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New Ternary Versions of Jordan Algebras*

Murray Bremner
Department of Mathematics and Statistics
University of Saskatchewan, Saskatoon, SK S7N 5E6, Canada
E-mail: bremner@math.usask.ca

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Abstract. This paper determines the homogeneous identities of degree ≤ 7 satisfied by the ternary anti-commutator $[a,b,c] := abc+acb+bac+bca+cab+cba$ in triple systems satisfying either total associativity $(abc)de \equiv a(bcd)e \equiv ab(cde)$ or partial associativity $(abc)de + a(bcd)e + ab(cde) \equiv 0$. These identities define new ternary versions of Jordan algebras different from Jordan triple systems.

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1 Introduction

Let F be a field of characteristic 0, A a vector space over F, and

$$t : A \otimes_F A \otimes_F A \longrightarrow A$$

a linear map. We call the pair (A,t) a *triple system* over F and usually write A instead of (A,t) and abc instead of $t(a \otimes b \otimes c)$. In this paper, we consider two different generalizations of associativity to the ternary case. We call A *totally associative* if it satisfies the identities

$$(abc)de \equiv a(bcd)e \equiv ab(cde),$$

and *partially associative* if it satisfies the identity

$$(abc)de + a(bcd)e + ab(cde) \equiv 0.$$

(For motivation, see Gnedbaye's papers [5–7].) In any triple system, we can define a new operation, the *ternary anti-commutator* (or *ternary Jordan product*):

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$$[a,b,c] := abc + acb + bac + bca + cab + cba.$$

(We usually omit the commas and write $[abc]$.) The ternary anti-commutator clearly satisfies the *ternary commutative* identity $[a,b,c] \equiv [\sigma(a),\sigma(b),\sigma(c)]$, where σ is any permutation of a,b,c.

The purpose of this paper is to determine the simplest identities of degree ≤ 7 satisfied by the ternary anti-commutator in totally or partially associative triple systems. (Similar results for the ternary commutator are presented in [2, 4].) This problem seems to have been raised originally by Kurosh in the late 1960s (see [8] and [1]). Little progress was made at that time, owing to the size of the matrices involved in the calculations and the relatively limited power of the computers then available.

In this paper, we show that every identity of degree ≤ 7 satisfied by the ternary anti-commutator in a totally associative triple system follows from ternary commutativity and the identity

$$3[[[aag]aa]aa] - 4[[[aaa]ag]aa] + 2[[[aaa]aa]ag] - [[aaa][aaa]g] \equiv 0,$$

and every identity of degree ≤ 7 satisfied by the ternary anti-commutator in a partially associative triple system follows from ternary commutativity and the identity $[[aaa]aa] \equiv 0$. These identities provide ternary generalizations of Jordan algebras different from Jordan triple systems (which are based on the ternary operation $abc + cba$).

2 The Binary Case

In this section, we review the familiar binary case to illustrate the methods used to study the ternary case.

Identities of degree 3. Let V denote the multilinear subspace of degree 3 in the free commutative (non-associative) algebra on three (free) generators. Then V has $\binom{3}{2} = 3$ monomials as a basis, i.e.,

$$[[ab]c], \qquad [[ac]b], \qquad [[bc]a].$$

Let W denote the multilinear subspace of degree 3 in the free associative algebra on three (free) generators. Then W has a basis consisting of six permutations of the monomial abc. Let $E : V \to W$ be the *expansion map*, i.e., the linear map which sends each basis monomial in V to the polynomial in W obtained by expanding each bracket $[ab]$ as the anti-commutator $ab + ba$. For example, $E([[ab]c]) = abc + bac + cab + cba$.

Proposition 2.1. *There is no multilinear identity of degree* 3 *satisfied by the anti-commutator in every associative algebra.*

Proof. The matrix representing the expansion map E and its row-reduced form are

$$
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
$$

respectively. The matrix has rank 3, so the nullspace is $\{0\}$. □

Table 1. The binary case.

The expansion matrix in degree 4:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
$$

The row-reduced form (zero rows omitted):

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
$$

Identities of degree 4. Now let V denote the multilinear subspace of degree 4 in the free commutative (non-associative) algebra on four (free) generators. Then V has the following $\binom{4}{2,1,1} + \frac{1}{2}\binom{4}{2,2} = 12 + 3 = 15$ monomials as a basis:

$$\begin{array}{lllll} [[[ab]c]d], & [[[ab]d]c], & [[[ac]b]d], & [[[ac]d]b], & [[[ad]b]c], \\ [[[ad]c]b], & [[[bc]a]d], & [[[bc]d]a], & [[[bd]a]c], & [[[bd]c]a], \\ [[[cd]a]b], & [[[cd]b]a], & [[ab][cd]], & [[ac][bd]], & [[ad][bc]]. \end{array}$$

Let W denote the multilinear subspace of degree 4 in the free associative algebra on four (free) generators. Then W has a basis consisting of 24 permutations of the monomial $abcd$. Let $E : V \to W$ be the expansion map defined as before by expanding each bracket as an anti-commutator. For example,

$$E([[[ab]c]d]) = abcd + bacd + cabd + cbad + dabc + dbac + dcab + dcba.$$

The matrix representing E and its row-reduced form are given in Table 1. Since we are discussing multilinear identities, the symmetric group S_4 acts on V and W by permuting the letters. Furthermore, the expansion map E is an S_4-module homomorphism, and hence, $\ker(E)$ is also an S_4-module.

Proposition 2.2. *The kernel of E is generated as an S_4-module by the linearization of the identity* $[[[aa]d]a] - [[aa][ad]]$.

Proof. The rank of the expansion matrix is 11, so the dimension of the space of identities is 4. The columns of the row-reduced form which do not contain a leading one are 10, 11, 12, 15. If we multiply these four columns by -1 and insert the rows of the 4×4 identity matrix in the given positions, then we obtain the following four basis vectors for $\ker(E)$:

$$\begin{array}{l} [1 \;\; -1 \;\; 0 \;\; -1 \;\; 0 \;\; 1 \;\; 0 \;\; -1 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0], \\ [0 \;\; -1 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 1 \;\; -1 \;\; 1 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0], \\ [0 \;\; -1 \;\; 1 \;\; -1 \;\; 1 \;\; 0 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0], \\ [0 \;\; -1 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 1 \;\; 1]. \end{array}$$

The first three basis vectors cannot generate $\ker(E)$ as an S_3-module since they contain only terms with the first bracket arrangement, whereas the last basis vector also contains terms with the second bracket arrangement. (The action of S_4 on the basis monomials for V does not affect the bracket arrangements.) It is easy to check that the last basis vector is an S_4-module generator of $\ker(E)$; the corresponding identity is the linearization of

$$-\frac{1}{2}[[aa]d]a + \frac{1}{2}[[aa][ad]].$$

We now multiply by -2 to obtain the identity in the statement of the proposition. □

The calculations in this section may seem like an unusual way to rediscover the Jordan identity, but a systematic linear-algebraic approach like this is necessary when studying the ternary case.

3 The Ternary Case: Total Associativity

Identities of degree 5. Let V denote the multilinear subspace of degree 5 in the free commutative triple system on five (free) generators. Then V has the following $\binom{5}{3} = 10$ monomials as a basis:

$$\begin{array}{ccccc} [[abc]de], & [[abd]ce], & [[abe]cd], & [[acd]be], & [[ace]bd], \\ [[ade]bc], & [[bcd]ae], & [[bce]ad], & [[bde]ac], & [[cde]ab]. \end{array}$$

Let W denote the multilinear subspace of degree 5 in the free totally associative triple system on five (free) generators. Then W has a basis consisting of 120 permutations of the monomial $abcde$.

Let $E : V \to W$ be the *ternary expansion map*, i.e., the linear map which sends each basis monomial in V to the polynomial in W obtained by expanding the brackets as ternary anti-commutators. For example,

$$\begin{aligned} & E([[abc]de]) \\ = \; & [abc]de + [abc]ed + d[abc]e + de[abc] + e[abc]d + ed[abc] \\ = \; & (abc + acb + bac + bca + cab + cba)de \\ & +(abc + acb + bac + bca + cab + cba)ed \\ & +d(abc + acb + bac + bca + cab + cba)e \\ & +de(abc + acb + bac + bca + cab + cba) \\ & +e(abc + acb + bac + bca + cab + cba)d \\ & +ed(abc + acb + bac + bca + cab + cba) \\ = \; & abcde + acbde + bacde + bcade + cabde + cbade \\ & +abced + acbed + baced + bcaed + cabed + cbaed \\ & +dabce + dacbe + dbace + dbcae + dcabe + dcbae \\ & +deabc + deacb + debac + debca + decab + decba \\ & +eabcd + eacbd + ebacd + ebcad + ecabd + ecbad \\ & +edabc + edacb + edbac + edbca + edcab + edcba. \end{aligned}$$

Theorem 3.1. *There is no multilinear identity of degree* 5 *satisfied by the ternary anti-commutator in every totally associative triple system.*

Proof. The matrix representing E has size 120×10, the (i,j)-entry is the coefficient of the ith totally associative monomial in the expansion of the jth commutative monomial. More precisely, let $[[x_1x_2x_3]x_4x_5]$ denote the jth commutative monomial. Then the (i,j)-entry of the expansion matrix is 1 if the ith associative monomial can be written in the form $\sigma(y)\sigma(x_4)\sigma(x_5)$

for some permutation σ of $\{y, x_4, x_5\}$, where $y = \tau(x_1)\tau(x_2)\tau(x_3)$ for some permutation τ of $\{x_1, x_2, x_3\}$, and 0 otherwise. The nullspace of this matrix can easily be computed using a symbolic algebra system such as Maple. The matrix has rank 10, so the nullspace is $\{0\}$. □

Identities of degree 7. Let V denote the multilinear subspace of degree 7 in the free commutative triple system on seven (free) generators, and W the multilinear subspace of degree 7 in the free totally associative triple system on seven (free) generators. The dimension of V is 280 (the number of monomials with the bracket arrangement $[[[abc]de]fg]$ is $\binom{7}{3,2,2} = 210$, and the number of monomials with the bracket arrangement $[[[abc][def]g]$ is $\frac{1}{2}\binom{7}{3,3,1} = 70$). A basis for W consists of 5040 permutations of the monomial $abcdefg$. Since we are dealing with multilinear identities, the vector spaces V and W are S_7-modules, the expansion map E is an S_7-module homomorphism, and $\ker(E)$ is also an S_7-module.

Theorem 3.2. *The kernel of E is generated as an S_7-module by the linearization of the identity*

$$3[[[aag]aa]aa] - 4[[[aaa]ag]aa] + 2[[[aaa]aa]ag] - [[aaa][aaa]g] \equiv 0.$$

Proof. Let $E : V \to W$ denote the expansion map in degree 7. As before, this is the linear map defined on commutative basis monomials by expanding each ternary anti-commutator. To compute the kernel of the expansion map, we need to find the nullspace of a matrix of size 5040×280.

Given the very large number of rows in the expansion matrix, it may not even be possible to store the entire matrix in the computer memory. One way of doing this computation is as follows:

(1) Generate the 280 commutative basis monomials.

(2) Expand the ternary anti-commutators in each of the 280 commutative basis monomials. Since each ternary anti-commutator has six terms, and each basis monomial has three pairs of brackets, each expansion is a list of $6^3 = 216$ terms. These terms are basis monomials from the free totally associative triple system.

(3) Choose a block size b (which should be a divisor of 5040) and partition the expansion matrix into blocks of size $b \times 280$. Initialize a working matrix of size $(280+b) \times 280$, and repeat the following for each of the $5040/b$ blocks: Search through the expansion lists to find those terms which correspond to entries in the current block of the expansion matrix, read these entries into the last b rows of the working matrix, and compute the row-reduced form of the working matrix.

(4) After all of the blocks have been processed, the working matrix contains the row-reduced form of the expansion matrix.

With a block size of $b = 72$, this computation required 11661.2 seconds (3 hours, 14 minutes, 21.2 seconds) of CPU time using Maple V.4 on a Sun Ultra Sparc. From this, we find that the rank of the expansion matrix is

273, so the kernel of the expansion map has dimension 7. From the row-reduced form of the expansion matrix, we can immediately determine a basis for $\ker(E)$. The columns of the row-reduced form which do not contain a leading one are 259, 263, 267, 268, 269, 270, 280. If we multiply these seven columns by -3 (to clear denominators) and insert the rows of three times the 7×7 identity matrix in the given positions, then we obtain the seven basis vectors for $\ker(E)$ displayed (as 14×20 matrices) in Table 2.

The six generating transpositions of S_7 act on these seven basis vectors as follows:

(ab) interchanges vectors 6 and 7, and fixes the others;
(bc) interchanges vectors 5 and 6, and fixes the others;
(cd) interchanges vectors 4 and 5, and fixes the others;
(de) interchanges vectors 3 and 4, and fixes the others;
(ef) interchanges vectors 2 and 3, and fixes the others;
(fg) interchanges vectors 1 and 2, and fixes the others.

Therefore, each of the seven basis vectors is a generator of $\ker(E)$ as an S_7-module.

We now analyze the identity corresponding to the first basis vector in Table 2. This identity contains 150 terms with coefficient -1, 60 terms with coefficient 2, and 10 terms with coefficient 3. The terms with coefficient -1 consist of $\binom{6}{2,2,2} = 90$ monomials with the first bracket arrangement and g in position 3, together with $\binom{6}{3,2,1} = 60$ monomials with the first bracket arrangement and g in position 7. These terms are the linearization of

$$-\frac{1}{8}[[[aag]aa]aa] - \frac{1}{12}[[[aaa]aa]ag].$$

The terms with coefficient 2 consist of $\binom{6}{3,1,2} = 60$ terms with g in position 5. These terms are the linearization of $\frac{1}{6}[[[aaa]ag]aa]$. The terms with coefficient 3 consist of $\frac{1}{2}\binom{6}{3,3,1} = 10$ monomials with the second bracket arrangement and g in position 7. These terms are the linearization of $\frac{1}{24}[[aaa][aaa]g]$. Adding together these four terms and multiplying it by -24, we obtain the identity in the statement of the theorem.

We need to do the calculations described above in order to find this identity, and to prove that it implies all the identities of degree 7. However, once we have the identity, there is an easy way to verify it directly. There are only seven possible associative terms in the expansions of the commutative monomials in the identity since g can occur in any of the seven positions and the other factors are all a. The monomials in the identity, together with the number of times each associative monomial occurs in each expansion, are listed below:

$[[[aag]aa]aa]$	8	24	48	56	48	24	8,
$[[[aaa]ag]aa]$	24	36	36	24	36	36	24,
$[[[aaa]aa]ag]$	72	36	0	0	0	36	72,
$[[aaa][aaa]g]$	72	0	0	72	0	0	72,

where column j gives the number of occurences of the associative monomial with g in position j. From this, one easily checks that the given linear combination of the commutative monomials does indeed expand to 0. □

4 The Ternary Case: Partial Associativity

Identities of degree 5. Let V denote the multilinear subspace of degree 5 in the free commutative triple system on five (free) generators.

The multilinear subspace of degree 5 in the (absolutely) free triple system on five (free) generators has a basis consisting of 360 monomials obtained by applying the 5! permutations of a, b, c, d, e to the three bracket arrangements $(abc)de$, $a(bcd)e$, and $ab(cde)$. Let W denote the multilinear subspace of degree 5 in the free partially associative triple system on five (free) generators. If we write the partial associativity identity in the form $ab(cde) \equiv -(abc)de - a(bcd)e$, then we see that W has a basis consisting of 240 monomials obtained by applying the 5! permutations of a, b, c, d, e to the two bracket arrangements $(abc)de$ and $a(bcd)e$.

Let $E : V \to W$ be the *expansion map*, i.e., the linear map which sends each basis monomial in V to the polynomial in W obtained by expanding the ternary anti-commutators and then applying the partial associativity identity. For example,

$$
\begin{aligned}
& E([[abc]de]) \\
= \; & [abc]de + [abc]ed + d[abc]e + de[abc] + e[abc]d + ed[abc] \\
= \; & (abc + acb + bac + bca + cab + cba)de \\
& +(abc + acb + bac + bca + cab + cba)ed \\
& +d(abc + acb + bac + bca + cab + cba)e \\
& +de(abc + acb + bac + bca + cab + cba) \\
& +e(abc + acb + bac + bca + cab + cba)d \\
& +ed(abc + acb + bac + bca + cab + cba) \\
= \; & (abc)de + (acb)de + (bac)de + (bca)de + (cab)de + (cba)de \\
& +(abc)ed + (acb)ed + (bac)ed + (bca)ed + (cab)ed + (cba)ed \\
& +d(abc)e + d(acb)e + d(bac)e + d(bca)e + d(cab)e + d(cba)e \\
& +de(abc) + de(acb) + de(bac) + de(bca) + de(cab) + de(cba) \\
& +e(abc)d + e(acb)d + e(bac)d + e(bca)d + e(cab)d + e(cba)d \\
& +ed(abc) + ed(acb) + ed(bac) + ed(bca) + ed(cab) + ed(cba)
\end{aligned}
$$

$$
\begin{aligned}
= {} & (abc)de + (acb)de + (bac)de + (bca)de + (cab)de + (cba)de \\
& +(abc)ed + (acb)ed + (bac)ed + (bca)ed + (cab)ed + (cba)ed \\
& +d(abc)e + d(acb)e + d(bac)e + d(bca)e + d(cab)e + d(cba)e \\
& -(dea)bc - (dea)cb - (deb)ac - (deb)ca - (dec)ab - (dec)ba \\
& -d(eab)c - d(eac)b - d(eba)c - d(ebc)a - d(eca)b - d(ecb)a \\
& +e(abc)d + e(acb)d + e(bac)d + e(bca)d + e(cab)d + e(cba)d \\
& -(eda)bc - (eda)cb - (edb)ac - (edb)ca - (edc)ab - (edc)ba \\
& -e(dab)c - e(dac)b - e(dba)c - e(dbc)a - e(dca)b - e(dcb)a.
\end{aligned}
$$

Theorem 4.1. *The only identity of degree* 5 *satisfied by the ternary anti-commutator in every partially associative triple system is* $[[aaa]aa] \equiv 0$.

Proof. The matrix representing E has size 240×10, whose (i,j)-entry is the coefficient of the ith partially associative monomial in the expansion of the jth commutative monomial. The row-reduced form of the expansion matrix (with zero rows omitted) is

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}.
$$

The rank of the expansion matrix is 9, and a basis for the nullspace consists of the single vector $(1,1,1,1,1,1,1,1,1,1)$, which corresponds to the identity

$$
\begin{aligned}
& [[abc]de] + [[abd]ce] + [[abe]cd] + [[acd]be] + [[ace]bd] \\
& +[[ade]bc] + [[bcd]ae] + [[bce]ad] + [[bde]ab] + [[cde]ab] \equiv 0.
\end{aligned}
$$

This is the sum of the ternary commutative monomials over all 3-2 shuffle permutations of a, b, c, d, e. Since F has characteristic 0, this shuffle identity is equivalent to the identity in the statement of the theorem. □

Identities of degree 7. Let V denote the multilinear subspace of degree 7 in the free commutative triple system on seven (free) generators, and W the multilinear subspace of degree 7 in the free partially associative triple system on seven (free) generators. A basis for V has already been described.

Lemma 4.2. *The* 7! *permutations of* a, b, c, d, e, f, g *applied to the four monomials* $((abc)de)fg$, $(a(bcd)e)fg$, $a(b(cde)f)g$, $(abc)(def)g$ *form a basis of* W. *Hence, the dimension of* W *is* 20160.

Proof. The multilinear subspace of degree 7 in the (absolutely) free triple system on seven (free) generators has a basis consisting of 60480 monomials obtained by applying the 7! permutations of a, b, c, d, e, f, g to the 12 bracket arrangements

$$\begin{array}{lll}
[1] = ((abc)de)fg, & [2] = (a(bcd)e)fg, & [3] = (ab(cde))fg, \\
[4] = a((bcd)ef)g, & [5] = a(b(cde)f)g, & [6] = a(bc(def))g, \\
[7] = ab((cde)fg), & [8] = ab(c(def)g), & [9] = ab(cd(efg)), \\
[10] = (abc)(def)g, & [11] = (abc)d(efg), & [12] = a(bcd)(efg).
\end{array}$$

Using partial associativity, we can left-normalize these monomials, i.e., rewrite any product with the bracket arrangement $vw(xyz)$ in terms of the two products $(vwx)yz$ and $v(wxy)z$. Five of the 12 monomials (numbers 1, 2, 4, 5, 10) are already normalized. For the others (omitting the monomial [9] for the moment), we have

$$\begin{aligned}
[3] &= (ab(cde))fg = -((abc)de)fg - (a(bcd)e)fg = -[1] - [2], \\
[6] &= a(bc(def))g = -a((bcd)ef)g - a(b(cde)f)g = -[4] - [5], \\
[7] &= ab((cde)fg) = -(ab(cde))fg - a(b(cde)f)g \\
&= ((abc)de)fg + (a(bcd)e)fg - a(b(cde)f)g = [1] + [2] - [5], \\
[8] &= ab(c(def)g) = -(abc)(def)g - a(bc(def))g \\
&= -(abc)(def)g + a((bcd)ef)g + a(b(cde)f)g = -[10] + [4] + [5], \\
[11] &= (abc)d(efg) = -((abc)de)fg - (abc)(def)g = -[1] - [10], \\
[12] &= a(bcd)(efg) = -(a(bcd)e)fg - a((bcd)ef)g = -[2] - [4].
\end{aligned}$$

The monomial $[9] = ab(cd(efg))$ can be normalized in two different ways, depending on whether we apply partial associativity first to the inner product $cd(efg)$ or to the outer product $ab(cdx)$, where $x = efg$, i.e.,

$$\begin{aligned}
[9] &= ab(cd(efg)) = -ab((cde)fg) - ab(c(def)g) \\
&= (ab(cde))fg + a(b(cde)f)g + (abc)(def)g + a(bc(def))g \\
&= -((abc)de)fg - (a(bcd)e)fg + a(b(cde)f)g \\
&\quad +(abc)(def)g - a((bcd)ef)g - a(b(cde)f)g \\
&= -((abc)de)fg - (a(bcd)e)fg + (abc)(def)g - a((bcd)ef)g \\
&= -[1] - [2] + [10] - [4], \\
[9] &= = ab(cd(efg)) = -(abc)d(efg) - a(bcd)(efg) \\
&= ((abc)de)fg + (abc)(def)g + (a(bcd)e)fg + a((bcd)ef)g \\
&= [1] + [10] + [2] + [4].
\end{aligned}$$

Equating these two normalizations, we find $2\big([1] + [2] + [4]\big) = 0$. This equation shows that a free partially associative triple system over $\mathbb{Z}$ is not a free $\mathbb{Z}$-module (see the reference [3] for details). Since F has characteristic 0, we can divide by 2 and obtain $[4] = -[1]-[2]$. Applying this to our previous results, we find

$$\begin{aligned}
[6] &= -[4] - [5] = [1] + [2] - [5], \\
[8] &= -[10] + [4] + [5] = -[1] - [2] + [5] - [10], \\
[9] &= -[1] - [2] + [10] - [4] = -[1] - [2] + [10] + [1] + [2] = [10], \\
[12] &= -[2] - [4] = -[2] + [1] + [2] = [1].
\end{aligned}$$

We have now shown that each of the 12 original monomials can be expressed as a linear combination of the monomials [1], [2], [5], and [10]. It is clear that there are no linear relations among monomials corresponding to different permutations of a, b, c, d, e, f, g since the partial associativity identity only involves one permutation of the letters. □

Theorem 4.3. *Every multilinear identity of degree 7 for the ternary anti-commutator in a partially associative triple system follows from the shuffle identity in degree 5.*

Proof. Let $E : V \to W$ denote the expansion map in degree 7. As before, this is the linear map defined on commutative basis monomials by expanding each ternary anti-commutator and then normalizing the terms using the partial associativity identity. Let $I = I(a, b, c, d, e)$ denote the shuffle identity, and U the subspace of V consisting of all the identities of degree 7 that follow from I. We clearly have the subspace containments $U \subseteq \ker(E) \subseteq V$. To prove the theorem, we need to show that the first containment is an equality. We do this by computing the dimensions of U and $\ker(E)$ and observing that they are equal.

Since the ternary anti-commutator and the identity $I(a, b, c, d, e)$ are both symmetric functions of their arguments, there are essentially only two distinct ways to lift I to degree 7:

$$I([a,b,c],d,e,f,g) \qquad \text{and} \qquad [I(a,b,c,d,e),f,g].$$

A spanning set for U consists of $\binom{7}{3} + \binom{7}{2} = 35 + 21 = 56$ distinct identities obtained by applying the 7! permutations of a, b, c, d, e, f, g to these two. To compute the dimension of U, we need to find the rank of a matrix of size 56×280, whose (i, j)-entry is the coefficient of the jth commutative monomial in the expansion of the ith spanning identity for U. The rank of this matrix, and hence, the dimension of U, is 56. This says that the spanning set is a basis.

We next compute the dimension of $\ker(E)$. By Lemma 4.2, this requires finding the dimension of the nullspace of a matrix of size 20160×280. This computation is very similar to that required for the identities of degree 7 in the totally associative case, except (i) when we compute the expansions of the commutative basis monomials, we must keep track of the bracket arrangements in the non-associative monomials, and (ii) we must search through each expansion list to find which non-associative monomials need to be normalized using the relations in Lemma 4.2, and then delete each non-normalized monomial and insert the corresponding normalized monomials.

After all of the blocks have been processed, the working matrix contains the row-reduced form of the expansion matrix. With a block size $b = 72$, this computation required 225155.09 seconds (62 hours, 32 minutes, 35.09 seconds) of CPU time using Maple V.4 on a Sun Ultra Sparc. The rank of the expansion matrix is 224, so the dimension of the nullspace is 56. This shows that $U = \ker(E)$ and the proof is complete. $\square$

Remark. The Maple procedures used to do the computations described in this paper are available by email from the author.

Table 2. The ternary case (total associativity).

Basis vectors for the module of identities in degree 7:

$$\begin{bmatrix}
-1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & 2 & -1 & 2 & 2 & -1 \\
-1 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & 2 \\
2 & 2 & 2 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 \\
-1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & 2 \\
2 & 2 & 2 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 \\
-1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & 2 \\
-1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
-1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 \\
2 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 \\
2 & -1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & 2 & -1 & 2 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
2 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 \\
2 & -1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & 2 & -1 & 2 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & -1 & -1 \\
2 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 \\
3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
2 & 2 & -1 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & 2 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
2 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 \\
-1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 \\
3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
-1 & 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 \\
2 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & -1 \\
-1 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 \\
2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 \\
2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & -1 \\
-1 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 \\
2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 \\
-1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 \\
-1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
3 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
2 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 \\
-1 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & 2 & 2 \\
-1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 \\
-1 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & 2 & 2 \\
-1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 \\
-1 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 \\
0 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 \\
-1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 \\
2 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 \\
-1 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 & -1 & -1 \\
-1 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 \\
-1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & 2 \\
0 & -1 & 2 & 2 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 \\
-1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 \\
2 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 \\
-1 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 \\
-1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 \\
2 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 2 \\
0 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3
\end{bmatrix}$$

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