COMMUNICATIONS IN ALGEBRA, 28(4), 2131–2145 (2000)

ON FREE PARTIALLY ASSOCIATIVE TRIPLE SYSTEMS

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ABSTRACT. A triple system is partially associative (by definition) if it satisfies the identity $(abc)de + a(bcd)e + ab(cde) \equiv 0$. This paper presents a computational study of the free partially associative triple system on one generator with coefficients in the ring $\mathbb{Z}$ of integers. In particular, the $\mathbb{Z}$-module structure of the homogeneous submodules of (odd) degrees ≤ 11 is determined, together with explicit generators for the free and torsion components in degrees ≤ 9. Elements of additive order 2 exist in degrees ≥ 7, and elements of additive order 6 exist in degrees ≥ 9. The most difficult case (degree 11) requires finding the row-reduced form over $\mathbb{Z}$ of a matrix of size 364×273. These computations were done with Maple V.4 on a Sun workstation.

PRELIMINARIES

A **triple system** is a $\mathbb{Z}$-module A together with a $\mathbb{Z}$-module homomorphism

$$t: A \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} A \longrightarrow A.$$

We will usually write abc instead of $t(a \otimes b \otimes c)$. There are two distinct generalizations of the familiar (binary) associative identity $(ab)c - a(bc) \equiv 0$ to the ternary case: **total associativity**, defined by the identities

$$(abc)de - a(bcd)e \equiv 0, \qquad a(bcd)e - ab(cde) \equiv 0,$$

and **partial associativity**, defined by the identity

1991 *Mathematics Subject Classification.* Primary 17A40, 17A50; Secondary 17-04.

$$I(a,b,c,d,e) := (abc)de + a(bcd)e + ab(cde) \equiv 0.$$

We call $I(a,b,c,d,e)$ the **ternary partial associator**. (Note that monomials in a triple system must have odd degree). For more information on total and partial associativity in k-ary rings, see the work of Gnedbaye [G1–3].

It is easy to motivate consideration of totally associative triple systems: the total associativity identities imply (by a straightforward extension of the inductive argument from the binary case) that any product $a_1 \cdots a_{2m+1}$ ($m \geq 2$) depends only on the order of the factors, not the placement of parentheses. There is also an interesting reason to study partially associative triple systems: the ternary commutator

$$[a,b,c] := \sum_{\sigma \in S_3} \mathrm{sign}(\sigma)\sigma(a)\sigma(b)\sigma(c) = abc - acb - bac + bca + cab - cba,$$

satisfies the quintic identity

$$\begin{aligned}&[[abc]de] - [[abd]ce] + [[abe]cd] + [[acd]be] - [[ace]bd]\\ &+ [[ade]bc] - [[bcd]ae] + [[bce]ad] - [[bde]ad] + [[cde]ab] \equiv 0,\end{aligned}$$

(a generalization of the Jacobi identity) in every partially associative triple system, but not in every totally associative triple system; see [G1–3] and [B1] for details. Thus in order to generalize Lie algebras to the ternary case, the natural setting is the variety of partially (not totally) associative triple systems. These two versions of associativity are quite different: one way of expressing this is to observe that a triple system which is both totally and partially associative obviously satisfies the identities $(abc)de \equiv a(bcd)e \equiv ab(cde) \equiv 0$. Another way of expressing the difference between these two kinds of associativity is to study the $\mathbb{Z}$-module structure of the corresponding free rings.

Proposition 1. *Any free totally associative triple system is a free $\mathbb{Z}$-module. The free totally associative triple system on one free generator is a free $\mathbb{Z}$-module of rank one in every (odd) degree.*

Proof. It is easy to see that the free totally associative triple system on g free generators is isomorphic (as a $\mathbb{Z}$-module) to the direct sum of the submodules of odd degree in the free associative (binary) ring on g free generators; the product is the obvious ternary product $abc := (ab)c = a(bc)$ in the associative (binary) ring. It follows that any free totally associative triple system is a free $\mathbb{Z}$-module, and in particular that the free ring on one generator has the stated structure. $\square$

On the other hand, we will see in this paper that a free partially associative triple system is *never* a free $\mathbb{Z}$-module; see Theorem 5 and Corollary 6.

Notation. Throughout this paper F denotes the (absolutely) free triple system on the single generator a, and R is the ideal of relations generated by the ternary partial associators $I(v, w, x, y, z)$ as v, w, x, y, z range over all elements of F; since I is a $\mathbb{Z}$-linear function of its arguments we may restrict v, w, x, y, z to be monomials. We let $P \approx F/R$ denote the free partially associative triple system on the single generator a (more precisely, $a + R$). We write F_n for the $\mathbb{Z}$-submodule of F spanned by the monomials of degree n; then F is the direct sum of all F_n as n ranges over the odd positive integers. We define $R_n := R \cap F_n$ and $P_n := F_n/R_n$; then we clearly have the direct sums

$$R = \sum_{m=1}^{\infty} R_{2m+1}, \qquad P = \sum_{m=1}^{\infty} P_{2m+1},$$

where the summation index m denotes the number of ternary operations in a ternary monomial of degree $n = 2m + 1$.

Lemma 2. *The dimension of F_n is the ternary Catalan number*

$$f_n = \frac{1}{m}\binom{3m}{m-1}, \qquad n = 2m + 1.$$

Proof. We use the following notation:

- W_m is the set of ternary words of degree $2m+1$ in a; each element of W_m contains $2m + 1$ occurrences of a, and m occurrences of the ternary operation (or $m - 1$ pairs of parentheses, since we omit the outermost pair);
- T_m is the set of complete rooted planar ternary trees with $2m + 1$ leaf nodes (denoted $\bullet$) and m nonleaf nodes, for a total of $3m + 1$ nodes;
- B_m is the set of all binary strings of length $3m + 1$ containing $2m + 1$ ones and m zeroes.

It is clear that the dimension of F_n is the number of elements in W_m. We define mappings

$$\alpha_m: W_m \longrightarrow T_m, \qquad \beta_m: T_m \longrightarrow B_m,$$

inductively as follows:

α_m: For $m = 0$ set $\alpha_0(a) = \bullet$ (the tree consisting of one leaf node and no nonleaf nodes); for $m \geq 1$ write a word w of length $2m+1$ as $w = w_1w_2w_3$ where w_i is a word of length $2m_i+1$ with $m_i < m$ ($i = 1,2,3$), and define $\alpha_m(w_1w_2w_3)$ to be the tree in which the root node is attached to the subtrees $\alpha_{m_1}(w_1)$ and $\alpha_{m_2}(w_2)$ and $\alpha_{m_3}(w_3)$ (in order from left to right);

β_m: For $m = 0$ set $\beta_0(\bullet) = 1$; for $m \geq 1$ consider a tree t with t_1, t_2, t_3 the subtrees of the root node (t_i is a tree with m_i nonleaf nodes), and define $\beta_m(t) = 0\beta_{m_1}(t_1)\beta_{m_2}(t_2)\beta_{m_3}(t_3)$; note the initial zero.

The mapping α_m is a bijection for all m; its inverse may be defined inductively as follows. For $m = 0$ we set $\alpha_0^{-1}(\bullet) = a$; for $m \geq 1$ we consider a tree t with subtrees t_i ($i = 1,2,3$) as in the definition of β_m, and then define $\alpha_m^{-1}(t) = \alpha_{m_1}^{-1}(t_1)\alpha_{m_2}^{-1}(t_2)\alpha_{m_3}^{-1}(t_3)$.

The mapping β_m is injective for all m but never surjective. For $m = 0$ the image of β_0 is $\{1\}$. For $m \geq 1$ we can describe the image of β_m as follows. The set B_m has $\binom{3m+1}{m}$ elements; we define an equivalence relation on this set by declaring two binary strings b and b' to be equivalent if b' may be obtained from b by applying a cyclic permutation. This equivalence relation partitions B_m into $\frac{1}{3m+1}\binom{3m+1}{m}$ equivalence classes each containing $3m+1$ elements. That there are $3m+1$ distinct elements in each equivalence class follows from the fact that m and $3m+1$ are relatively prime, so there can be no periodicity in any of the strings in B_m. In each equivalence class there is a unique string having the least numerical value in the sense of an integer expressed in base 2; we call this string the representative of that class. The image of β_m consists of the set of representatives of the equivalence classes.

From the above considerations it is clear that the number of elements in W_m is the same as the number of equivalence classes in B_m; and since $\frac{1}{3m+1}\binom{3m+1}{m} = \frac{1}{m}\binom{3m}{m-1}$ this completes the proof. □

Remark. A straightforward generalization of the above proof shows that the number of distinct monomials of degree n in the free k-ary ring on one generator is

$$\frac{1}{m}\binom{km}{m-1}, \qquad \text{where } n = (k-1)m + 1.$$

See [K], especially exercise 2.3.4.4-11 (p. 396) and the solution (p. 584).

Example. The proof of Lemma 2 may be clarified by an example. The composition $\beta_m \circ \alpha_m$ for $m \geq 1$ applied to the word w may be described simply as follows: start with 0 (this corresponds to the omitted pair of outermost parentheses), then moving from left to right convert each left parenthesis in w into 0, and a into 1; the right parentheses are ignored. In the cases $m \leq 3$ we have the following lists of ternary words w and the corresponding binary strings $(\beta_m \circ \alpha_m)(w)$:

$$a \mapsto 1, \qquad aaa \mapsto 0111,$$

$$(aaa)aa \mapsto 0011111, \qquad a(aaa)a \mapsto 0101111, \qquad aa(aaa) \mapsto 0110111,$$

$$((aaa)aa)aa \mapsto 0001111111, \qquad (a(aaa)a)aa \mapsto 0010111111,$$

$$(aa(aaa))aa \mapsto 0011011111, \qquad a((aaa)aa)a \mapsto 0100111111,$$

$$a(a(aaa)a)a \mapsto 0101011111, \qquad a(aa(aaa))a \mapsto 0101101111,$$

$$aa((aaa)aa) \mapsto 0110011111, \qquad aa(a(aaa)a) \mapsto 0110101111,$$

$$aa(aa(aaa)) \mapsto 0110110111, \qquad (aaa)(aaa)a \mapsto 0011101111,$$

$$(aaa)a(aaa) \mapsto 0011110111, \qquad a(aaa)(aaa) \mapsto 0101110111.$$

Each of these binary strings has the property that any cyclic permutation results in a string with strictly greater numerical value. □

Monomial basis. The monomial basis W_m for F_n may be constructed recursively. We begin with $W_0 := \{a\}$. For $m \geq 1$ we consider the compositions (i.e. ordered partitions) of $n = 2m+1$ into 3 odd parts. For each such composition $n = i+j+k$ we form all the words $w = xyz$ where x, y, z have degrees i, j, k respectively. Some examples: when $m = 1$ we have $n = 3$ and the single composition $3 = 1 + 1 + 1$ giving the word aaa. When $m = 2$ we have $n = 5$ and the compositions

$$5 = 3 + 1 + 1 = 1 + 3 + 1 = 1 + 1 + 3,$$

giving the three words $(aaa)aa$, $a(aaa)a$, $aa(aaa)$. When $m = 3$ we have $n = 7$ and the compositions

$$7 = 5+1+1 = 1+5+1 = 1+1+5 = 3+3+1 = 3+1+3 = 1+3+3,$$

giving the twelve words displayed in the previous example. This construction also totally orders the set of monomials, since we use the reverse lexicographical order on the partitions ($n = p_1 + \cdots + p_r$ precedes $n = q_1 + \cdots + q_s$ if and only if there exists j with $0 \le j < \min(r,s)$ such that $p_i = q_i$ for $1 \le i \le j$ and $p_{j+1} > q_{j+1}$) and the lexicographical order on the permutations of a given partition (if $n = p_{\sigma.1} + \cdots + p_{\sigma.r}$ and $n = p_{\tau.1} + \cdots + p_{\tau.r}$ are two distinct compositions of n then the first precedes the second if and only if there exists j with $0 \le j < r$ such that $\sigma.i = \tau.i$ for $1 \le i \le j$ and $\sigma(j+1) < \tau(j+1)$); together with the previously computed total orders on the monomials of degrees $< m$ this gives a total order on the monomials of degree m.

Remark. This construction of W_m gives a recursive formula for f_n:

$$f_1 = 1, \qquad f_n = \sum_{\substack{i+j+k=n \\ i,j,k \text{ odd}}} f_i f_j f_k \quad \text{for } n \ge 3, \text{ odd}.$$

The corresponding generating function and functional equation are

$$\phi(x) := \sum_{m \ge 0} f_{2m+1} x^m, \qquad x\phi(x)^3 - \phi(x) + 1 \equiv 0.$$

From this we find that $\phi(x) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3} \mid \frac{3}{2} \mid \frac{27}{4}x\right)$ is a hypergeometric function. This contrasts with the binary case, where the generating function is elementary; see [C], pp. 60-61.

Spanning relations. There is a beautiful way to obtain a set of spanning relations (i.e. $\mathbb{Z}$-module generators) for R_n from the set of basis monomials for F_n. Recall (from the proof of Lemma 2) that any word w in W_m can be represented as a complete rooted planar ternary tree $\alpha_m(w)$. Consider an edge e of this tree which joins two nonleaf nodes. These two nonleaf nodes have altogether five subtrees: two from the node closer to the root (not counting the subtree attached by the edge e) and three from the other node. Denote these five subtrees by t_i $(1 \le i \le 5)$ from left to right; the corresponding subwords of w are $x_i := \alpha_{m_i}^{-1}(t_i)$ $(1 \le i \le 5)$. In the word w replace the five consecutive subwords x_1, x_2, x_3, x_4, x_5 by the ternary partial associator $I(x_1, x_2, x_3, x_4, x_5)$. The result, call it $\gamma(w)$, is a polynomial with three terms which is an element of R_n since it is homogeneous of degree n and is clearly in the ideal generated by the ternary partial associators. Furthermore, since $I(u,v,x,y,z)$ is a $\mathbb{Z}$-linear function of its arguments, any element of R_n is a $\mathbb{Z}$-linear combination of the $\gamma(w)$ for $w \in W_m$. This construction also totally orders the spanning relations since we use the previously defined total order on the set of basis monomials and then use the total order on the set of edges joining two nonleaf nodes defined by $e < e'$ if and only if either (i) e is closer to the root than e' or (ii) e and e' are the same distance from the root but e lies to the left of e'. At each stage of the construction of the spanning relations we ignore relations that have already been computed at an earlier stage. We call the ordered set of polynomials $\gamma(w)$ the **standard spanning relations** for R_n.

Lemma 3. *The number of standard spanning relations for R_n is*

$$r_n = \frac{m-1}{3} f_n = \binom{3m-1}{m-2}, \qquad n = 2m+1.$$

Proof. Given $w \in W_m$ the tree $\alpha_m(w)$ has m nonleaf nodes and $n = 2m+1$ leaf nodes. Each nonleaf node has three outgoing edges for a total of $3m$ edges. Of these edges $2m+1$ join a leaf node to a nonleaf node, so there are $m-1$ edges e which join two nonleaf nodes. The subwords x_i $(1 \le i \le 5)$ of w which correspond to the five subtrees attached to e can appear in any one of the three parenthesizations $(x_1x_2x_3)x_4x_5$ or $x_1(x_2x_3x_4)x_5$ or $x_1x_2(x_3x_4x_5)$, but each of these three subwords becomes $I(x_1,x_2,x_3,x_4,x_5)$ in $\gamma(w)$. Thus from each word in W_m we obtain $m-1$ relations, but each relation is obtained three times; hence the formula for r_n. □

Example. Here is the ordered set of 8 spanning relations in degree 7 as computed by the algorithm described above. Column 1 gives the basis monomial, column 2 gives the list of subwords x_1, x_2, x_3, x_4, x_5 (the subscripts on the a's indicate positions in the basis monomial), and column 3 gives the resulting spanning relation:

1: $((aaa)aa)aa$	$a_1a_2a_3, a_4, a_5, a_6, a_7$	R1: $I(aaa,a,a,a,a)$
	a_1, a_2, a_3, a_4, a_5	R2: $I(a,a,a,a,a)aa$
2: $(a(aaa)a)aa$	$a_1, a_2a_3a_4, a_5, a_6, a_7$	R3: $I(a,aaa,a,a,a)$
	a_1, a_2, a_3, a_4, a_5	same as R2

3: $(aa(aaa))aa$	$a_1, a_2, a_3a_4a_5, a_6, a_7$	R4: $I(a, a, aaa, a, a)$
	a_1, a_2, a_3, a_4, a_5	same as R2
4: $a((aaa)aa)a$	$a_1, a_2a_3a_4, a_5, a_6, a_7$	same as R3
	a_2, a_3, a_4, a_5, a_6	R5: $aI(a, a, a, a, a)a$
5: $a(a(aaa)a)a$	$a_1, a_2, a_3a_4a_5, a_6, a_7$	same as R4
	a_2, a_3, a_4, a_5, a_6	same as R5
6: $a(aa(aaa))a$	$a_1, a_2, a_3, a_4a_5a_6, a_7$	R6: $I(a, a, a, aaa, a)$
	a_2, a_3, a_4, a_5, a_6	same as R5
7: $aa((aaa)aa)$	$a_1, a_2, a_3a_4a_5, a_6, a_7$	same as R4
	a_3, a_4, a_5, a_6, a_7	R7: $aaI(a, a, a, a, a)$
8: $aa(a(aaa)a)$	$a_1, a_2, a_3, a_4a_5a_6, a_7$	same as R6
	a_3, a_4, a_5, a_6, a_7	same as R7
9: $aa(aa(aaa))$	$a_1, a_2, a_3, a_4, a_5a_6a_7$	R8: $I(a, a, a, a, aaa)$
	a_3, a_4, a_5, a_6, a_7	same as R7
10: $(aaa)(aaa)a$	$a_1, a_2, a_3, a_4a_5a_6, a_7$	same as R6
	$a_1a_2a_3, a_4, a_5, a_6, a_7$	same as R1
11: $(aaa)a(aaa)$	$a_1a_2a_3, a_4, a_5, a_6, a_7$	same as R1
	$a_1, a_2, a_3, a_4, a_5a_6a_7$	same as R8
12: $a(aaa)(aaa)$	$a_1, a_2a_3a_4, a_5, a_6, a_7$	same as R3
	$a_1, a_2, a_3, a_4, a_5a_6a_7$	same as R8

Table. Here are some values of the two functions defined in Lemmas 2 and 3:

n	1	3	5	7	9	11	13	15	17	19	21
f_n	1	1	3	12	55	273	1428	7752	43263	246675	1430715
r_n	–	0	1	8	55	364	2380	15504	100947	657800	4292145

Computational methods. The purpose of this paper is to determine the $\mathbb{Z}$-module structure of P_n for $1 \leq n \leq 11$. In order to do this we first form the **relation matrix**, the integer matrix of size $r_n \times f_n$ which expresses each spanning polynomial for R_n as a $\mathbb{Z}$-linear combination of the basis monomials for F_n. We then compute the reduced row-echelon form over $\mathbb{Z}$ of the relation matrix; from this we can read off the structure of the quotient module $P_n = F_n/R_n$. (For details see any discussion of modules over a principal ideal domain, for example [J], Ch. 3). The difference between elementary row operations over $\mathbb{Q}$ and over $\mathbb{Z}$ is that over $\mathbb{Z}$ we can only multiply a row by ± 1 since these are the only units. This means that the leading entry of a nonzero row can be any positive integer (say ℓ); we then normalize the entries above that leading entry by using elementary row operations to convert them to representatives (say $0, \ldots, \ell - 1$) of the residue classes modulo ℓ. In order to determine what the leading entry in a row should be, we need to compute the greatest common divisor of the nonzero entries below that position, and then use elementary row operations to put the gcd in that row as the leading entry.

Most of the computations described in this paper were done using Maple V.4, which has a very useful linear algebra package (linalg). However it does not have a procedure for computing the row-reduced form over $\mathbb{Z}$ of an integer matrix; Maple procedures to do this are available by email from the author. All computations were executed on a Sun workstation (either an LX or an UltraSparc).

The methods of this paper are a further development of those in my earlier work on free semialternative rings [B2] and on free assosymmetric rings [B3].

Proposition 4. *As $\mathbb{Z}$-modules we have*

$$P_1 \approx \mathbb{Z}, \qquad P_3 \approx \mathbb{Z}, \qquad P_5 \approx \mathbb{Z}^2.$$

Proof. It is clear that F_1 has basis a, that F_3 has basis aaa, and that F_5 has basis

$$(aaa)aa, \quad a(aaa)a, \quad aa(aaa).$$

Since $R_1 = R_3 = \{0\}$ we get $P_1 = F_1$ and $P_3 = F_3$. Since R_5 is spanned by $I(a,a,a,a,a)$ the relation matrix in degree 5 is just

$$(1 \quad 1 \quad 1);$$

this is already in row-reduced form. A basis for P_5 consists of (the cosets modulo R of) the two words $a(aaa)a$ and $aa(aaa)$. $\square$

DEGREE SEVEN

Theorem 5. *As $\mathbb{Z}$-modules we have*

$$P_7 \approx \mathbb{Z}^4 \oplus \mathbb{Z}_2.$$

Proof. We have seen that an ordered basis for F_7 consists of the twelve words

$$\begin{array}{llll} ((aaa)aa)aa, & (a(aaa)a)aa, & (aa(aaa))aa, & a((aaa)aa)a, \\ a(a(aaa)a)a, & a(aa(aaa))a, & aa((aaa)aa), & aa(a(aaa)a), \\ aa(aa(aaa)), & (aaa)(aaa)a, & (aaa)a(aaa), & a(aaa)(aaa); \end{array}$$

and an ordered spanning set for R_7 consists of the eight polynomials

$$\begin{array}{llll} I(aaa,a,a,a,a), & I(a,a,a,a,a)aa, & I(a,aaa,a,a,a), & I(a,a,aaa,a,a), \\ aI(a,a,a,a,a)a, & I(a,a,a,aaa,a), & aaI(a,a,a,a,a), & I(a,a,a,a,aaa). \end{array}$$

Expanding these eight relations in terms of the twelve basis words gives the relation matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

We now apply the following 19 elementary row operations (expressed as Maple commands):

```
addrow(",1,2,-1):   addrow(",2,3,-1):   mulrow(",3,-1):
addrow(",3,4,-1):   addrow(",3,2,-1):   addrow(",4,5,-1):
addrow(",4,2,-1):   addrow(",4,3):      addrow(",5,6,-1):
addrow(",6,7,-1):   addrow(",6,2):      addrow(",6,3,-1):
addrow(",6,4,-1):   addrow(",6,5):      addrow(",7,8,-1):
addrow(",8,3):      addrow(",8,4):      addrow(",8,6,-1):
addrow(",8,7);
```

The meaning of these commands is as follows:

addrow(",i,j,m):	add m times row i to row j (m omitted if $m = 1$)
mulrow(",i,m):	multiply row i by m
swaprow(",i,j):	interchange row i and row j

The result is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \end{pmatrix}$$

This the row-reduced form over $\mathbb{Z}$ of the relation matrix, which explains row 8 and column 10: we cannot divide row 8 by 2 (since 2 is not a unit in $\mathbb{Z}$), and so we normalize column 10 by making the entries 0 or 1. A basis for the free component of P_7 is given by (the cosets of) the basis monomials for F_7 corresponding to columns 5, 8, 11, 12 of the row-reduced form; these are the columns which do not contain a leading entry of a row. The corresponding basis monomials are

$$a(a(aaa)a)a, \qquad aa(a(aaa)a), \qquad (aaa)a(aaa), \qquad a(aaa)(aaa).$$

A generator for the torsion component of P_7 is (the coset of) the polynomial corresponding to one-half of row 8, namely

$$T_7 := (aaa)(aaa)a + (aaa)a(aaa) + a(aaa)(aaa).$$

This completes the proof. $\square$

Remark. The CPU time for these computations was 7.49 seconds on a Sun Sparcstation LX.

Corollary 6. *A free partially associative triple system is never a free $\mathbb{Z}$-module.*

Proof. The free ring on one generator is a subring of any free ring. $\square$

Corollary 7. *In any partially associative triple system over a coefficient ring which contains $\frac{1}{2}$ we have the identity*

$$(aaa)(aaa)a + (aaa)a(aaa) + a(aaa)(aaa) \equiv 0.$$

Proof. Over $\mathbb{Z}$ we have $2T_7 \equiv 0$; hence over a coefficient ring containing $\frac{1}{2}$ have $T_7 \equiv 0$. $\square$

DEGREE NINE

Theorem 8. *As $\mathbb{Z}$-modules we have*

$$P_9 \approx \mathbb{Z}^5 \oplus \mathbb{Z}_2^8 \oplus \mathbb{Z}_6.$$

Proof. The 55 monomials forming an ordered basis for F_9 are

$(((aaa)aa)aa)aa_1$	$((a(aaa)a)aa)aa_2$	$((aa(aaa))aa)aa_3$
$(a((aaa)aa)a)aa_4$	$(a(a(aaa)a)a)aa_5$	$(a(aa(aaa))a)aa_6$
$(aa((aaa)aa))aa_7$	$(aa(a(aaa)a))aa_8$	$(aa(aa(aaa)))aa_9$
$((aaa)(aaa)a)aa_{10}$	$((aaa)a(aaa))aa_{11}$	$(a(aaa)(aaa))aa_{12}$
$a(((aaa)aa)aa)a_{13}$	$a((a(aaa)a)aa)a_{14}$	$a((aa(aaa))aa)a_{15}$
$a(a((aaa)aa)a)a_{16}$	$a(a(a(aaa)a)a)a_{17}$	$a(a(aa(aaa))a)a_{18}$
$a(aa((aaa)aa))a_{19}$	$a(aa(a(aaa)a))a_{20}$	$a(aa(aa(aaa)))a_{21}$
$a((aaa)(aaa)a)a_{22}$	$a((aaa)a(aaa))a_{23}$	$a(a(aaa)(aaa))a_{24}$
$aa(((aaa)aa)aa)_{25}$	$aa((a(aaa)a)aa)_{26}$	$aa((aa(aaa))aa)_{27}$
$aa(a((aaa)aa)a)_{28}$	$aa(a(a(aaa)a)a)_{29}$	$aa(a(aa(aaa))a)_{30}$
$aa(aa((aaa)aa))_{31}$	$aa(aa(a(aaa)a))_{32}$	$aa(aa(aa(aaa)))_{33}$
$aa((aaa)(aaa)a)_{34}$	$aa((aaa)a(aaa))_{35}$	$aa(a(aaa)(aaa))_{36}$
$((aaa)aa)(aaa)a_{37}$	$(a(aaa)a)(aaa)a_{38}$	$(aa(aaa))(aaa)a_{39}$
$((aaa)aa)a(aaa)_{40}$	$(a(aaa)a)a(aaa)_{41}$	$(aa(aaa))a(aaa)_{42}$
$(aaa)((aaa)aa)a_{43}$	$(aaa)(a(aaa)a)a_{44}$	$(aaa)(aa(aaa))a_{45}$
$(aaa)a((aaa)aa)_{46}$	$(aaa)a(a(aaa)a)_{47}$	$(aaa)a(aa(aaa))_{48}$
$a((aaa)aa)(aaa)_{49}$	$a(a(aaa)a)(aaa)_{50}$	$a(aa(aaa))(aaa)_{51}$
$a(aaa)((aaa)aa)_{52}$	$a(aaa)(a(aaa)a)_{53}$	$a(aaa)(aa(aaa))_{54}$
$(aaa)(aaa)(aaa)_{55}$.		

The ordered set of 55 spanning relations for R_9 are listed below; the superscripts indicate the basis monomial from which the relation comes (by the procedure described previously):

$I((aaa)aa,a,a,a,a)_1^1$	$I(aaa,a,a,a,a)aa_2^1$	$(I(a,a,a,a,a)aa)aa_3^1$
$I(a(aaa)a,a,a,a,a)_4^2$	$I(a,aaa,a,a,a)aa_5^2$	$I(aa(aaa),a,a,a,a)_6^3$
$I(a,a,aaa,a,a)aa_7^3$	$I(a,(aaa)aa,a,a,a)_8^4$	$(aI(a,a,a,a,a)a)aa_9^4$
$I(a,a(aaa)a,a,a,a)_{10}^5$	$I(a,aa(aaa),a,a,a)_{11}^6$	$I(a,a,a,(aaa),a)aa_{12}^6$
$I(a,a,(aaa)aa,a,a)_{13}^7$	$(aaI(a,a,a,a,a))aa_{14}^7$	$I(a,a,a(aaa)a,a,a)_{15}^8$
$I(a,a,aa(aaa),a,a)_{16}^9$	$I(a,a,a,a,aaa)aa_{17}^9$	$I(aaa,aaa,a,a,a)_{18}^{10}$
$I(aaa,a,aaa,a,a)_{19}^{11}$	$I(a,aaa,aaa,a,a)_{20}^{12}$	$aI(aaa,a,a,a,a)a_{21}^{13}$
$a(I(a,a,a,a,a)aa)a_{22}^{13}$	$aI(a,aaa,a,a,a)a_{23}^{14}$	$aI(a,a,aaa,a,a)a_{24}^{15}$
$a(aI(a,a,a,a,a)a)a_{25}^{16}$	$aI(a,a,a,aaa,a)a_{26}^{18}$	$I(a,a,a,(aaa)aa,a)_{27}^{19}$
$a(aaI(a,a,a,a,a))a_{28}^{19}$	$I(a,a,a,a(aaa)a,a)_{29}^{20}$	$I(a,a,a,aa(aaa),a)_{30}^{21}$
$aI(a,a,a,a,aaa)a_{31}^{21}$	$I(a,aaa,a,aaa,a)_{32}^{23}$	$I(a,a,aaa,aaa,a)_{33}^{24}$
$aaI(aaa,a,a,a,a)_{34}^{25}$	$aa(I(a,a,a,a,a)aa)_{35}^{25}$	$aaI(a,aaa,a,a,a)_{36}^{26}$
$aaI(a,a,aaa,a,a)_{37}^{27}$	$aa(aI(a,a,a,a,a)a)_{38}^{28}$	$aaI(a,a,a,aaa,a)_{39}^{30}$
$I(a,a,a,a,(aaa)aa)_{40}^{31}$	$aa(aaI(a,a,a,a,a))_{41}^{31}$	$I(a,a,a,a,a(aaa)a)_{42}^{32}$
$I(a,a,a,a,aa(aaa))_{43}^{33}$	$aaI(a,a,a,a,aaa)_{44}^{33}$	$I(a,a,aaa,a,aaa)_{45}^{35}$
$I(a,a,a,aaa,aaa)_{46}^{36}$	$I(aaa,a,a,aaa,a)_{47}^{37}$	$I(a,a,a,a,a)(aaa)a_{48}^{37}$
$I(aaa,a,a,a,aaa)_{49}^{40}$	$I(a,a,a,a,a)a(aaa)_{50}^{40}$	$I(a,aaa,a,a,aaa)_{51}^{41}$
$(aaa)I(a,a,a,a,a)a_{52}^{43}$	$(aaa)aI(a,a,a,a,a)_{53}^{46}$	$aI(a,a,a,a,a)(aaa)_{54}^{49}$
$a(aaa)I(a,a,a,a,a)_{55}^{52}$		

Expanding the relations in terms of the basis monomials we obtain the 55×55 relation matrix. Determining the row-reduced form over $\mathbb{Z}$ of the relation matrix required 596 elementary row operations. The row-reduced form has rank 50; hence the rank of the free component of P_9 is 5 (the difference between the rank and the number of columns of the relation matrix). A basis for the free component consists of (the cosets of) the monomials corresponding to columns 17, 29, 47, 50 and 54:

$$a(a(a(aaa)a)a)a, \quad aa(a(a(aaa)a)a), \quad (aaa)a(a(aaa)a),$$
$$a(a(aaa)a)(aaa), \quad a(aaa)(aa(aaa)).$$

Rows 21, 32, 37, 42, 43, 45, 47 and 49 have leading entry 2, which gives the torsion submodule $\mathbb{Z}_2^8$. Generators for the eight $\mathbb{Z}_2$ components are (the cosets of)

$$T_9^1 := a((aaa)(aaa)a)a+2(aaa)(aaa)(aaa),$$
$$T_9^2 := aa((aaa)(aaa)a)+2(aaa)(aaa)(aaa),$$
$$T_9^3 := (aa(aaa))(aaa)a+2(aaa)(aaa)(aaa),$$
$$T_9^4 := (aaa)(a(aaa)a)a-(aaa)a(a(aaa)a)+a(aaa)(aa(aaa))+2(aaa)(aaa)(aaa),$$
$$T_9^5 := (aaa)(aa(aaa))a+(aaa)a(a(aaa)a)-a(aaa)(aa(aaa))+2(aaa)(aaa)(aaa),$$
$$T_9^6 := (aaa)a(aa(aaa))+a(aaa)(aa(aaa))+(aaa)(aaa)(aaa),$$
$$T_9^7 := a(aa(aaa))(aaa)+2(aaa)(aaa)(aaa),$$
$$T_9^8 := a(aaa)(a(aaa)a)+a(aaa)(aa(aaa))+(aaa)(aaa)(aaa).$$

Row 50 has leading entry 6; the coset of

$$U_9 := (aaa)(aaa)(aaa),$$

generates the torsion submodule $\mathbb{Z}_6$. □

Remark. The CPU time for these computations was 683.62 seconds (11 minutes, 23.62 seconds) on a Sun Sparcstation LX.

Corollary 9. *In any partially associative triple system over a coefficient ring which contains $\frac{1}{6}$, we have the identity*

$$(aaa)(aaa)(aaa) \equiv 0.$$

Proof. See the proof of Corollary 7. □

DEGREE ELEVEN

Theorem 10. *As $\mathbb{Z}$-modules we have*

$$P_{11} \approx \mathbb{Z}^6 \oplus \mathbb{Z}_2^{35} \oplus \mathbb{Z}_6.$$

Proof. Since the computations in this case are so large, only a brief summary will be given. The relation matrix in degree 11 has size 364 × 273; computing the row-reduced form over $\mathbb{Z}$ required 12753 elementary row operations. The row-reduced matrix has rank 267, so the rank of the free component of P_{11} is 6; the free generators correspond to columns 72, 127, 206, 218, 251 and 261. The rows which have a leading entry other than 1 are

76	93	98	99	130	147	152	153	171	175	192	197
199	208	217	219	221	226	231	233	238	239	241	242
247	249	250	255	257	258	260	261	263	264	266	267

Of these 36 rows, the first 35 have leading entry 2 and the last has leading entry 6. □

Remark. The CPU time for these computations was 116450.97 seconds (32 hours, 20 minutes, 50.97 seconds) on a Sun UltraSparc; this computation would have taken an unreasonably long time on the Sparcstation LX used for degrees 7 and 9.

ACKNOWLEDGEMENTS

This work was supported by NSERC, the Natural Sciences and Engineering Research Council of Canada, through grant OGP0153128.

REFERENCES

[B1] M. Bremner, *Identities for the ternary commutator*, J. Algebra **206** (1998), 615-623.
[B2] ———, *On the $\mathbb{Z}$-module structure of a free semialternative ring*, Comm. Algebra **27** (1999), 1951-1965.
[B3] ———, *Torsion in free assosymmetric rings*, preprint (December 1998).
[C] P. J. Cameron, *Combinatorics: Topics, Techniques, Algorithms*, University Press, Cambridge, 1994.
[G1] A. V. Gnedbaye, *Opérades des algèbres k-aires*, Thèse (Seconde partie), Université Louis Pasteur, Institut de recherche mathématique avancée, Strasbourg, 1995.
[G2] ———, *Les algèbres k-aires et leurs opérades*, C. R. Acad. Sci. Paris, Série I **321** (1995), 147-152.
[G3] ———, *Opérades des algèbres $(k+1)$-aires*, Operads: Proceedings of Renaissance Conferences (Hartford/Luminy, 1995), Contemp. Math. **202**, Amer. Math. Soc., Providence, 1997, pp. 83-113.
[J] N. Jacobson, *Basic Algebra I*, W. H. Freeman, San Francisco, 1974.
[K] D.E. Knuth, *The Art of Computer Programming, Vol. 1: Fundamental Algorithms*, second edition, Addison-Wesley, Reading, Massachusetts, 1973.

Received: December 1998

Revised: May 1999