International Journal of Mathematics, Game Theory and Algebra
Volume 12, Number 1, pp. 23-37
ISSN 1060-9881

Additive Structure of Free Left-symmetric and Assosymmetric Rings

Murray Bremner
Department of Mathematics and Statistics,
University of Saskatchewan, McLean Hall, 106 Wiggins Road
Saskatoon, SK, S7N 5E6, Canada
e-mail address: bremner@math.usask.ca

Abstract. This paper describes a computational study of the homogeneous submodules of degree $0 \leq n \leq 8$ in the free left-symmetric ring L (over $\mathbb{Z}$ on one generator) and the free assosymmetric ring A (over $\mathbb{Z}$ on one generator). It is shown that L is a free $\mathbb{Z}$-module for $0 \leq n \leq 8$ but that A has torsion elements of order 2 for $5 \leq n \leq 8$ and order 6 for $6 \leq n \leq 8$. The existence of these torsion elements in A proves the necessity of the restriction on the characteristic in the theorem of Hentzel, Jacobs and Peresi, which gives a uniform construction of a basis for the free assosymmetric algebra on a finite number of generators over a field of characteristic $\neq 2, 3$. The longer computations in this paper were executed on a Sun Ultra 5 workstation using procedures written in Maple V.5.1.

Introduction

We define a **ring** (not necessarily associative) to be a $\mathbb{Z}$-module R together with a $\mathbb{Z}$-bilinear map $m: R \times R \to R$. We write ab for $m(a,b)$ and define the **associator** by $(a,b,c) = (ab)c - a(bc)$. We say that R is **left-symmetric** if it satisfies the identity

$$\ell(a,b,c) \equiv 0 \qquad \text{where} \qquad \ell(a,b,c) = (a,b,c) - (b,a,c),$$

and **right-symmetric** if it satisfies the identity

$$r(a,b,c) \equiv 0 \qquad \text{where} \qquad r(a,b,c) = (a,b,c) - (a,c,b).$$

We say that R is **assosymmetric** if it is both left-symmetric and right-symmetric; equivalently, if the associator is invariant under all permutations of the three arguments. We have the properties

$$\ell(a,b,c) = -\ell(b,a,c), \quad \ell(a,a,b) = 0, \quad r(a,b,c) = -r(a,c,b), \quad r(a,b,b) = 0.$$

The opposite ring of a right-symmetric ring is a left-symmetric ring; for this reason we restrict attention to left-symmetric rings and assosymmetric rings.

Left-symmetric and assosymmetric rings are closely related to Lie rings. We say that R is **Lie-admissible** if the commutator $[a,b] = ab - ba$ gives R the structure of a Lie ring, denoted R^-.

1991 *Mathematics Subject Classification.* Primary 17A30, 17A50; Secondary 17-04, 17-08.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Proposition. *A ring R is left-symmetric if and only if R is Lie-admissible and left multiplication makes R into a left R^--module. A ring R is assosymmetric if and only if R is Lie-admissible and left (resp. right) multiplication makes R into a left (resp. right) R^--module.*

Proof. Expanding the associators in ℓ and r gives

$$\ell(a,b,c) = (ab)c - (ba)c - a(bc) + b(ac) = [a,b]c - [L_a, L_b]c,$$
$$r(a,b,c) = (ab)c - (ac)b - a(bc) + a(cb) = a[R_b, R_c] - a[b,c],$$

where $L_a b = ab = aR_b$. Expanding the commutators in the Jacobi identity

$$[[a,b],c] + [[b,c],a] + [[c,a],b]$$

gives the alternating sum of the associators:

$$(a,b,c) - (a,c,b) - (b,a,c) + (b,c,a) + (c,a,b) - (c,b,a).$$

From these two observations the result follows. □

A recent search (1 February 2001) on MathSciNet obtained 27 papers on left-symmetric or right-symmetric rings and algebras, and 5 papers on assosymmetric rings and algebras.

Results on free semialternative rings and free partially associative triple systems obtained by methods similar to those of the present paper appear in [Br1] and [Br2].

Additive Structure of the Free Left-Symmetric Ring

In this section

- F denotes the free nonassociative ring with 1 over $\mathbb{Z}$ on one generator a;
- I denotes the ideal in F generated by all $\ell(x,y,z)$ as x,y,z range over all elements of F (since ℓ is multilinear we may assume that x,y,z are monomials);
- L denotes the quotient F/I, the free left-symmetric ring over $\mathbb{Z}$ on one generator (also denoted a since this should not cause any confusion).

All these objects are graded by the degree $n \geq 0$; the homogeneous submodules of degree n will be denoted F_n, I_n and $L_n = F_n/I_n$. The $\mathbb{Z}$-module F_n is free of rank equal to the Catalan number $\frac{1}{n}\binom{2n-2}{n-1}$. Since any submodule of a free module over a PID is also free, I_n is a free $\mathbb{Z}$-module. The purpose of this section is to determine the $\mathbb{Z}$-module structure of the quotient module $L_n = F_n/I_n$ for $n \leq 8$. We will use the theory of canonical forms for matrices over $\mathbb{Z}$. For the general theory of canonical forms for matrices over a PID, see [AW] (Ch. 5, esp. §3).

Theorem (Degree ≤ 4). *We have the following isomorphisms of $\mathbb{Z}$-modules:*

$$L_0 \approx \mathbb{Z}, \qquad L_1 \approx \mathbb{Z}, \qquad L_2 \approx \mathbb{Z}, \qquad L_3 \approx \mathbb{Z}^2, \qquad L_4 \approx \mathbb{Z}^4.$$

The free left-symmetric ring L has no torsion in degree ≤ 4.

Proof. Since the left-symmetric identity is homogeneous and of degree 3, we have $I_n = \{0\}$ and $L_n \approx F_n$ for $0 \leq n \leq 2$; in fact this holds for any number of generators. In degree 3, since L is generated by the single element a, we see that I_3 is spanned as a $\mathbb{Z}$-module by the relation $\ell(a,a,a)$ which is 0. Hence $I_3 = \{0\}$ and $L_3 \approx F_3$. The $\mathbb{Z}$-module F_4 has an ordered free basis consisting of 5 monomials:

$$(a^2a)a, \qquad (aa^2)a, \qquad a^2a^2, \qquad a(a^2a), \qquad a(aa^2).$$

We have a single non-zero spanning relation for I_4:

$$\ell(a^2,a,a) = (a^2a)a - (aa^2)a - a^2a^2 + a(a^2a).$$

This gives the relation matrix

	$(a^2a)a$	$(aa^2)a$	a^2a^2	$a(a^2a)$	$a(aa^2)$
$\ell(a^2,a,a)$	1	-1	-1	1	0

From this we see that the quotient module $L_4 \approx F_4/I_4$ is free of rank 4. □

Theorem (Degree 5). *We have the isomorphism of $\mathbb{Z}$-modules*

$$L_5 \approx \mathbb{Z}^9.$$

There is no torsion in degree 5.

Proof. The $\mathbb{Z}$-module F_5 has an ordered free basis consisting of 14 monomials:

$$\begin{array}{llll} M_1 = ((a^2a)a)a, & M_2 = ((aa^2)a)a, & M_3 = (a^2a^2)a, & M_4 = (a(a^2a))a, \\ M_5 = (a(aa^2))a, & M_6 = (a^2a)a^2, & M_7 = (aa^2)a^2, & M_8 = a^2(a^2a), \\ M_9 = a^2(aa^2), & M_{10} = a((a^2a)a), & M_{11} = a((aa^2)a), & M_{12} = a(a^2a^2), \\ M_{13} = a(a(a^2a)), & M_{14} = a(a(aa^2)). & & \end{array}$$

We obtain a set of spanning relations for the submodule I_5 from the basis monomials for F_5 by replacing each subword $(xy)z$ or $x(yz)$ in each basis monomial by $\ell(x,y,z)$. This gives a redundant but complete list of the $\mathbb{Z}$-module generators of I_5. Since each basis monomial of degree 5 contains four occurrences of the binary operation, and each subword $(xy)z$ or $x(yz)$ contains two nested occurrences of this operation, each basis monomial for F_5 gives rise to three spanning relations for I_5, for a total of 42 relations, but only 5 of these are non-redundant:

$$\begin{array}{lllll} ((a^2a)a)a & \longrightarrow & \ell(a^2a,a,a) = R_1, & \ell(a^2,a,a)a = R_2, & (\ell(a,a,a)a)a = 0, \\ ((aa^2)a)a & \longrightarrow & \ell(aa^2,a,a) = R_3, & \ell(a,a^2,a)a = -R_2, & (\ell(a,a,a)a)a = 0, \\ (a^2a^2)a & \longrightarrow & \ell(a^2,a^2)a = 0, & \ell(a,a,a^2)a = 0, & \ell(a^2,a,a)a = R_2, \\ (a(a^2a))a & \longrightarrow & \ell(a,a^2a,a) = -R_1, & \ell(a,a^2,a)a = -R_2, & (a\ell(a,a,a))a = 0, \\ (a(aa^2))a & \longrightarrow & \ell(a,aa^2,a) = -R_3, & \ell(a,a,a^2)a = 0, & (a\ell(a,a,a))a = 0, \\ (a^2a)a^2 & \longrightarrow & \ell(a^2,a,a^2) = R_4, & \ell(a,a,a)a^2 = 0, & \ell(a^2a,a,a) = R_1, \end{array}$$

$$\begin{array}{llll}
(aa^2)a^2 & \longrightarrow \ \ell(a,a^2,a^2) = -R_4, & \ell(a,a,a)a^2 = 0, & \ell(aa^2,a,a) = R_3, \\
a^2(a^2a) & \longrightarrow \ \ell(a,a,a^2a) = 0, & \ell(a^2,a^2,a) = 0, & a^2\ell(a,a,a) = 0, \\
a^2(aa^2) & \longrightarrow \ \ell(a,a,aa^2) = 0, & \ell(a^2,a,a^2) = R_4, & a^2\ell(a,a,a) = 0, \\
a((a^2a)a) & \longrightarrow \ \ell(a,a^2a,a) = -R_1, & a\ell(a^2,a,a) = R_5, & a(\ell(a,a,a)a) = 0, \\
a((aa^2)a) & \longrightarrow \ \ell(a,aa^2,a) = -R_3, & a\ell(a,a^2,a) = -R_5, & a(\ell(a,a,a)a) = 0, \\
a(a^2a^2) & \longrightarrow \ \ell(a,a^2,a^2) = R_4, & a\ell(a,a,a^2) = 0, & a\ell(a^2,a,a) = R_5, \\
a(a(a^2a)) & \longrightarrow \ \ell(a,a,a^2a) = 0, & a\ell(a,a^2,a) = -R_5, & a(a\ell(a,a,a)) = 0, \\
a(a(aa^2)) & \longrightarrow \ \ell(a,a,aa^2) = 0, & a\ell(a,a,a^2) = 0, & a(a\ell(a,a,a)) = 0.
\end{array}$$

Expanding the relations R_i for $1 \le i \le 5$ in terms of the basis monomials for F_5 we obtain

$$\begin{aligned}
R_1 &= ((a^2a)a)a - (a(a^2a))a - (a^2a)a^2 + a((a^2a)a) = M_1 - M_4 - M_6 + M_{10}, \\
R_2 &= ((a^2a)a)a - ((aa^2)a)a - (a^2a^2)a + (a(a^2a))a = M_1 - M_2 - M_3 + M_4, \\
R_3 &= ((aa^2)a)a - (a(aa^2))a - (aa^2)a^2 + a((aa^2)a) = M_2 - M_5 - M_7 + M_{11}, \\
R_4 &= (a^2a)a^2 - (aa^2)a^2 - a^2(aa^2) + a(a^2a^2) \qquad = M_6 - M_7 - M_9 + M_{12}, \\
R_5 &= a((a^2a)a) - a((aa^2)a) - a(a^2a^2) + a(a(a^2a)) = M_{10} - M_{11} - M_{12} + M_{13}.
\end{aligned}$$

From this we obtain the relation matrix in which the ij-entry is the coefficient of M_j in the expansion of R_i:

$$\begin{pmatrix}
1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0
\end{pmatrix}$$

We now apply 9 elementary row operations (expressed in Maple):

```
addrow(",1,2,-1);   mulrow(",2,-1);      addrow(",2,3,-1);
mulrow(",3,-1);     addrow(",3,2,-1);    addrow(",4,1);
addrow(",4,3);      addrow(",5,1,-1);    addrow(",5,3,-1);
```

The meaning of these commands is:

addrow(",i,j,m);	add m times row i to row j (omit m if $m = 1$)
mulrow(",i,m);	multiply row i by m
swaprow(",i,j);	interchange row i and row j

The result is the row-canonical form over $\mathbb{Z}$ of the relation matrix:

$$\begin{pmatrix}
1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 2 & -1 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0
\end{pmatrix}$$

From the leading entries of the rows we see that the quotient F_5/I_5 is free of rank $14-5=9$ □

Remark. It is possible to do the next case, degree 6, more or less "by hand" (one row operation at a time with the help of a computer) as in degree 5; however, since the relation matrix in degree 6 has size 23×42 and requires 144 operations to get the row-canonical form, at this stage Maple procedures were written to do the computations in the general case. These procedures are available from the author.

Theorem (Degree 6). *We have the isomorphism of $\mathbb{Z}$-modules*

$$L_6 \approx \mathbb{Z}^{20}.$$

There is no torsion in degree 6.

Proof. The monomial basis of F_6 has 42 elements:

$$
\begin{array}{lll}
M_1 = (((a^2a)a)a)a, & M_2 = (((aa^2)a)a)a, & M_3 = ((a^2a^2)a)a, \\
M_4 = ((a(a^2a))a)a, & M_5 = ((a(aa^2))a)a, & M_6 = ((a^2a)a^2)a, \\
M_7 = ((aa^2)a^2)a, & M_8 = (a^2(a^2a))a, & M_9 = (a^2(aa^2))a, \\
M_{10} = (a((a^2a)a)a, & M_{11} = (a((aa^2)a)a, & M_{12} = (a(a^2a^2))a, \\
M_{13} = (a(a(a^2a))a, & M_{14} = (a(a(aa^2))a, & M_{15} = ((a^2a)a)a^2, \\
M_{16} = ((aa^2)a)a^2, & M_{17} = (a^2a^2)a^2, & M_{18} = (a(a^2a))a^2, \\
M_{19} = (a(aa^2))a^2, & M_{20} = (a^2a)(a^2a), & M_{21} = (a^2a)(aa^2), \\
M_{22} = (aa^2)(a^2a), & M_{23} = (aa^2)(aa^2), & M_{24} = a^2((a^2a)a), \\
M_{25} = a^2((aa^2)a), & M_{26} = a^2(a^2a^2), & M_{27} = a^2(a(a^2a)), \\
M_{28} = a^2(a(aa^2)), & M_{29} = a(((a^2a)a)a), & M_{30} = a(((aa^2)a)a), \\
M_{31} = a((a^2a^2)a), & M_{32} = a((a(a^2a))a), & M_{33} = a((a(aa^2))a), \\
M_{34} = a((a^2a)a^2), & M_{35} = a((aa^2)a^2), & M_{36} = a(a^2(a^2a)), \\
M_{37} = a(a^2(aa^2)), & M_{38} = a(a((a^2a)a), & M_{39} = a(a((aa^2)a), \\
M_{40} = a(a(a^2a^2)), & M_{41} = a(a(a(a^2a)), & M_{42} = a(a(a(aa^2)).
\end{array}
$$

Each of these monomials gives rise to 4 spanning relations in I_6, but only 23 of these are non-redundant:

$$
\begin{array}{lll}
R_1 = \ell((a^2a)a, a, a), & R_2 = \ell(a^2a, a, a)a, & R_3 = (\ell(a^2, a, a)a)a, \\
R_4 = \ell((aa^2)a, a, a), & R_5 = \ell(aa^2, a, a)a, & R_6 = \ell(a^2a^2, a, a), \\
R_7 = \ell(a(a^2a), a, a), & R_8 = \ell(a(aa^2), a, a), & R_9 = \ell(a^2a, a^2, a), \\
R_{10} = \ell(a^2, a, a^2)a, & R_{11} = \ell(aa^2, a^2, a), & R_{12} = (a\ell(a^2, a, a))a, \\
R_{13} = \ell(a^2a, a, a^2), & R_{14} = \ell(a^2, a, a)a^2, & R_{15} = \ell(aa^2, a, a^2),
\end{array}
$$

$R_{16} = \ell(a^2, a, a^2a)$, $R_{17} = \ell(a^2, a, aa^2)$, $R_{18} = a^2\ell(a^2, a, a)$,
$R_{19} = a\ell(a^2a, a, a)$, $R_{20} = a(\ell(a^2, a, a)a)$, $R_{21} = a\ell(aa^2, a, a)$,
$R_{22} = a\ell(a^2, a, a^2)$, $R_{23} = a(a\ell(a^2, a, a))$.

Expanding these relations in terms of the basis monomials for F_6 we obtain:

$R_1 = M_1 - M_{10} - M_{15} + M_{29}$, $R_2 = M_1 - M_4 - M_6 + M_{10}$,
$R_3 = M_1 - M_2 - M_3 + M_4$, $R_4 = M_2 - M_{11} - M_{16} + M_{30}$,
$R_5 = M_2 - M_5 - M_7 + M_{11}$, $R_6 = M_3 - M_{12} - M_{17} + M_{31}$,
$R_7 = M_4 - M_{13} - M_{18} + M_{32}$, $R_8 = M_5 - M_{14} - M_{19} + M_{33}$,
$R_9 = M_6 - M_8 - M_{20} + M_{24}$, $R_{10} = M_6 - M_7 - M_9 + M_{12}$,
$R_{11} = M_7 - M_9 - M_{22} + M_{25}$, $R_{12} = M_{10} - M_{11} - M_{12} + M_{13}$,
$R_{13} = M_{15} - M_{18} - M_{21} + M_{34}$, $R_{14} = M_{15} - M_{16} - M_{17} + M_{18}$,
$R_{15} = M_{16} - M_{19} - M_{23} + M_{35}$, $R_{16} = M_{20} - M_{22} - M_{27} + M_{36}$,
$R_{17} = M_{21} - M_{23} - M_{28} + M_{37}$, $R_{18} = M_{24} - M_{25} - M_{26} + M_{27}$,
$R_{19} = M_{29} - M_{32} - M_{34} + M_{38}$, $R_{20} = M_{29} - M_{30} - M_{31} + M_{32}$,
$R_{21} = M_{30} - M_{33} - M_{35} + M_{39}$, $R_{22} = M_{34} - M_{35} - M_{37} + M_{40}$,
$R_{23} = M_{38} - M_{39} - M_{40} + M_{41}$.

From this we construct the 23×42 relation matrix in which the ij-entry is the coefficient of M_j in the expansion of R_i. To obtain the row-canonical form of the relation matrix we must perform 144 elementary row operations. The nonzero entries in the row-canonical form are listed below, where $[i, j] = k$ means that the ij-entry is k:

$[1,1] = 1$, $[1,11] = 1$, $[1,12] = -4$, $[1,13] = 4$, $[1,14] = -2$,
$[1,18] = -1$, $[1,22] = -1$, $[1,25] = 1$, $[1,28] = -2$, $[1,32] = 1$,
$[1,37] = 6$, $[1,40] = -6$, $[1,41] = 2$,
$[2,2] = 1$, $[2,11] = 1$, $[2,12] = -3$, $[2,13] = 3$, $[2,14] = -2$,
$[2,19] = -1$, $[2,22] = -1$, $[2,25] = 1$, $[2,28] = -1$, $[2,33] = 1$,
$[2,37] = 3$, $[2,40] = -3$, $[2,41] = 1$,
$[3,3] = 1$, $[3,12] = -1$, $[3,18] = -2$, $[3,19] = 1$, $[3,28] = -1$,
$[3,32] = 2$, $[3,33] = -1$, $[3,37] = 3$, $[3,40] = -3$, $[3,41] = 1$,
$[4,4] = 1$, $[4,13] = -1$, $[4,18] = -1$, $[4,32] = 1$,
$[5,5] = 1$, $[5,14] = -1$, $[5,19] = -1$, $[5,33] = 1$,
$[6,6] = 1$, $[6,12] = -5$, $[6,13] = 6$, $[6,14] = -2$, $[6,22] = -1$,
$[6,25] = 1$, $[6,28] = -2$, $[6,37] = 6$, $[6,40] = -6$, $[6,41] = 2$,
$[7,7] = 1$, $[7,12] = -3$, $[7,13] = 3$, $[7,14] = -1$, $[7,22] = -1$,
$[7,25] = 1$, $[7,28] = -1$, $[7,37] = 3$, $[7,40] = -3$, $[7,41] = 1$,

$[8,8]=1, \quad [8,12]=-5, \quad [8,13]=6, \quad [8,14]=-2, \quad [8,26]=-1,$
$[8,27]=2, \quad [8,28]=-2, \quad [8,36]=-1, \quad [8,37]=6, \quad [8,40]=-6,$
$[8,41]=2,$
$[9,9]=1, \quad [9,12]=-3, \quad [9,13]=3, \quad [9,14]=-1, \quad [9,28]=-1,$
$[9,37]=3, \quad [9,40]=-3, \quad [9,41]=1,$
$[10,10]=1, \quad [10,11]=1, \quad [10,12]=-4, \quad [10,13]=4, \quad [10,14]=-2,$
$[10,22]=-1, \quad [10,23]=1, \quad [10,25]=1, \quad [10,28]=-1, \quad [10,35]=-2,$
$[10,37]=3, \quad [10,39]=1, \quad [10,40]=-3, \quad [10,41]=1,$
$[11,11]=2, \quad [11,12]=-3, \quad [11,13]=3, \quad [11,14]=-2, \quad [11,22]=-1,$
$[11,23]=1, \quad [11,25]=1, \quad [11,28]=-1, \quad [11,35]=-2, \quad [11,37]=3,$
$[11,39]=1, \quad [11,40]=-3, \quad [11,41]=1,$
$[12,15]=1, \quad [12,18]=-1, \quad [12,23]=-1, \quad [12,28]=-1, \quad [12,35]=1,$
$[12,37]=2, \quad [12,40]=-1,$
$[13,16]=1, \quad [13,19]=-1, \quad [13,23]=-1, \quad [13,35]=1,$
$[14,17]=1, \quad [14,18]=-2, \quad [14,19]=1, \quad [14,28]=-1, \quad [14,37]=2,$
$[14,40]=-1,$
$[15,20]=1, \quad [15,22]=-1, \quad [15,27]=-1, \quad [15,36]=1,$
$[16,21]=1, \quad [16,23]=-1, \quad [16,28]=-1, \quad [16,37]=1,$
$[17,24]=1, \quad [17,25]=-1, \quad [17,26]=-1, \quad [17,27]=1,$
$[18,29]=1, \quad [18,32]=-1, \quad [18,35]=-1, \quad [18,37]=-1, \quad [18,39]=1,$
$[18,40]=2, \quad [18,41]=-1,$
$[19,30]=1, \quad [19,33]=-1, \quad [19,35]=-1, \quad [19,39]=1,$
$[20,31]=1, \quad [20,32]=-2, \quad [20,33]=1, \quad [20,37]=-1, \quad [20,40]=2,$
$[20,41]=-1,$
$[21,34]=1, \quad [21,35]=-1, \quad [21,37]=-1, \quad [21,40]=1,$
$[22,38]=1, \quad [22,39]=-1, \quad [22,40]=-1, \quad [22,41]=1$

The row-canonical form of the relation matrix has 22 nonzero rows, 21 of which have a leading 1. The exception (row 11) has a leading 2, but also contains odd entries, so we can apply elementary column operations to change this leading 2 into a leading 1. Therefore the quotient module is free of rank $42-22=20$. □

Remark. If all the entries of row 11 were even, it would not be possible to change the leading 2 into a leading 1 by column operations; this would indicate the existence of 2-torsion in the quotient module. We will see an example of this in the next section when we study the free assosymmetric ring.

Theorem (Degree 7). *We have the isomorphism of $\mathbb{Z}$-modules*

$$L_7 \approx \mathbb{Z}^{48}.$$

There is no torsion in degree 7.

Proof. The monomial basis of F_7 contains 132 monomials, and each monomial gives rise to 5 left spanning relations, for total of 660; only 96 of these are non-redundant. Computing the row-canonical form of the 96×132 relation matrix requires 1421 elementary row operations. The row-canonical form of the relation matrix has 84 nonzero rows, 81 of which have a leading 1. The remaining 3 (rows 32, 45, 73) have a leading 2 but also contain odd entries, so we can apply elementary column operations to change each leading 2 into a leading 1. Therefore the quotient module is free of rank $132 - 84 = 48$. □

Theorem (Degree 8). *We have the isomorphism of $\mathbb{Z}$-modules*

$$L_8 \approx \mathbb{Z}^{115}.$$

There is no torsion in degree 8.

Proof. The monomial basis of F_8 contains 429 monomials, and each monomial gives rise to 6 left spanning relations, for total of 2574; only 392 of these are non-redundant. Computing the row-canonical form of the 392×429 relation matrix requires 19128 elementary row operations. The row-canonical form of the relation matrix has 314 nonzero rows, 303 of which have a leading 1. The remaining 11 (rows 101, 103, 104, 108, 148, 171, 172, 219, 262, 275, 303) have leading entries 2, 12, 2, 3, 2, 2, 2, 2, 2, 2, 2 but also contain entries relatively prime to the leading entry, so we can apply elementary column operations to change each leading entry into a leading 1. Therefore the quotient module is free of rank $429 - 314 = 115$. □

Additive Structure of the Free Assosymmetric Ring

In this section

- F denotes the free nonassociative ring over $\mathbb{Z}$ on one generator a;
- J denotes the ideal in F generated by all $\ell(x,y,z)$ and $r(x,y,z)$ as x,y,z range over all elements of F (since ℓ and r are multilinear we may assume that x,y,z are monomials);
- A denotes the quotient F/J, the free assosymmetric ring over $\mathbb{Z}$ on one generator (also denoted a since this should not cause any confusion).

All these objects are graded by the degree $n \geq 0$ as in the previous section. The purpose of this section is to determine the $\mathbb{Z}$-module structure of the quotient module $A_n = F_n/J_n$ for $n \leq 8$.

Theorem (Degrees ≤ 4). *We have the following isomorphisms of $\mathbb{Z}$-modules:*

$$A_0 \approx \mathbb{Z}, \qquad A_1 \approx \mathbb{Z}, \qquad A_2 \approx \mathbb{Z}, \qquad A_3 \approx \mathbb{Z}^2, \qquad A_4 \approx \mathbb{Z}^3.$$

There is no torsion in degrees ≤ 4.

Proof. We have $A_n \approx F_n$ for $0 \leq n \leq 2$ as in the previous section. In degree 3 we see that J_3 is spanned as a $\mathbb{Z}$-module by the relations $\ell(a,a,a)$ and $r(a,a,a)$ which are both

0. Hence $J_3 = \{0\}$ and $A_3 \approx F_3$. In degree 4, we have two non-zero spanning relations for J_4:

$$\ell(a^2, a, a) = (a^2a)a - (aa^2)a - a^2a^2 + a(a^2a),$$
$$r(a, a^2, a) = (aa^2)a - a^2a^2 - a(a^2a) + a(aa^2).$$

This gives the relation matrix

	$(a^2a)a$	$(aa^2)a$	a^2a^2	$a(a^2a)$	$a(aa^2)$
$\ell(a^2, a, a)$	1	−1	−1	1	0
$r(a, a^2, a)$	0	1	−1	−1	1

which has the row-canonical form

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

From this we see that the quotient module $A_4 \approx F_4/J_4$ is free of rank 3. □

Remark. It follows from Theorem 1 on page 308 of [HJP] that the dimension of the homogeneous subspace of degree n in the free assosymmetric algebra over $\mathbb{Q}$ on one free generator is $n - 1$ for all $n \geq 2$. Since that algebra is the tensor product (over $\mathbb{Z}$) of A with $\mathbb{Q}$, it follows that the free submodule of A_n is isomorphic to $\mathbb{Z}^{n-1}$ for all $n \geq 2$.

In the next theorem we see our first example of torsion in a quotient of $\mathbb{Z}$-modules. We write $\mathbb{Z}_k$ for the quotient module $\mathbb{Z}/k\mathbb{Z}$.

Theorem (Degree 5). *We have the isomorphism of* $\mathbb{Z}$*-modules*

$$A_5 \approx \mathbb{Z}^4 \oplus \mathbb{Z}_2.$$

A generator for the torsion submodule is

$$(a, a, (a, a, a)).$$

Proof. In the previous section we saw how to obtain the 5 spanning relations for I_5 involving $\ell(x, y, z)$ from the basis monomials for F_5. From these relations we can obtain the spanning relations for J_5 involving $r(x, y, z)$ by replacing each occurrence of $\ell(x, y, z)$ by $r(z, x, y)$; note the cyclic permutation of the arguments. This gives an additional 5 spanning relations for J_5:

$$R_6 = r(a, a^2a, a), \quad R_7 = r(a, a^2, a)a, \quad R_8 = r(a, aa^2, a), \quad R_9 = r(a^2, a^2, a),$$
$$R_{10} = ar(a, a^2, a).$$

Expanding these additional relations in terms of the basis monomials for F_5 we obtain

$$R_6 = (a(a^2a))a - a^2(a^2a) - a((a^2a)a) + a(a(a^2a)) = M_4 - M_8 - M_{10} + M_{13},$$

$$\begin{aligned}
R_7 &= ((aa^2)a)a - (a^2a^2)a - (a(a^2a))a + (a(aa^2))a = M_2 - M_3 - M_4 + M_5,\\
R_8 &= (a(aa^2))a - a^2(aa^2) - a((aa^2)a) + a(a(aa^2)) = M_5 - M_9 - M_{11} + M_{14},\\
R_9 &= (a^2a^2)a - (a^2a)a^2 - a^2(a^2a) + a^2(aa^2) \qquad = M_3 - M_6 - M_8 + M_9,\\
R_{10} &= a((aa^2)a) - a(a^2a^2) - a(a(a^2a)) + a(a(aa^2)) = M_{11} - M_{12} - M_{13} + M_{14},
\end{aligned}$$

Combining the 5 relations involving $\ell(x,y,z)$ from the previous section with these 5 additional relations we obtain the relation matrix

$$\begin{pmatrix}
1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0\\
0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1\\
0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1
\end{pmatrix}$$

We now apply 42 elementary row operations:

```
addrow(",1,2,-1);  mulrow(",2,-1);     addrow(",2,3,-1);  addrow(",2,7,-1);
mulrow(",3,-1);    addrow(",3,2,-1);   addrow(",3,7,2);   addrow(",3,9,-1);
swaprow(",4,6);    addrow(",4,1);      addrow(",4,3,2);   addrow(",4,7,3);
addrow(",4,9,-2);  swaprow(",5,8);     addrow(",5,2);     addrow(",5,3,-1);
addrow(",5,7,-3);  addrow(",5,9);      addrow(",6,1);     addrow(",6,3);
addrow(",6,7);     addrow(",7,1);      addrow(",7,2);     addrow(",7,6);
addrow(",7,9);     swaprow(",8,9);     mulrow(",8,-1);    addrow(",8,1,2);
addrow(",8,2,2);   addrow(",8,3);      addrow(",8,4);     addrow(",8,6,2);
addrow(",8,7,2);   addrow(",9,8,-1);   addrow(",10,1);    addrow(",10,2);
addrow(",10,3);    addrow(",10,4);     addrow(",10,5);    addrow(",10,6);
addrow(",10,7);    addrow(",10,9);
```

The result is the row-canonical form over $\mathbb{Z}$ of the relation matrix:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & -1 & 1 & 2\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 0 & 0 & -2 & 0 & 3\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -1 & 0 & 2\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & -2 & -1 & 3\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 2\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -3 & 0 & 0 & -1 & 0 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 & -2 & 0 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 & -2 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1
\end{pmatrix}$$

Since there are 10 nonzero rows the free rank of the quotient module is $14 - 10 = 4$. Since every entry in row 8 is a multiple of 2 there is 2-torsion in the quotient module. From row

8 we see that the coset in A_5 of the element

$$T_5 = M_8 - M_9 - M_{13} + M_{14} = a^2(a^2a) - a^2(aa^2) - a(a(a^2a)) + a(a(aa^2))$$
$$= (a, a, (a, a, a)),$$

is a torsion element of order 2, since $2T_5 \in J_5$ but $T_5 \notin J_5$. □

Corollary. *The free assosymmetric ring over $\mathbb{Z}$ on any number of generators is not a free $\mathbb{Z}$-module. The free assosymmetric ring over any coefficient ring containing $\frac{1}{2}$ on any number of generators satisfies the identity $(a, a, (a, a, a)) \equiv 0$.*

Proof. The first assertion follows from the fact that A is a subring of every free assosymmetric ring over $\mathbb{Z}$. The second assertion follows from the fact that the identity $2(a, a, (a, a, a)) \equiv 0$ holds in the free assosymmetric ring over $\mathbb{Z}$, and so the identity $(a, a, (a, a, a)) \equiv 0$ holds over any coefficient ring containing $\frac{1}{2}$. □

Theorem (Degree 6). *We have the isomorphism of $\mathbb{Z}$-modules*

$$A_6 \approx \mathbb{Z}^5 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_6.$$

Generators for the torsion submodules are

order 2: $a(a, a, (a, a, a))$ *and* $(a, a, (a, a, a^2)) + (a, a, a(a, a, a))$,
order 6: $(a, a, a(a, a, a))$.

Proof. From the 23 spanning relations for I_6 involving $\ell(x, y, z)$ from the previous section we obtain an additional 23 spanning relations for J_6 involving $r(x, y, z)$:

$$R_{24} = r(a, (a^2a)a, a), \quad R_{25} = r(a, a^2a, a)a, \quad R_{26} = (r(a, a^2, a)a)a,$$
$$R_{27} = r(a, (aa^2)a, a), \quad R_{28} = r(a, aa^2, a)a, \quad R_{29} = r(a, a^2a^2, a),$$
$$R_{30} = r(a, a(a^2a), a), \quad R_{31} = r(a, a(aa^2), a), \quad R_{32} = r(a, a^2a, a^2),$$
$$R_{33} = r(a^2, a^2, a)a, \quad R_{34} = r(a, aa^2, a^2), \quad R_{35} = (ar(a, a^2, a))a,$$
$$R_{36} = r(a^2, a^2a, a), \quad R_{37} = r(a, a^2, a)a^2, \quad R_{38} = r(a^2, aa^2, a),$$
$$R_{39} = r(a^2a, a^2, a), \quad R_{40} = r(a^2a, a^2, a2), \quad R_{41} = a^2r(a, a^2, a),$$
$$R_{42} = ar(a, a^2a, a), \quad R_{43} = a(r(a, a^2, a)a), \quad R_{44} = ar(a, aa^2, a),$$
$$R_{45} = ar(a^2, a^2, a), \quad R_{46} = a(ar(a, a^2, a)).$$

Expanding these additional spanning relations in terms of the basis monomials we obtain

$$R_{24} = M_{10} - M_{24} - M_{29} + M_{38}, \quad R_{25} = M_4 - M_8 - M_{10} + M_{13},$$
$$R_{26} = M_2 - M_3 - M_4 + M_5, \quad R_{27} = M_{11} - M_{25} - M_{30} + M_{39},$$
$$R_{28} = M_5 - M_9 - M_{11} + M_{14}, \quad R_{29} = M_{12} - M_{26} - M_{31} + M_{40},$$

$R_{30} = M_{13} - M_{27} - M_{32} + M_{41}$, $R_{31} = M_{14} - M_{28} - M_{33} + M_{42}$,
$R_{32} = M_{18} - M_{22} - M_{34} + M_{36}$, $R_{33} = M_3 - M_6 - M_8 + M_9$,
$R_{34} = M_{19} - M_{23} - M_{35} + M_{37}$, $R_{35} = M_{11} - M_{12} - M_{13} + M_{14}$,
$R_{36} = M_8 - M_{20} - M_{24} + M_{27}$, $R_{37} = M_{16} - M_{17} - M_{18} + M_{19}$,
$R_{38} = M_9 - M_{21} - M_{25} + M_{28}$, $R_{39} = M_6 - M_{15} - M_{20} + M_{21}$,
$R_{40} = M_7 - M_{16} - M_{22} + M_{23}$, $R_{41} = M_{25} - M_{26} - M_{27} + M_{28}$,
$R_{42} = M_{32} - M_{36} - M_{38} + M_{41}$, $R_{43} = M_{30} - M_{31} - M_{32} + M_{33}$,
$R_{44} = M_{33} - M_{37} - M_{39} + M_{42}$, $R_{45} = M_{31} - M_{34} - M_{36} + M_{37}$,
$R_{46} = M_{39} - M_{40} - M_{41} + M_{42}$.

From this we construct the relation matrix: the 46×42 matrix in which the i, j-entry is the coefficient of M_j in the expansion of R_i. To obtain the row-canonical form of the relation matrix we must perform 762 elementary row operations. The nonzero entries in the row-canonical form are listed below, where $[i, j] = k$ means that the ij-entry is k:

$[1,1]=1$, $[1,27]=2$, $[1,28]=-6$, $[1,37]=-2$, $[1,42]=5$,
$[2,2]=1$, $[2,26]=1$, $[2,27]=1$, $[2,28]=-5$, $[2,37]=-3$,
$[2,40]=-1$, $[2,41]=1$, $[2,42]=5$,
$[3,3]=1$, $[3,26]=1$, $[3,27]=2$, $[3,28]=-6$, $[3,37]=-2$,
$[3,40]=-2$, $[3,42]=6$,
$[4,4]=1$, $[4,27]=5$, $[4,28]=-7$, $[4,37]=-3$, $[4,40]=-1$,
$[4,41]=-3$, $[4,42]=8$,
$[5,5]=1$, $[5,28]=-2$, $[5,37]=-2$, $[5,40]=-2$, $[5,41]=2$,
$[5,42]=3$,
$[6,6]=1$, $[6,27]=5$, $[6,28]=-8$, $[6,37]=-2$, $[6,41]=-4$,
$[6,42]=8$,
$[7,7]=1$, $[7,28]=-2$, $[7,36]=1$, $[7,37]=-4$, $[7,42]=4$,
$[8,8]=1$, $[8,27]=2$, $[8,28]=-4$, $[8,36]=1$, $[8,37]=-2$,
$[8,40]=-2$, $[8,41]=-2$, $[8,42]=6$,
$[9,9]=1$, $[9,26]=1$, $[9,27]=1$, $[9,28]=-4$, $[9,36]=1$,
$[9,37]=-2$, $[9,40]=-2$, $[9,41]=-2$, $[9,42]=6$,
$[10,10]=1$, $[10,27]=2$, $[10,28]=-3$, $[10,37]=-3$, $[10,40]=-1$,
$[10,41]=-1$, $[10,42]=5$,
$[11,11]=1$, $[11,26]=1$, $[11,27]=1$, $[11,28]=-3$, $[11,36]=1$,
$[11,37]=-3$, $[11,40]=-3$, $[11,41]=-1$, $[11,42]=6$,
$[12,12]=1$, $[12,26]=1$, $[12,27]=2$, $[12,28]=-4$, $[12,37]=-2$,
$[12,40]=-2$, $[12,41]=-2$, $[12,42]=6$,

$[13,13]=1,\quad [13,27]=5,\quad [13,28]=-6,\quad [13,36]=1,\quad [13,37]=-2,$
$[13,40]=-2,\quad [13,41]=-6,\quad [13,42]=9,$
$[14,14]=1,\quad [14,28]=-1,\quad [14,37]=-1,\quad [14,40]=-1,\quad [14,41]=-1,$
$[14,42]=3,$
$[15,15]=1,\quad [15,28]=-3,\quad [15,37]=-2,\quad [15,41]=2,\quad [15,42]=2,$
$[16,16]=1,\quad [16,28]=-2,\quad [16,37]=-3,\quad [16,41]=2,\quad [16,42]=2,$
$[17,17]=1,\quad [17,28]=-2,\quad [17,37]=-2,\quad [17,40]=-1,\quad [17,41]=2,$
$[17,42]=2,$
$[18,18]=1,\quad [18,28]=-1,\quad [18,37]=-3,\quad [18,40]=-1,\quad [18,41]=2,$
$[18,42]=2,$
$[19,19]=1,\quad [19,28]=-1,\quad [19,37]=-2,\quad [19,40]=-2,\quad [19,41]=2,$
$[19,42]=2,$
$[20,20]=1,\quad [20,27]=5,\quad [20,28]=-7,\quad [20,36]=1,\quad [20,37]=-2,$
$[20,41]=-6,\quad [20,42]=8,$
$[21,21]=1,\quad [21,28]=-2,\quad [21,36]=1,\quad [21,37]=-2,\quad [21,42]=2,$
$[22,22]=1,\quad [22,28]=-1,\quad [22,37]=-2,\quad [22,42]=2,$
$[23,23]=1,\quad [23,28]=-1,\quad [23,36]=1,\quad [23,37]=-3,\quad [23,42]=2,$
$[24,24]=1,\quad [24,27]=2,\quad [24,28]=-3,\quad [24,40]=-2,\quad [24,41]=-2,$
$[24,42]=4,$
$[25,25]=1,\quad [25,26]=1,\quad [25,27]=1,\quad [25,28]=-3,\quad [25,40]=-2,$
$[25,41]=-2,\quad [25,42]=4,$
$[26,26]=2,\quad [26,27]=2,\quad [26,28]=-4,\quad [26,40]=-2,\quad [26,41]=-2,$
$[26,42]=4,$
$[27,27]=6,\quad [27,28]=-6,\quad [27,41]=-6,\quad [27,42]=6,$
$[28,29]=1,\quad [28,37]=-3,\quad [28,40]=-1,\quad [28,41]=1,\quad [28,42]=2,$
$[29,30]=1,\quad [29,36]=1,\quad [29,37]=-3,\quad [29,40]=-2,\quad [29,42]=3,$
$[30,31]=1,\quad [30,37]=-2,\quad [30,40]=-1,\quad [30,42]=2,$
$[31,32]=1,\quad [31,36]=1,\quad [31,37]=-2,\quad [31,40]=-2,\quad [31,41]=-1,$
$[31,42]=3,$
$[32,33]=1,\quad [32,37]=-1,\quad [32,40]=-1,\quad [32,41]=-1,\quad [32,42]=2,$
$[33,34]=1,\quad [33,36]=1,\quad [33,37]=-3,\quad [33,40]=-1,\quad [33,42]=2,$
$[34,35]=1,\quad [34,36]=1,\quad [34,37]=-2,\quad [34,40]=-2,\quad [34,42]=2,$
$[35,36]=2,\quad [35,37]=-2,\quad [35,41]=-2,\quad [35,42]=2,$
$[36,38]=1,\quad [36,40]=-2,\quad [36,42]=1,$
$[37,39]=1,\quad [37,40]=-1,\quad [37,41]=-1,\quad [37,42]=1.$

Since there are 37 nonzero rows the free rank of the quotient module is $42 - 37 = 5$. Since every entry in rows 26 and 35 is a multiple of 2 there is 2-torsion in the quotient module, and since every entry in row 27 is a multiple of 6 there is 6-torsion in the quotient module. From rows 26 and 35 we see that the cosets in A_6 of the elements

$$M_{26} + M_{27} - 2M_{28} - M_{40} - M_{41} + 2M_{42} = (a, a, (a, a, a^2)) + (a, a, a(a, a, a)),$$
$$M_{36} - M_{37} - M_{41} + M_{42} = a(a, a, (a, a, a)),$$

are torsion elements of order 2. From row 27 we see that the coset in A_6 of the element

$$M_{27} - M_{28} - M_{41} + M_{42} = (a, a, a(a, a, a))$$

is a torsion element of order 6. □

Remark. Since the last theorem explicitly constructs torsion elements with orders involving the primes 2 and 3, it follows that the restrictions on the characteristic in Theorem 1 of [HJP] are necessary. That is, there can be no uniform construction in all characteristics of a basis for free assosymmetric algebras over a field; the best we can expect is a construction which works for all characteristics $\neq 2, 3$.

Theorem (Degree 7). *We have the isomorphism of $\mathbb{Z}$-modules*

$$A_7 \approx \mathbb{Z}^6 \oplus \mathbb{Z}_2^6 \oplus \mathbb{Z}_6.$$

Proof. We have a total of 192 spanning relations for J_7. Computing the row-reduced form of the 192×132 relation matrix requires 10484 elementary row operations. The row-canonical form of the relation matrix has 126 nonzero rows, 119 of which have a leading 1. The remaining 7 (rows 84, 85, 88, 89, 115, 116, 124) have leading entries 2, 2, 2, 2, 2, 6, 2, and every entry in each of these rows is a multiple of the leading entry. Therefore the free rank of the quotient is $132 - 126 = 6$ and the torsion submodule has the given structure. □

Theorem (Degree 8). *We have the isomorphism of $\mathbb{Z}$-modules*

$$A_8 \approx \mathbb{Z}^7 \oplus \mathbb{Z}_2^{11} \oplus \mathbb{Z}_6.$$

Proof. We have a total of 784 spanning relations for J_8. Computing the row-reduced form of the 784×429 relation matrix requires 144828 elementary row operations. The row-canonical form of the relation matrix has 422 nonzero rows, 410 of which have a leading 1. The remaining 12 (rows 281, 283, 291, 295, 296, 380, 381, 384, 385, 411, 412, 420) have leading entries 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 6, 2, and every entry in each of these rows is a multiple of the leading entry. Therefore the free rank of the quotient is $429 - 422 = 7$ and the torsion submodule has the given structure. □

ACKNOWLEDGEMENT

This work was supported by a research grant from the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

[AW] W. A. Adkins, S. H. Weintraub, *Algebra: an approach via module theory*, Springer-Verlag, New York, 1992.

[Br1] M. Bremner, *On the $\mathbb{Z}$-module structure of a free semialternative ring*, Comm. Algebra **27** (1999), 1951-1965.

[Br2] ———, *On free partially associative triple systems*, Comm. Algebra **28** (2000), 2131-2145.

[HJP] I.R. Hentzel, D.P. Jacobs, L.A. Peresi, *A basis for free assosymmetric algebras*, J. Algebra **183** (1996), 306-318.