation other than the natural representation. We study the simplest case, namely the adjoint representation of  $\mathfrak{sl}_2(\mathbb{C})$ .

We consider the free Lie algebra  $L$  generated by the ordered set  $A = \{a, b, c\}$  with  $a < b < c$ . The generators  $a, b$  and  $c$  are the Hall words of degree 1; they form a basis of the subspace  $L_1$ . We regard the 3-dimensional space  $L_1$  as the adjoint representation of  $\mathfrak{sl}_2(\mathbb{C})$  by identifying  $a, b, c$  with  $x, h, y$  respectively. The

matrices for adjoint representation of  $\mathfrak{sl}_2(\mathbb{C})$  are as follows:

$$\text{ad}(x) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{ad}(h) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \text{ad}(y) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

This gives the action of  $\mathfrak{sl}_2(\mathbb{C})$  on  $L_1$ :

$$\begin{aligned} x.a &= 0, & x.b &= -2a, & x.c &= b, \\ h.a &= 2a, & h.b &= 0, & h.c &= -2c, \\ y.a &= -b, & y.b &= 2b, & y.c &= 0. \end{aligned}$$

**Lemma 12.** *We have  $L_j^w = \text{span}\{t \mid 2\#a(t) - 2\#c(t) = w\}$ .*

*Proof.* The  $h$ -action shows that  $a, b, c$  have weights 2, 0,  $-2$  respectively.  $\square$

**Lemma 13.** *The Lie invariants of degree  $j$  for  $\mathfrak{sl}_2(\mathbb{C})$  in the adjoint representation are the kernel of the linear  $x$ -action map  $X: L_j^0 \rightarrow L_j^2$  defined by  $X(Z) = x.Z$ .*

*Proof.* Similar to the proof of Lemma 9.  $\square$

We remark that the dimension formulas of Wever [21, 22] and Burrow [6, 7] do not apply in this case, since they apply only to the natural representation. The Lie invariants for the adjoint representation appear not to have been studied before.

**Lemma 14.** *There are no Lie invariants for the adjoint representation of  $\mathfrak{sl}_2(\mathbb{C})$  in degree  $\leq 4$ .*

*Proof.* In degree 1, there is one Hall word  $b$  of weight 0 and one Hall word  $a$  of weight 2. Since  $x.b = -2a$ , the kernel of the  $x$ -action is zero. In degree 2, there is one Hall word  $[c, a]$  of weight 0 and one Hall word  $[b, a]$  of weight 2. Since  $x.[c, a] = [b, a]$ , the kernel of the  $x$ -action is zero. In degree 3, there are two Hall words of weight 0, namely  $[[b, a], c]$  and  $[[c, a], b]$ , and two of weight 2, namely  $[[b, a], b]$  and  $[[c, a], a]$ . We have  $x. [[b, a], c] = [[b, a], b]$  and  $x. [[c, a], b] = [[b, a], b] - 2[[c, a], a]$ , and again the kernel of the  $x$ -action is zero. In degree 4, there are four Hall words of weight 0, and four of weight 2,

$$\begin{aligned} &[[c, b], [b, a]], \quad [[[b, a], b], c], \quad [[[c, a], a], c], \quad [[[c, a], b], b]; \\ &[[c, a], [b, a]], \quad [[[b, a], a], c], \quad [[[b, a], b], b], \quad [[[c, a], a], b]. \end{aligned}$$

The  $x$ -action is given by these equations:

$$\begin{aligned} x. [[c, b], [b, a]] &= -2[[c, a], [b, a]], & x. [[[b, a], b], c] &= -2[[[b, a], a], c] + [[[b, a], b], b], \\ x. [[[c, a], a], c] &= [[[b, a], a], c] + [[[c, a], a], b], \\ x. [[[c, a], b], b] &= -2[[c, a], [b, a]] + [[[b, a], b], b] - 4[[[c, a], a], b]. \end{aligned}$$

The  $x$ -action matrix has full rank, and so there are no invariants. We remark that these results also follow from Lemma 5 on the surjectivity of the  $x$ -action map, since in every case the numbers of Hall words for weights 0 and 2 are equal.  $\square$

**Theorem 15.** *For the adjoint representation of  $\mathfrak{sl}_2(\mathbb{C})$ , the space of invariants in degree 5 has dimension 1 and basis*

$$I_5 = [[[[b, a], b], [c, b]] + 4[[[b, a], c], [c, a]] + 2[[[c, a], a], [c, b]] - 2[[[c, a], b], [c, a]] - 2[[[c, a], c], [b, a]] - [[[c, b], b], [b, a]].$$

*Proof.* In degree 5, there are 10 Hall words of weight 0,

$$\begin{aligned} & [[[ba]b][cb]], \quad [[[ba]c][ca]], \quad [[[ca]a][cb]], \quad [[[ca]b][ca]], \quad [[[ca]c][ba]], \\ & [[[cb]b][ba]], \quad [[[ba]a]c], \quad [[[ba]b]b]c, \quad [[[ca]a]b]c, \quad [[[ca]b]b]b, \end{aligned}$$

and 9 of weight 2,

$$\begin{aligned} & [[[ba]a][cb]], \quad [[[ba]b][ca]], \quad [[[ba]c][ba]], \quad [[[ca]a][ca]], \quad [[[ca]b][ba]], \\ & [[[ba]a]b]c, \quad [[[ba]b]b]b, \quad [[[ca]a]a]c, \quad [[[ca]a]b]b. \end{aligned}$$

We calculate the  $x$ -action on the words of weight 0, and express the results as linear combinations of the words of weight 2. We obtain the following matrix  $[X]$  representing the map  $X: L_5^0 \rightarrow L_5^2$ :

$$[X] = \begin{bmatrix} -2 & . & 1 & . & . & . & 1 & . & . & . \\ -2 & 1 & . & 1 & . & . & . & . & . & 4 \\ . & 1 & . & . & 1 & 2 & . & . & . & . \\ . & . & -2 & -2 & . & . & . & . & . & . \\ . & . & . & 1 & 1 & -4 & . & . & . & -6 \\ . & . & . & . & . & . & 2 & -4 & 1 & . \\ . & . & . & . & . & . & . & 1 & . & 1 \\ . & . & . & . & . & . & . & . & -2 & . \\ . & . & . & . & . & . & . & . & 1 & -6 \end{bmatrix}$$

We compute the row canonical form of this matrix, and find that the rank is 9, so the nullspace has dimension 1. A basis of the nullspace is the invariant  $I_5$ .  $\square$

**Example 16.** We can give an explicit direct proof that  $I_5$  is an invariant. We calculate as follows, using Hall's algorithm to simplify the results:

$$\begin{aligned} x. [[[ba]b][cb]] &= -2[[[ba]a][cb]] - 2[[[ba]b][ca]], \\ x. [[[ba]c][ca]] &= [[[ba]b][ca]] + [[[ba]c][ba]], \\ x. [[[ca]a][cb]] &= [[[ba]a][cb]] - 2[[[ca]a][ca]], \\ x. [[[ca]b][ca]] &= [[[ba]b][ca]] - 2[[[ca]a][ca]] + [[[ca]b][ba]], \\ x. [[[ca]c][ba]] &= [[[ba]c][ba]] + [[[ca]b][ba]], \\ x. [[[cb]b][ba]] &= 2[[[ba]c][ba]] - 4[[[ca]b][ba]]. \end{aligned}$$

The linear combination with coefficients 1, 4, 2, -2, -2, -1 gives 0 as required.

**Theorem 17.** *For the adjoint representation of  $\mathfrak{sl}_2(\mathbb{C})$ , the space of invariants in degree 6 has dimension 1 and basis*

$$\begin{aligned} I_6 &= 4[[[cb]b][ba]] - 2[[[cb][ba]][ca]] - 4[[[ba]a]c][cb] - [[[ba]b]b][cb] + 4[[[ba]c]c][ba]] \\ &\quad - 8[[[ca]a]c][ca] - 2[[[ca]b]b][ca] - 4[[[ca]b]c][ba] - [[[cb]b]b][ba]]. \end{aligned}$$

*Proof.* In degree 6, there are 22 Hall words of weight 0 and 21 of weight 2. The  $x$ -action has a 1-dimensional kernel with  $I_6$  as its basis.  $\square$

**Remark 18.** Since there are no invariants in degree  $\leq 4$ , only one invariant in degree 5 and only one in degree 6, it follows that the smallest degree for a non-primitive invariant is 11. That is, every invariant in degree  $\leq 10$  is primitive.

In degree 7, there are 56 Hall words of weight 0, and 51 of weight 2. Hence the  $x$ -action matrix  $[X]$  has size  $51 \times 56$ ; we use Maple to compute that its rank is 51, and so its nullspace has dimension 5. If we compute the row canonical form of  $[X]$  over  $\mathbb{Q}$ , most of the entries of the RCF are integers, but some are fractions with denominator 2. We extract the canonical basis of the nullspace, and clear denominators where necessary to obtain a basis for the nullspace consisting of vectors with relatively prime integer components. The squared Euclidean norms of these vectors are 15, 23, 514, 690, 3218. We can obtain better results by computing an integral basis for the nullspace using the Hermite normal form combined with the LLL algorithm for lattice basis reduction.

**Definition 19.** The  $m \times n$  matrix  $H$  over  $\mathbb{Z}$  is in *Hermite Normal Form (HNF)* if there exists  $r \in \mathbb{Z}$ ,  $0 \leq r \leq m$ , and integers  $1 \leq j_1 < j_2 < \cdots < j_r \leq n$  such that

- $H_{ij} = 0$  for  $1 \leq i \leq r$  and  $1 \leq j < j_i$ ,
- $H_{i,j_i} \geq 1$  for  $1 \leq i \leq r$ ,
- $0 \leq H_{k,j_i} < H_{i,j_i}$  for  $1 \leq i \leq r$  and  $1 \leq k < i$ ,
- $H_{ij} = 0$  for  $r+1 \leq i \leq m$  and  $1 \leq j \leq n$ .

**Proposition 20.** *If  $A$  is an  $m \times n$  matrix over  $\mathbb{Z}$ , then there is a unique  $m \times n$  matrix  $H$  over  $\mathbb{Z}$  in HNF such that  $UA = H$  for some  $m \times m$  matrix  $U$  over  $\mathbb{Z}$  with  $\det(U) = \pm 1$ . (The matrix  $U$  is in general not unique.)*

*Proof.* Adkins and Weintraub [1, §5.2]. □

**Proposition 21.** *Let  $A$  be an  $m \times n$  integer matrix, let  $H$  be the Hermite normal form of the transpose  $A^t$ , and let  $U$  be an  $n \times n$  matrix with  $\det(U) = \pm 1$  such that  $UA^t = H$ . If  $r$  is the rank of  $H$ , then the last  $n - r$  rows of  $U$  form a lattice basis for the integer nullspace of  $A$ .*

*Proof.* Cohen [8, Proposition 2.4.9]. □

In degree 7, if we use this command from Maple's `LinearAlgebra` package,

```
HermiteForm( xactionmatrix, output='U' );
```

and extract the last 5 rows of the result, then we obtain an integer basis for the nullspace with the following (sorted) squared Euclidean norms: 130977928, 264952077, 483975356, 571555922, 1935778474. At this point the results are much worse than those obtained using the RCF over  $\mathbb{Q}$ .

**Theorem 22.** *For the adjoint representation of  $\mathfrak{sl}_2(\mathbb{C})$ , the space of invariants in degree 7 has dimension 5 and the following basis:*

$$\begin{aligned}
I_7^{(1)} &= 2[[[ca][ba]][[ca]c] + [[[ca][ba]][[cb]b]] + 2[[[cb][ba]][[ba]c]] - [[[cb][ba]][[ca]b]] \\
&\quad - [[[cb][ca]][[ba]b]] - 2[[[cb][ca]][[ca]a]], \\
I_7^{(2)} &= [[[ca][ba]][[cb]b]] + [[[cb][ba]][[ba]c]] - [[[cb][ba]][[ca]b]] - 2[[[cb][ca]][[ca]a]] \\
&\quad + [[[[ba]a][cb]][cb]] - [[[[ba]b][ca]][cb]] + 2[[[[ca]a][ca]][cb]] + [[[[ca]b][ba]][cb]] \\
&\quad - 2[[[[ca]b][ca]][ca]] + 2[[[[ca]c][ba]][ca]] - [[[[cb]b][ba]][ca]] + [[[[cb]c][ba]][ba]], \\
I_7^{(3)} &= 2[[[ca][ba]][[ca]c]] - 2[[[ca][ba]][[cb]b]] + 3[[[cb][ba]][[ca]b]] - [[[cb][ca]][[ba]b]] \\
&\quad - [[[[ba]a]b][[cb]c]] + 2[[[[ba]a]c][[cb]b]] + [[[[ba]b]b][[ca]c]] - 3[[[[ba]b]c][[ca]b]] \\
&\quad + 2[[[[ba]c]c][[ba]b]] - 2[[[[ca]a]a][[cb]c]] + 2[[[[ca]a]b][[ca]c]] - [[[[ca]a]b][[cb]b]]
\end{aligned}$$

$$\begin{aligned}
& - 4[[[ca]a]c][[ba]c] + 2[[[ca]a]c][[ca]b] - [[[ca]b]b][[ba]c] + 2[[[ca]b]b][[ca]b] \\
& - [[[[ca]b]c][[ba]b] - 2[[[[ca]b]c][[ca]a] + 2[[[[ca]c]c][[ba]a] - [[[[cb]b]b][[ca]a] \\
& + [[[[cb]b]c][[ba]a],
\end{aligned}$$

$$\begin{aligned}
I_7^{(4)} &= 8[[[ca][ba]][[ca]c] - [[[ca][ba]][[cb]b] + [[[cb][ba]][[ba]c] + 5[[[cb][ba]][[ca]b] \\
& + 2[[[cb][ca]][[ca]a] - 2[[[[ba]a]b][[cb]c] - 8[[[[ba]a]c][[ca]c] - [[[[ba]b]b][[cb]b] \\
& - 6[[[[ba]b]c][[ca]b] - 8[[[[ba]c]c][[ca]a] - 4[[[[ca]a]a][[cb]c] + 4[[[[ca]a]b][[ca]c] \\
& - 2[[[[ca]a]b][[cb]b] + 8[[[[ca]a]c][[ba]c] - 4[[[[ca]a]c][[ca]b] + 2[[[[ca]b]b][[ba]c] \\
& + 2[[[[ca]b]b][[ca]b] + 2[[[[ca]b]c][[ba]b] + 4[[[[ca]b]c][[ca]a] + 4[[[[ca]c]c][[ba]a] \\
& + [[[[cb]b]b][[ba]b] + 2[[[[cb]b]c][[ba]a] + [[[[ba]a][cb]][[cb] - [[[[ba]b][ca]][[cb] \\
& + 2[[[[ca]a][ca]][[cb] + [[[[ca]b][ba]][[cb] - 2[[[[ca]b][ca]][[ca] + 2[[[[ca]c][ba]][[ca] \\
& - [[[[cb]b][ba]][[ca] + [[[[cb]c][ba]][[ba]],
\end{aligned}$$

$$\begin{aligned}
I_7^{(5)} &= 2[[[ca][ba]][[cb]b] - 8[[[cb][ba]][[ba]c] + 2[[[cb][ba]][[ca]b] - 6[[[cb][ca]][[ba]b] \\
& - 4[[[cb][ca]][[ca]a] + 4[[[[ba]a][cb]][[cb] + 6[[[[ba]b][ca]][[cb] + 8[[[[ba]c][ba]][[cb] \\
& + 32[[[[ba]c][ca]][[ca] + 12[[[[ca]a][ca]][[cb] - 12[[[[ca]b][ca]][[ca] - 20[[[[ca]c][ba]][[ca] \\
& - 8[[[[cb]b][ba]][[ca] - 4[[[[cb]c][ba]][[ba] + 4[[[[ba]a]b]c][[cb] + 16[[[[ba]a]c]c][[ca] \\
& + [[[[ba]b]b]b][[cb] + 4[[[[ba]b]b]c][[ca] + 4[[[[ba]b]c]c][[ba] + 8[[[[ca]a]a]c][[cb] \\
& + 2[[[[ca]a]b]b][[cb] - 8[[[[ca]a]b]c][[ca] - 8[[[[ca]a]c]c][[ba] - 2[[[[ca]b]b]b][[ca] \\
& - 6[[[[ca]b]b]c][[ba] - [[[[cb]b]b]b][[ba]].
\end{aligned}$$

*Proof.* If we use the Maple command

```
HermiteForm( xactionmatrix, output='U', method='integer[reduced]' );
```

and extract the last 5 rows of the result, then we obtain an integer basis for the nullspace with the following (sorted) squared Euclidean norms: 15, 24, 82, 446, 2574. This is a substantial improvement over the results obtained using the RCF. For further details on the application of HNF and LLL to problems in nonassociative algebra, see Bremner and Peresi [5].  $\square$

The invariants of degree 8 are calculated explicitly in Hu [11]. The matrix  $[X]$  has size  $127 \times 136$ ; by Lemma 5, its rank is 127 and so its nullspace has dimension 9. Using the canonical integral basis of the nullspace obtained from the row canonical form, we obtain the following sorted list of squared norms of the coefficient vectors of the invariants: 83, 95, 95, 143, 147, 150, 5030, 18490, 63770. Using the Hermite normal form with lattice basis reduction gives these squared norms: 32, 47, 47, 62, 83, 143, 1058, 2791, 31295. Again we have found significantly simpler invariants: the first four invariants using HNF and LLL have coefficient vectors shorter than the first invariant using the RCF.

We extended these calculations to degree 12, and obtained the results in Table 1. The invariants  $I_5$  and  $I_6$  imply that there is a 1-dimensional space of non-primitive invariants in degree 11; the invariants  $I_5$  and  $I_7^{(1)}, \dots, I_7^{(5)}$  imply that there is a 5-dimensional space of non-primitive invariants in degree 12. For degree  $\geq 9$  we used modular arithmetic with  $p = 101$  in order to ensure that each matrix entry would only use one byte of memory during the calculation of the row canonical form of the  $x$ -action matrix  $[X]$ .

degree	weight 0	weight 2	invariants	primitive
1	1	1	0	0
2	1	1	0	0
3	2	2	0	0
4	4	4	0	0
5	10	9	1	1
6	22	21	1	1
7	56	51	5	5
8	136	127	9	9
9	348	323	25	25
10	890	835	55	55
11	2332	2188	144	143
12	6136	5798	338	333

TABLE 1. Dimensions of invariants in the adjoint representation of  $\mathfrak{sl}_2(\mathbb{C})$ 5. LIE INVARIANTS IN THE NATURAL REPRESENTATION OF  $\mathfrak{sl}_3(\mathbb{C})$ 

In this section, we present explicit invariants of minimal degree for  $\mathfrak{sl}_3(\mathbb{C})$  in the natural representation. As implied by Wever [21, 22], there are no invariants in degree  $\leq 9$ , and a 4-dimensional space of invariants in degree 9. We find the first explicit basis for this 4-dimensional space.

We use the standard ordered basis of the 8-dimensional simple Lie algebra  $\mathfrak{sl}_3(\mathbb{C})$ :

$$\begin{aligned}
 x_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & x_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, & x_3 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
 h_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & h_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\
 y_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & y_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, & y_3 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
 \end{aligned}$$

We consider the free Lie algebra  $L$  generated by the ordered set  $A = \{a, b, c\}$  with  $a < b < c$ . The generators  $a, b, c$  are the Hall words of degree 1; they form a basis of the subspace  $L_1$ . We regard the 3-dimensional space  $L_1$  as the natural representation of  $\mathfrak{sl}_3(\mathbb{C})$  by identifying  $a, b, c$  with column vectors as follows:

$$a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The natural action of  $\mathfrak{sl}_3(\mathbb{C})$  on  $L_1$  by matrix-vector multiplication gives

$$\begin{aligned}
 x_1.a &= 0, & x_1.b &= a, & x_1.c &= 0, & x_2.a &= 0, & x_2.b &= 0, & x_2.c &= b, \\
 x_3.a &= 0, & x_3.b &= a, & x_3.c &= a, & h_1.a &= a, & h_1.b &= -b, & h_1.c &= 0, \\
 h_2.a &= 0, & h_2.b &= b, & h_2.c &= -c, & y_1.a &= b, & y_1.b &= 0, & y_1.c &= 0, \\
 y_2.a &= 0, & y_2.b &= c, & y_2.c &= 0, & y_3.a &= c, & y_3.b &= 0, & y_3.c &= 0.
 \end{aligned}$$

We define the action of  $\mathfrak{sl}_3(\mathbb{C})$  on the Hall words of degree  $j$ , which form a basis of  $L_j$ , inductively by degree as in the previous sections. Details about representations of  $\mathfrak{sl}_3(\mathbb{C})$  may be found in Fulton and Harris [9, Lecture 12].

**Definition 23.** By the *weight*  $(w_1, w_2)$  of the Hall word  $t$ , we mean the ordered pair of its eigenvalues with respect to the action of  $h_1$  and  $h_2$ . The *weight space*  $L_j^{(w_1, w_2)}$  is the subspace of  $L_j$  spanned by the Hall words of weight  $(w_1, w_2)$ .

**Lemma 24.** *We have*

$$L_j^{(w_1, w_2)} = \text{span}\{t \mid \#a(t) - \#b(t) = w_1, \#b(t) - \#c(t) = w_2\}.$$

*Proof.* From the natural action of  $\mathfrak{sl}_3(\mathbb{C})$  on  $L_1$  we see that

$$\text{weight}(a) = (1, 0), \quad \text{weight}(b) = (-1, 1), \quad \text{weight}(c) = (0, -1).$$

Therefore the weight of an arbitrary Hall word  $t$  equals

$$\#a(t)(1, 0) + \#b(t)(-1, 1) + \#c(t)(0, -1) = (\#a(t) - \#b(t), \#b(t) - \#c(t)).$$

This completes the proof.  $\square$

The root system of  $\mathfrak{sl}_3(\mathbb{C})$  with respect to the Cartan subalgebra spanned by  $h_1, h_2$  is the same as the weight system of the adjoint representation. The adjoint action of  $h_1$  and  $h_2$  on the simple root vectors  $x_1$  and  $x_2$  is as follows:

$$[h_1, x_1] = 2x_1, \quad [h_1, x_2] = -x_2, \quad [h_2, x_1] = -x_1, \quad [h_2, x_2] = 2x_2,$$

From this we see that the weights of  $x_1$  and  $x_2$  are  $(2, -1)$  and  $(-1, 2)$  respectively. It follows that if  $M$  is any representation of  $\mathfrak{sl}_3(\mathbb{C})$  and  $v \in M$  has weight  $(0, 0)$ , then  $x_1.v$  and  $x_2.v$  have weights  $(2, -1)$  and  $(-1, 2)$  respectively.

Our problem is to determine which linear combinations of the Hall words of degree  $j$  are invariant under the natural action of  $\mathfrak{sl}_3(\mathbb{C})$ . That is, we want to find the elements  $Z \in L_j$  which satisfy  $D.Z = 0$  for all  $D \in \mathfrak{sl}_3(\mathbb{C})$ .

**Lemma 25.** *The Lie invariants of degree  $j$  for  $\mathfrak{sl}_3(\mathbb{C})$  in the natural representation are the kernel of the linear map*

$$X: L_j^{(0,0)} \rightarrow L_j^{(2,-1)} \oplus L_j^{(-1,2)}, \quad X(Z) = (x_1.Z, x_2.Z).$$

*Proof.* Since  $L_j$  is finite-dimensional and  $\mathfrak{sl}_3(\mathbb{C})$  is a semisimple Lie algebra, Weyl's Theorem implies that  $L_j$  is isomorphic to a direct sum of simple highest weight modules. If  $Z \in L_j$  satisfies  $D.Z = 0$  for all  $D \in \mathfrak{sl}_3(\mathbb{C})$ , then clearly  $h_1.Z = h_2.Z = 0$ , and so  $Z \in L_j^{(0,0)}$ , the weight space of weight  $(0, 0)$ . If  $D \neq \text{span}(h_1, h_2)$ , then it suffices to consider  $D = x_i$  and  $D = y_i$  for  $i = 1, 2, 3$ . If  $h_i.Z = x_i.Z = 0$  then also  $y_i.Z = 0$ , as we saw in the proof of Lemma 9; so it suffices to consider  $x_i$  ( $i = 1, 2, 3$ ). But in  $\mathfrak{sl}_3(\mathbb{C})$  we have  $[x_1, x_2] = x_3$ , and so if  $x_1.Z = x_2.Z = 0$  then also  $x_3.Z = 0$ . This completes the proof.  $\square$

To compute explicit invariants, we consider separately the linear maps

$$X_1: L_j^{(0,0)} \rightarrow L_j^{(2,-1)}, \quad X_2: L_j^{(0,0)} \rightarrow L_j^{(-1,2)}.$$

The matrices  $[X_1]$  and  $[X_2]$  represent the linear maps  $X_1$  and  $X_2$ . We stack  $[X_1]$  on top of  $[X_2]$  to get the matrix  $[X]$  representing the linear map  $X$  of Lemma 25. The coefficient vectors of the invariants are then the vectors in the nullspace of  $[X]$ .

By Wever [21, 22], we know that there are no invariants in degree  $j$  unless  $j$  is a multiple of 3. We can also see this as follows: Lemma 24 implies that any Hall word  $t$  of weight  $(0, 0)$  must satisfy  $\#a(t) = \#b(t) = \#c(t)$  and hence its degree  $j = \#a(t) + \#b(t) + \#c(t)$  must be a multiple of 3. The next two results confirm Wever's observations for  $q = 3$  and  $d = 3, 6$ .

$[[[ca]b][[ba]c]]^9$	$[[[ca]c][[ba]b]]^{12}$	$[[[cb]b][[ca]a]]^{19}$	$[[[cb]c][[ba]a]]^{22}$	$[[[ca][ba]][cb]]^{31}$
$[[[cb][ba]][ca]]^{33}$	$[[[[ba]a]c][cb]]^{45}$	$[[[[ba]b]c][ca]]^{50}$	$[[[[ba]c]c][ba]]^{52}$	$[[[[ca]a]b][cb]]^{60}$
$[[[[ca]b]b][ca]]^{65}$	$[[[[ca]b]c][ba]]^{67}$	$[[[[ba]a]b]c]c]^{90}$	$[[[[ca]a]b]b]c]^{104}$	
$[[[ca]a][[ba]c]]^6$	$[[[ca]b][[ca]a]]^{10}$	$[[[ca]c][[ba]a]]^{11}$	$[[[ca][ba]][ca]]^{30}$	$[[[[ba]a]c][ca]]^{44}$
$[[[[ca]a]a][cb]]^{57}$	$[[[[ca]a]b][ca]]^{59}$	$[[[[ca]a]c][ba]]^{61}$	$[[[[ba]a]a]c]c]^{87}$	$[[[[ca]a]a]b]c]^{101}$
$[[[ba]c][[ba]b]]^3$	$[[[ca]b][[ba]b]]^8$	$[[[cb]b][[ba]a]]^{16}$	$[[[cb][ba]][ba]]^{32}$	$[[[[ba]a]b][cb]]^{42}$
$[[[[ba]b]b][ca]]^{47}$	$[[[[ba]b]c][ba]]^{49}$	$[[[[ca]b]b][ba]]^{64}$	$[[[[ba]a]b]b]c]^{89}$	$[[[[ca]a]b]b]b]^{103}$

TABLE 2. Hall words of weights  $(0, 0)$ ,  $(2, -1)$ ,  $(-1, 2)$  in degree 6
$$\begin{bmatrix} 1 & . & . & . & . & . & . & . & . & . & -1 & 1 & . & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . \\ 1 & 2 & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & -1 & . & . & 2 & . & . & . \\ . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . & 2 & . & . & . & . & . & . & 1 & . & 1 & . & . \\ . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & 1 & 1 & . & . \\ . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 2 & . \\ 1 & . & . & . & . & . & . & . & . & . & . & 1 & . & 2 & . & . & . & . & . & 1 & 1 \end{bmatrix}$$
TABLE 3. The transpose of the matrix  $[X]$  for  $\mathfrak{sl}_3(\mathbb{C})$  in degree 6

**Lemma 26.** *For the natural representation of  $\mathfrak{sl}_3(\mathbb{C})$ , there are no Lie invariants in degree 3.*

*Proof.* In degree 3, there are two Hall words of weight  $(0, 0)$ , namely  $[[b, a], c]$  and  $[[c, a], b]$ . There is one Hall word of weight  $(2, -1)$ , namely  $[[c, a], a]$ , and one Hall word of weight  $(-1, 2)$ , namely  $[[b, a], b]$ . We calculate that

$$x_1 \cdot [[b, a], c] = 0, \quad x_1 \cdot [[c, a], b] = [[c, a], a], \quad x_2 \cdot [[b, a], c] = [[b, a], b], \quad x_2 \cdot [[c, a], b] = [[b, a], b].$$

From this we obtain the matrices

$$[X_1] = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad [X_2] = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad [X] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

It is clear that  $[X]$  has rank 2 and hence its nullspace is  $\{0\}$ .  $\square$

**Lemma 27.** *For the natural representation of  $\mathfrak{sl}_3(\mathbb{C})$ , there are no Lie invariants in degree 6.*

*Proof.* In degree 6, there are 14 Hall words in  $a, b, c$  for weight  $(0, 0)$ , and 10 Hall words for each weight  $(2, -1)$  and  $(-1, 2)$ ; see Table 2. For each Hall word of weight  $(0, 0)$  we compute the actions of  $x_1$  and  $x_2$  and use Hall's algorithm (Figure

1) to reduce the results to linear combinations of Hall words of weights  $(2, -1)$  and  $(-1, 2)$  respectively. The coefficients give the entries of the  $20 \times 14$  matrix  $[X]$ ; see Table 3 (but note that we have displayed the transpose). This matrix has rank 14, and hence its nullspace is  $\{0\}$ .  $\square$

Wever [21] found an invariant in degree 9 for the natural representation of  $\mathfrak{sl}_3(\mathbb{C})$ :

$$[[[ab][ac]][[ab][bc]]c] + [[[[bc][ba]][[bc][ca]]a] + [[[[ca][cb]][[ca][ab]]b].$$

Our next result presents the first complete explicit basis for the space of Lie invariants for the natural representation of  $\mathfrak{sl}_3(\mathbb{C})$  in the minimal degree 9. We note that the terms of Wever's invariant are not Hall words; the Hall form of his invariant is just the negative of  $I_9^{(1)}$  below. (It would be interesting to see if the other invariants  $I_9^{(2)}, \dots, I_9^{(4)}$  can be expressed more compactly without using Hall words.)

**Theorem 28.** *For the natural representation of  $\mathfrak{sl}_3(\mathbb{C})$ , the space of invariants in degree 9 has dimension 4 and the following basis:*

$$\begin{aligned} I_9^{(1)} = & [[[[ba]a][cb]][[cb][ca]]] - [[[[ba]b][ca]][[cb][ca]]] + [[[[ba]c][ba]][[cb][ca]]] \\ & - [[[[ba]c][cb]][[ca][ba]]] - [[[[ca]a][cb]][[cb][ba]]] + [[[[ca]b][ca]][[cb][ba]]] \\ & + [[[[ca]b][cb]][[ca][ba]]] - [[[[ca]c][ba]][[cb][ba]]] - [[[[cb]b][ca]][[ca][ba]]] \\ & + [[[[cb]c][ba]][[ca][ba]]], \end{aligned}$$

$$\begin{aligned} I_9^{(2)} = & [[[[ca][ba][ba]][[cb]c]] - [[[[ca][ba][ca]][[cb]b]] - 2[[[[ca][ba][cb]][[ba]c]] \\ & + [[[[ca][ba][cb]][[ca]b]] - [[[[cb][ba][ba]][[ca]c]] + [[[[cb][ba][ca]][[ba]c]] \\ & + [[[[cb][ba][ca]][[ca]b]] - [[[[cb][ba][cb]][[ca]a]] - [[[[cb][ca][ca]][[ba]b]] \\ & + [[[[cb][ca][cb]][[ba]a]], \end{aligned}$$

$$\begin{aligned} I_9^{(3)} = & [[[[ca][ba][ba]][[cb]c]] - [[[[ca][ba][ca]][[cb]b]] + [[[[ca][ba][cb]][[ba]c]] \\ & - 2[[[[ca][ba][cb]][[ca]b]] - [[[[cb][ba][ba]][[ca]c]] - [[[[cb][ba][ca]][[ba]c]] \\ & + 2[[[[cb][ba][ca]][[ca]b]] + [[[[cb][ba][cb]][[ca]a]] - [[[[ba]a]a][cb]][[cb]c]] \\ & + [[[[ba]a]b][ca]][[cb]c]] + [[[[ba]a]b][cb]][[ca]c]] - [[[[ba]a]c][ba]][[cb]c]] \\ & - 2[[[[ba]a]c][cb]][[ba]c]] + [[[[ba]a]c][cb]][[ca]b]] - [[[[ba]b]b][ca]][[ca]c]] \\ & + [[[[ba]b]c][ba]][[ca]c]] + 2[[[[ba]b]c][ca]][[ba]c]] - [[[[ba]b]c][ca]][[ca]b]] \\ & - [[[[ba]b]c][cb]][[ca]a]] - 2[[[[ba]c]c][ba]][[ba]c]] + [[[[ba]c]c][ba]][[ca]b]] \\ & + [[[[ba]c]c][cb]][[ba]a]] + [[[[ca]a]a][cb]][[cb]b]] - [[[[ca]a]b][ca]][[cb]b]] \\ & + [[[[ca]a]b][cb]][[ba]c]] - 2[[[[ca]a]b][cb]][[ca]b]] + [[[[ca]a]c][ba]][[cb]b]] \\ & + [[[[ca]a]c][cb]][[ba]b]] - [[[[ca]b]b][ca]][[ba]c]] + 2[[[[ca]b]b][ca]][[ca]b]] \\ & + [[[[ca]b]b][cb]][[ca]a]] + [[[[ca]b]c][ba]][[ba]c]] - 2[[[[ca]b]c][ba]][[ca]b]] \\ & - [[[[ca]b]c][ca]][[ba]b]] - [[[[ca]b]c][cb]][[ba]a]] + [[[[ca]c]c][ba]][[ba]b]] \\ & - [[[[cb]b]b][ca]][[ca]a]] + [[[[cb]b]c][ba]][[ca]a]] + [[[[cb]b]c][ca]][[ba]a]] \\ & - [[[[cb]c]c][ba]][[ba]a]], \end{aligned}$$

$$\begin{aligned} I_9^{(4)} = & [[[[ba]a][ba]][[cb]c]c] - [[[[ba]a][ca]][[cb]b]c] - [[[[ba]a][cb]][[ba]c]c] \\ & + [[[[ba]a][cb]][[ca]b]c] - [[[[ba]b][ba]][[ca]c]c] + [[[[ba]b][ca]][[ca]b]c] \\ & - [[[[ba]b][cb]][[ca]a]c] - [[[[ba]c][ba]][[cb]ca]] + 2[[[[ba]c][ba]][[ba]c]c] \end{aligned}$$

$$\begin{aligned}
& - [[[[ba]c][ba]][[[ca]b]c]] + 2[[[[ba]c][ca]][[cb][ba]]] - 2[[[[ba]c][ca]][[[ba]b]c]] \\
& + [[[[ba]c][ca]][[[ca]b]b]] - 2[[[[ba]c][cb]][[ca][ba]]] + 2[[[[ba]c][cb]][[[ba]a]c]] \\
& - [[[[ba]c][cb]][[[ca]a]b]] - [[[[ca]a][ba]][[cb]b]c]] + [[[[ca]a][ca]][[[cb]b]b]] \\
& - [[[[ca]a][cb]][[cb][ba]]] + [[[[ca]a][cb]][[ba]b]c]] - [[[[ca]a][cb]][[[ca]b]b]] \\
& - [[[[ca]b][ba]][[[ba]c]c]] + 2[[[[ca]b][ba]][[[ca]b]c]] - 2[[[[ca]b][ca]][[cb][ba]]] \\
& + [[[[ca]b][ca]][[[ba]b]c]] - 2[[[[ca]b][ca]][[[ca]b]b]] + 2[[[[ca]b][cb]][[ca][ba]]] \\
& - [[[[ca]b][cb]][[[ba]a]c]] + 2[[[[ca]b][cb]][[[ca]a]b]] + [[[[ca]c][ba]][[cb][ba]]] \\
& - [[[[ca]c][ba]][[[ba]b]c]] + [[[[ca]c][ca]][[[ba]b]b]] - [[[[ca]c][cb]][[[ba]a]b]] \\
& - [[[[cb]b][ba]][[[ca]a]c]] + [[[[cb]b][ca]][[ca][ba]]] + [[[[cb]b][ca]][[[ca]a]b]] \\
& - [[[[cb]b][cb]][[[ca]a]a]] - [[[[cb]c][ba]][[ca][ba]]] + [[[[cb]c][ba]][[[ba]a]c]] \\
& - [[[[cb]c][ca]][[[ba]a]b]] + [[[[cb]c][cb]][[[ba]a]a]].
\end{aligned}$$

*Proof.* We use Maple to generate the 2184 Hall words in  $a, b, c$  in degree 9. There are 186 for weight  $(0, 0)$ , and 140 each for weights  $(2, -1)$  and  $(-1, 2)$ . We apply  $x_1$  and  $x_2$ , compute the normal forms of the results, and store the coefficients in the  $280 \times 186$  matrix  $[X]$ . We compute the row canonical form of this matrix, and find that the rank is 182; hence the nullspace has dimension 4. The canonical basis vectors for the nullspace have integer components, and their (sorted) squared norms are 10, 13, 64, 79. Using the Hermite normal form with lattice basis reduction provides a little improvement in the longest coefficient vector: we obtain an integral basis for the nullspace consisting of vectors with (sorted) squared norms 10, 13, 64, 71. These coefficient vectors correspond to the stated invariants.  $\square$

Since the minimal degree of the (primitive) invariants is 9, it follows that the minimal degree of the non-primitive invariants is 18. Hence all the invariants in degrees 12 and 15 are primitive.

**Theorem 29.** *For the natural representation of  $\mathfrak{sl}_3(\mathbb{C})$ , the space of invariants in degree 12 has dimension 35.*

*Proof.* In degree 12, there are 44220 Hall words; 2880 of weight  $(0, 0)$  and 2310 of each weight  $(2, -1)$  and  $(-1, 2)$ . Hence the matrix  $[X]$  has size  $4620 \times 2880$ . Using modular arithmetic with  $p = 101$  we use the Maple command `RowReduce` from the `LinearAlgebra[Modular]` package to compute the row canonical form. The rank is 2845 and hence the nullspace has dimension 35. We use the command `Basis` to extract the canonical basis of the nullspace; the sorted list of the number of terms in each basis invariant is as follows: 22, 24, 30, 32, 34, 34, 36, 36, 41, 42, 44, 48, 64, 79, 84, 89, 96, 109, 133, 137, 153, 155, 158, 161, 163, 165, 169, 173, 174, 174, 216, 244, 334, 399, 425. The simplest invariant is displayed in Table 4.  $\square$

## 6. CONCLUSION

**6.1. Generalized Witt dimension formula.** The formula of Witt [23] gives the dimension of the space of homogeneous elements of degree  $n$  in the free Lie algebra on  $q$  generators each of degree 1. The generalized Witt formula of Kang and Kim [14] and Jurisich [13] gives the dimension of the space of homogeneous elements of vector degree  $\mathbf{n} \in \mathbb{Z}^k$  in the free Lie algebra on any set of generators with given vector degrees in  $\mathbb{Z}^k$ . In particular, we can use this more general formula to find

$$\begin{aligned}
& [[[[ca]c][[ba]b]][[[ca]b][[ba]c]] + [[[[ca]c][[ba]c]][[ba]c][[ba]b]] - [[[[ca]c][[ca]b]][[[ca]b][[ba]b]]] \\
& - [[[[cb]b][[ba]a]][[[ca]c][[ba]c]] + [[[[cb]b][[ba]a]][[[ca]c][[ca]b]]] - [[[[cb]b][[ba]b]][[[ca]c][[ca]a]]] \\
& + [[[[cb]b][[ba]c]][[[ca]a][[ba]c]] + [[[[cb]b][[ba]c]][[[ca]b][[ca]a]]] + [[[[cb]b][[ba]c]][[[ca]c][[ba]a]]] \\
& + [[[[cb]b][[ca]a]][[[ca]b][[ba]c]] - [[[[cb]b][[ca]b]][[[ca]a][[ba]c]]] - [[[[cb]b][[ca]c]][[[ca]b][[ba]a]]] \\
& + [[[[cb]c][[ba]a]][[[ca]b][[ba]c]] - [[[[cb]c][[ba]b]][[[ca]a][[ba]c]]] - [[[[cb]c][[ba]c]][[[ca]b][[ba]a]]] \\
& + [[[[cb]c][[ca]a]][[[ba]c][[ba]b]] - [[[[cb]c][[ca]a]][[[ca]b][[ba]b]]] - [[[[cb]c][[ca]b]][[[ba]c][[ba]a]]] \\
& + [[[[cb]c][[ca]b]][[[ca]a][[ba]b]] + [[[[cb]c][[ca]b]][[[ca]b][[ba]a]]] + [[[[cb]c][[ca]c]][[[ba]b][[ba]a]]] \\
& - [[[[cb]c][[cb]b]][[[ca]a][[ba]a]]]
\end{aligned}$$

TABLE 4. The simplest invariant for  $\mathfrak{sl}_3(\mathbb{C})$  in degree 12

the dimension of the space of homogeneous elements of degree  $n$  in the free Lie algebra on  $q_i$  generators in degree  $i$ .

Let  $L$  be the free Lie algebra on  $q$  generators over a field  $F$ , and let  $G$  be any simple Lie algebra over  $F$  with an irreducible representation of dimension  $q$ . The action of  $G$  extends naturally to all of  $L$ , and we can consider the free Lie subalgebra  $J \subseteq L$  of  $G$ -invariants. We have decompositions of both  $L$  and  $J$  into direct sums of homogeneous subspaces:

$$L = \bigoplus_{n \geq 1} L_n, \quad J = \bigoplus_{n \geq 1} J_n, \quad J_n = J \cap L_n.$$

Up to a certain degree, all of the invariants will be primitive; that is, not in the Lie subalgebra generated by known invariants of lower degree. Suppose that we have a set of primitive invariants which forms a basis for the space of all invariants of degree  $\leq n$ . We can then use the generalized Witt formula to compute the dimension of the invariants in all degrees  $> n$  which belong to the Lie subalgebra generated by the known primitive invariants. We can then compare this dimension to the dimension of the space of all invariants in the least degree  $N > n$  for which invariants are known to exist. The difference between these two dimensions will be the number of new primitive invariants in degree  $N$ .

A closely related problem is to find an inverse of the Witt dimension formula:

**Open Problem 1.** Let  $L = \bigoplus_{n \geq 1} L_n$  be a free Lie algebra, and let  $J = \bigoplus_{n \geq 1} J_n$  be a homogeneous subalgebra; that is  $J_n = J \cap L_n$  for all  $n \geq 1$ . Suppose that we know  $\dim J_n$  for all  $n \geq 1$ . We know that  $J$  is a free Lie algebra. Determine the degrees of the elements of a free generating set for  $J$ .

A solution to this problem would give a way of determining the dimensions of the spaces of primitive invariants, given the dimensions of the spaces of all invariants.

**6.2. Dimension formula of Thrall and Brandt.** We can compute the dimension of the space of all invariants in degree  $N$  using the computational methods of this paper, at least for relatively small values of  $N$ . However, in the case of the natural representation of  $\mathfrak{sl}_n(\mathbb{C})$ , this dimension is given by a formula of Thrall [19] and Brandt [3]; see also Reutenauer [16, §8.6.1]. The dimension of  $J_N$  where  $N = mq$  equals the multiplicity in the  $S_N$ -module  $M_N$  of the simple  $S_N$ -module

corresponding to the partition  $m^q$ ; here  $S_N$  is the symmetric group and  $M_N$  is the space of multilinear Lie polynomials of degree  $N$ .

**Open Problem 2.** Generalize the formula of Thrall and Brandt to irreducible representations of  $\mathfrak{sl}_n(\mathbb{C})$  other than the natural  $n$ -dimensional representation, and then to arbitrary irreducible representations of arbitrary simple Lie algebras.

We outline one possible approach to this problem. Let  $L_n^{\mathbf{0}}$  be the span of the Hall words of degree  $n$  and weight  $\mathbf{0}$ , and let  $L_n^{\mathbf{w}_i}$  be the span of the Hall words of degree  $n$  and weight  $\mathbf{w}_i$ , where  $\mathbf{w}_i$  is the weight of the simple root vector  $x_i$  in the adjoint representation of the simple Lie algebra  $A$ . (For example, consider the computations for  $A = \mathfrak{sl}_3(\mathbb{C})$  in Section 5.) Lemma 5 shows that the  $x_i$ -action maps  $X_i: L_n^{\mathbf{0}} \rightarrow L_n^{\mathbf{w}_i}$  are surjective, and it may be possible to use this fact to compute the dimension of the kernel of the map  $X: L_n^{\mathbf{0}} \rightarrow \bigoplus_i L_n^{\mathbf{w}_i}$ . To illustrate, we consider the three cases studied in this paper; the formula of Thrall and Brandt can be applied to the first and third cases, but not to the second:

- $\mathfrak{sl}_2(\mathbb{C})$  in the natural representation:  $\deg(a) = (1, 1)$ ,  $\deg(b) = (1, -1)$ . The generalized Witt formula gives the dimension of the space of degree  $(n, w)$  spanned by the Hall words of degree  $n = \#a + \#b$ , weight  $w = \#a - \#b$ .
- $\mathfrak{sl}_2(\mathbb{C})$  in the adjoint representation:  $\deg(a) = (1, 2)$ ,  $\deg(b) = (1, 0)$ ,  $\deg(c) = (1, -2)$ . The space of degree  $(n, w)$  is spanned by the Hall words of degree  $n = \#a + \#b + \#c$ ,  $w = 2\#a - 2\#c$ .
- $\mathfrak{sl}_3(\mathbb{C})$  in the natural representation:  $\deg(a) = (1, 1, 0)$ ,  $\deg(b) = (1, -1, 1)$ ,  $\deg(c) = (1, 0, -1)$ . The subspace of degree  $(n, w_1, w_2)$  is spanned by the Hall words of degree  $n = \#a + \#b + \#c$ , weight  $w_1 = \#a - \#b$ ,  $w_2 = \#b - \#c$ .

**6.3. Finite generation problem.** The computations in this paper suggest that the dimensions of the spaces of primitive invariants grow rapidly as a function of the degree. This raises the following question.

**Open Problem 3.** Let  $L$  be the free Lie algebra on  $q$  generators over a field  $F$ , and let  $A$  be any simple Lie algebra over  $F$  with an irreducible representation of dimension  $q$ . Let  $j \subseteq L$  be the subalgebra of Lie invariants for the action of  $A$  on  $L$ . Prove or disprove that  $J$  is finitely generated.

One way to approach this problem would be to use the dimension formulas to study asymptotic estimates for the number of invariants in each degree.

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