On a Lie algebra of vector fields on a complex torus

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This paper discusses a modification of the Krichever–Novikov Lie algebra of vector fields on the torus in which the two specified points where singularities can occur are points of order 1 and 2 rather than points of infinite order.

I. INTRODUCTION

Krichever and Novikov[1] initiated the study of Lie algebras of meromorphic vector fields, on compact Riemann surfaces of arbitrary genus, which are holomorphic except that poles are allowed at two specified points. In genus zero, the surface is the Riemann sphere, and one conventionally takes the two points to be 0 and ∞, thus obtaining the (centerless) Virasoro algebra, which has been studied in great detail. In genus 1, one represents a complex torus Σ as the quotient of the complex plane $\mathbf{C}$ by a lattice

$$\Lambda := \mathbf{Z}(2\omega_1) \oplus \mathbf{Z}(2\omega_2) \quad (\omega_1,\omega_2 \in \mathbf{C}, \quad \operatorname{Im} \omega_1/\omega_2 > 0),$$

chooses a point z_0 in the fundamental parallelogram, and sets $P_\pm := \pm z_0$. (We shall write z for both a point in $\mathbf{C}$ and its image in Σ.) Using the theory of elliptic functions, together with the Riemann–Roch theorem, one determines that the Lie algebra in this case has a basis $E_i := e_i(z)(d/dz)$, $i \in \mathbf{Z} + \frac{1}{2}$, where $e_i(z)$ is defined using the Weierstrass σ function:

$$e_i(z) := \sigma^{i-1/2}(z - z_0)\sigma(z + 2iz_0)/\sigma^{i+1/2}(z + z_0). \tag{1}$$

Thus $e_i(z)$ has a zero of order $i - \frac{1}{2}$ at P_+, a pole of order $i + \frac{1}{2}$ at P_-, and (as required by double periodicity) another zero, this time simple, at $-2iz_0$. The structure constants of this Lie algebra can then be expressed in terms of the Weierstrass ζ function.

One sees easily from (1) that the choice of z_0 is not completely free: For example, if z_0 is a torsion point, say $nz_0 = 0$, then $e_{n-1/2}(z) = e_{n+1/2}(z)$. There is no redundancy of this type in (1) if and only if z_0 is a point of infinite order.

However, one can take both P_+ and P_- to be torsion points, but the resulting Lie algebra will not have the form described by Krichever and Novikov. The simplest example occurs for $P_+ = 0, P_- = \omega_1 + \omega_2$; this is the case that will be considered in the remainder of this paper.

II. BASIS AND COMMUTATION RELATIONS

Write $\mathscr{K}$ for the Lie algebra of meromorphic vector fields on $\Sigma := \mathbf{C}/\Lambda$, which are holomorphic on $\Sigma \setminus \{P_\pm\}$. Let $\wp(z)$ be the Weierstrass $\wp$ function for the lattice Λ, and set $\omega := \omega_1 + \omega_2$, $\rho := \wp(\omega)$. Now $\wp(z) - \rho$ has a double pole at $z = 0$, a double zero at $z = \omega$ [since $\wp'(\omega) = 0$], and no other zeros or poles.[2] Thus $\mathscr{K}$ must include the vector fields

$$K_a := k_a(z)\frac{d}{dz}, \quad k_a(z) := (\wp(z) - \rho)^a, \quad a \in \mathbf{Z}. \tag{2}$$

Calculating commutators of these gives

$$[K_a, K_b] = (b - a)(\wp(z) - \rho)^{a+b-1}\wp'(z)\frac{d}{dz},$$

and thus $\mathscr{K}$ must also contain

$$J_a := j_a(z)\frac{d}{dz}, \quad j_a(z) := (\wp(z) - \rho)^{a-1}\wp'(z), \quad a \in \mathbf{Z}. \tag{3}$$

Proposition: The vector fields J_a and K_a for $a \in \mathbf{Z}$ form a basis of $\mathscr{K}$.

Proof: On Σ the vector field d/dz has no zeros or poles. Thus we have a vector space isomorphism from $\mathscr{F}$, the field of meromorphic functions on Σ, holomorphic on $\Sigma \setminus \{P_\pm\}$, to $\mathscr{K}$, given by $f(z) \mapsto f(z)(d/dz)$. So it suffices to show that the functions $j_a(z)$, $k_a(z)$ for $a \in \mathbf{Z}$, form a basis of $\mathscr{F}$. This is an immediate corollary of the Riemann–Roch theorem,[3] as follows. Consider the positive divisor $D := cP_- + dP_+$ ($c,d \in \mathbf{Z}_{>0}$) on Σ, and let $\mathscr{L}(D)$ be the finite-dimensional vector space of meromorphic functions $f(z)$ on Σ such that $\operatorname{div}(f) \geqslant -D$, together with the zero function. Then $\dim \mathscr{L}(D) = c + d$. One checks that $k_a(z)$ has a pole of order $2a$ at $z = 0$, a zero of order $2a$ at $z = \rho$, and no other zeros or poles, and that $j_a(z)$ has a pole of order $2a + 1$ at $z = 0$, a zero of order $2a - 1$ at $z = \rho$, and simple zeros at $z = \omega_1$ and $z = \omega_2$. Thus

$$\{k_{-[d/2]},\dots,k_0 \equiv 1,\dots,k_{[c/2]}\} \cup \{j_{-[(d-1)/2]},\dots j_0 \dots j_{[(c-1)/2]}\},$$

is a set of $c + d$ linearly independent functions $f(z)$ satisfying $\operatorname{div} f \geqslant -D$. Q.E.D.

The $\wp$ function satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \quad g_i = g_i(\omega_1,\omega_2), \tag{4}$$

which by differentiation gives

$$\wp''(z) = 6\wp(z)^2 - \tfrac{1}{2}g_2. \tag{5}$$

We also have the addition formula

$$\wp(z_1 + z_2) = \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'_1(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2 - \wp(z_1) - \wp(z_2), \quad z_1 \neq z_2. \tag{6}$$

Proposition: The commutation relations among the J_a and K_a are as follows:

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$$[J_a,K_b] = (b-a+\tfrac{1}{2})(12\rho^2-g_2)K_{a+b-1} + (b-a)12\rho K_{a+b} + 4(b-a-\tfrac{1}{2})K_{a+b+1}. \quad (9)$$

$$[J_a,J_b] = (b-a)((12\rho^2-g_2)J_{a+b-1} + 12\rho J_{a+b} + 4J_{a+b+1}), \quad (7)$$

$$[K_a,K_b] = (b-a)J_{a+b}, \quad (8)$$

Proof: We only give the details for (9); (7) and (8) are simpler. Using (4) and (5) we have

$$[J_a,K_b] = (\wp(z)-\rho)^{a-1}\wp'(z)\frac{d}{dz}(\wp(z)-\rho)^b\frac{d}{dz} - (\wp(z)-\rho)^b\frac{d}{dz}(\wp(z)-\rho)^{a-1}\wp'(z)\frac{d}{dz}$$

$$= (b-a+1)(\wp(z)-\rho)^{a+b-2}\wp'(z)^2\frac{d}{dz} - (\wp(z)-\rho)^{a+b-1}\wp''(z)\frac{d}{dz}$$

$$= (b-a+1)(\wp(z)-\rho)^{a+b-2}(4(\wp(z)-\rho)^3 + 12\rho(\wp(z)-\rho)^2 + (12\rho^2-g_2)(\wp(z)-\rho))\frac{d}{dz}$$

$$- (\wp(z)-\rho)^{a+b-1}\left(6(\wp(z)-\rho)^2 + 12\rho(\wp(z)-\rho) + \left(6\rho^2 - \frac{1}{2}g_2\right)\right)\frac{d}{dz}$$

$$= 4(b-a-\tfrac{1}{2})(\wp(z)-\rho)^{a+b+1}\frac{d}{dz} + 12\rho(b-a)(\wp(z)-\rho)^{a+b}\frac{d}{dz}$$

$$+ (12\rho^2-g_2)(b-a+\tfrac{1}{2})(\wp(z)-\rho)^{a+b-1}\frac{d}{dz},$$

which gives (9). Q.E.D.

The automorphism of Σ given by $z \mapsto z+\omega$ interchanges P_+ and P_-, and induces an automorphism of $\mathscr{K}$, namely $\sigma: f(z)(d/dz) \mapsto f(z+\omega)(d/dz)$.

Proposition: The automorphism σ takes the explicit form $\sigma(J_a) = -\beta^{-2a}J_{-a}$, $\sigma(K_a) = \beta^{-2a}K_{-a}$, where β is a square root of $1/(3\rho^2 - \frac{1}{4}g_2)$.

Proof: Note that for $f(X) := 4X^3 - g_2X - g_3$, we have $f'(X) = 4(3X^2 - \frac{1}{4}g_2)$. By assumption $f(X)$ has distinct roots, and so $f(\rho)=0$ implies $f'(\rho) \neq 0$. Thus $3\rho^2 - \frac{1}{4}g_2 \neq 0$ and β is well defined. Using (4) and (6) we obtain

$$\wp(z+\omega) - \rho = \frac{1}{4}\frac{\wp'(z)^2}{(\wp(z)-\rho)^2} - \wp(z) - 2\rho$$

$$= \frac{1}{4}\frac{1}{(\wp(z)-\rho)^2}((12\rho^2-g_2) \times (\wp(z)-\rho) + (4\rho^3 - g_2\rho - g_3))$$

$$= \beta^{-2}\frac{1}{(\wp(z)-\rho)}.$$

Differentiating gives

$$\wp'(z+\omega) = -\beta^{-2}[\wp'(z)/(\wp(z)-\rho)^2].$$

Thus

$$\sigma(J_a) = (\wp(z+\omega)-\rho)^{a-1}\wp'(z+\omega)\frac{d}{dz}$$

$$= -\beta^{-2a}(\wp(z)-\rho)^{-a-1}\wp'(z)\frac{d}{dz}$$

$$= -\beta^{-2a}J_{-a}.$$

The calculation of $\sigma(K_a)$ is similar. Q.E.D.

Thus if we define $J'_a := \beta^a J_a$, $K'_a := \beta^a K_a$, then σ takes the more convenient form $\sigma(J'_a) = -J'_{-a}$, $\sigma(K'_a) = K'_{-a}$. If we further define $J''_a := -\frac{1}{4}\beta J'_a$, $K''_a := -\frac{1}{2}\beta K'_a$, we still have $\sigma(J''_a) = -J''_a$, $\sigma(K''_a) = K''_{-a}$, and the commutation relations take a very simple and symmetric form. Here, we omit the double-prime superscripts, and set $\gamma := 3\rho\beta$:

$$[J_a,J_b] = (a-b)(J_{a+b-1} + \gamma J_{a+b} + J_{a+b+1}), \quad (10a)$$

$$[K_a,K_b] = (a-b)\beta J_{a+b}, \quad (10b)$$

$$[J_a,K_b] = (a-b-\tfrac{1}{2})K_{a+b-1} + (a-b)\gamma K_{a+b} + (a-b+\tfrac{1}{2})K_{a+b+1}. \quad (10c)$$

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