ENVELOPING ALGEBRAS OF SOLVABLE MALCEV ALGEBRAS OF DIMENSION FIVE

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Abstract. We study the universal enveloping algebras of the one-parameter family of solvable 5-dimensional non-Lie Malcev algebras. We explicitly determine the universal nonassociative enveloping algebras (in the sense of Pérez-Izquierdo and Shestakov) and the centers of the universal enveloping algebras. We also determine the universal alternative enveloping algebras.

1. Introduction

In this paper we study the 5-dimensional solvable (non-nilpotent non-Lie) Malcev algebras and their universal enveloping algebras. Over a field of characteristic 0, the 5-dimensional Malcev algebras were classified by Kuzmin [6]: there is one nilpotent algebra, solvable algebras of five different types, and one non-solvable algebra. Except for a finite number of special cases, the solvable algebras belong to a family whose structure constants involve a non-zero parameter γ. Kuzmin omitted the details of the classification in the solvable case, but Gavrilov [5] has recently recovered these results using Malcev cocycles.

In 2004, Pérez-Izquierdo and Shestakov [8] extended the Poincaré-Birkhoff-Witt (PBW) theorem from Lie algebras to Malcev algebras. For any Malcev algebra M over a field of characteristic 0 or $p > 3$, they constructed a universal nonassociative enveloping algebra $U(M)$ which shares many properties of the universal associative enveloping algebras of Lie algebras. In general $U(M)$ is not alternative, and so it is interesting to determine its alternator ideal $I(M)$ and its maximal alternative quotient $A(M) = U(M)/I(M)$, which is the universal alternative enveloping algebra of M. This produces new examples of infinite dimensional alternative algebras. The details have been worked out by Bremner, Hentzel, Peresi and Usefi [1, 3] for the 4-dimensional solvable algebra and the 5-dimensional nilpotent algebra. See also the survey article [2].

The goal of this paper is to compute explicit structure constants for $U(\mathbb{M})$ and $A(\mathbb{M})$ where $\mathbb{M} = \mathbb{M}_\gamma$ belongs to the one-parameter family of 5-dimensional solvable Malcev algebras. We also determine the center of $U(\mathbb{M})$ which is non-trivial if and only if the parameter γ is rational.

We recall the structure constants of $\mathbb{M} = \operatorname{span}\{e_0, e_1, e_2, e_3, e_4\}$ from [6]:

$$e_1e_2 = e_3, \quad e_0e_1 = e_1, \quad e_0e_2 = e_2, \quad e_0e_3 = -e_3, \quad e_0e_4 = \gamma' e_4 \ (\gamma' \neq 0).$$

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Thus e_1, e_2, e_3 span a 3-dimensional nilpotent Lie algebra, and e_1, e_2, e_3, e_4 span the direct sum of this Lie algebra with a 1-dimensional abelian Lie algebra. The basis element e_0 acts diagonally on this 4-dimensional nilpotent Lie algebra, producing a 5-dimensional solvable Malcev algebra; the parameter enters only into the action of e_0 on e_4. We change notation, replacing e_0 by $-a$, e_1 by b, e_2 by c, e_3 by $2d$, e_4 by e (and γ' by $-\gamma$), and obtain the following structure constants for $\mathbb{M}$:

$$[b,c]=2d, \quad [a,b]=-b, \quad [a,c]=-c, \quad [a,d]=d, \quad [a,e]=\gamma e \ (\gamma\neq 0). \tag{1}$$

The span of a, b, c, d is the 4-dimensional solvable (non-Lie) Malcev algebra; see [1].

2. Preliminaries

Definition 2.1. The **generalized alternative nucleus** of a nonassociative algebra A over a field F is the subspace

$$N_{\text{alt}}(A) = \{\, s\in A \mid (s,x,y) = -(x,s,y) = (x,y,s),\ \forall\, x,y\in A \,\}.$$

In general $N_{\text{alt}}(A)$ is not a subalgebra of A, but it is a subalgebra of A^- (it is closed under the commutator) and is in fact a Malcev algebra.

Theorem 2.2. (Pérez-Izquierdo and Shestakov [8]) *For every Malcev algebra M over a field F of characteristic $\neq 2,3$ there exists a nonassociative algebra $U(M)$ and an injective algebra morphism $\iota\colon M\to U(M)^-$ such that $\iota(M)\subseteq N_{\text{alt}}(U(M))$; furthermore, $U(M)$ is a universal object with respect to such morphisms.*

The algebra $U(M)$ is constructed as follows. Let $F(M)$ be the unital free nonassociative algebra on a basis of M. Let $R(M)$ be the ideal of $F(M)$ generated by the following elements for all $s,t\in M$ and all $x,y\in F(M)$:

$$st-ts-[s,t], \qquad (s,x,y)+(x,s,y), \qquad (x,s,y)+(x,y,s).$$

Define $U(M)=F(M)/R(M)$ with the natural mapping

$$\iota\colon M\to N_{\text{alt}}(U(M))\subseteq U(M), \qquad s\mapsto \iota(s)=\overline{s}=s+R(M).$$

Since ι is injective, we may identify M with $\iota(M)\subseteq U(M)$. We fix a basis $B=\{a_i \mid i\in\mathcal{I}\}$ of M and a total order $<$ on $\mathcal{I}$, and define

$$\Omega = \{\,(i_1,\dots,i_n) \mid n\geq 0,\ i_1\leq\cdots\leq i_n\,\}.$$

For $n=0$ we have $\overline{a}_\emptyset = 1\in U(M)$, and for $n\geq 1$ the n-tuple $(i_1,\dots,i_n)\in\Omega$ defines a left-tapped monomial

$$\overline{a}_I = \overline{a}_{i_1}(\overline{a}_{i_2}(\cdots(\overline{a}_{i_{n-1}}\overline{a}_{i_n})\cdots)), \qquad |\overline{a}_I| = n.$$

In [8] it is shown that the set of all $\overline{a}_I$ for $I\in\Omega$ is a basis of $U(M)$. It follows that there is a linear isomorphism $\phi\colon U(M)\to P(M)$ which is the identity on M, where $P(M)$ is the polynomial algebra on M. Since $M\subseteq N_{\text{alt}}(U(M))$, for any $s,t\in M$ and $x\in U(M)$ we have

$$(s,t,x) = \tfrac{1}{6}[[x,s],t] - \tfrac{1}{6}[[x,t],s] - \tfrac{1}{6}[x,[s,t]].$$

This equation implies the following lemma, which is implicit in [8].

Lemma 2.3. *Let x be a basis monomial of $U(M)$ with $|x|\geq 2$ and write $x=ty$ with $t\in M$. Then for any $s\in M$ we have*

$$[x,s] = [t,s]y + t[y,s] + \tfrac{1}{2}[[y,s],t] - \tfrac{1}{2}[[y,t],s] - \tfrac{1}{2}[y,[s,t]], \tag{2}$$

$$sx = t(sy) + [s,t]y - \tfrac{1}{3}[[y,s],t] + \tfrac{1}{3}[[y,t],s] + \tfrac{1}{3}[y,[s,t]]. \tag{3}$$

	$D_{a,b}$	$D_{a,c}$	$D_{a,d}$	$D_{a,e}$	$D_{b,c}$	$D_{b,d}$	$D_{b,e}$	$D_{c,d}$	$D_{c,e}$	$D_{d,e}$
a	$-b$	$-c$	$-d$	$-\gamma^2 e$	d	.	.	.	.	.
b	.	$-d$	.	.	.	.	.	.	.	.
c	d	.	.	.	.	.	.	.	.	.
d	.	.	.	.	.	.	.	.	.	.
e	.	.	.	.	.	.	.	.	.	.

Table 1. Derivations of the one-parameter family of Malcev algebras

Let y be a basis monomial of $U(M)$ with $|y| \geq 2$ and write $y = sx$ with $s \in M$. Then for any basis monomial z of $U(M)$ we have

$$yz = (sx)z = 2s(xz) - x(sz) - x[z,s] + [xz,s]. \tag{4}$$

Definition 2.4. In the nonassociative algebra A, we write L_s and R_s for the operators of left and right multiplication by s, and set $\mathrm{ad}_s = R_s - L_s$. We define

$$D_{s,t} = [L_s, L_t] + [L_s, R_t] + [R_s, R_t].$$

We note that $D_{t,s} = -D_{s,t}$ and $D_{s,s} = 0$.

In Table 1 we record the values of the derivations $D_{s,t}$ on the one-parameter family $\mathbb{M}_\gamma$ of 5-dimensional solvable Malcev algebras (dot indicates zero).

The following lemma is proved by Morandi, Pérez-Izquierdo and Pumplün [7, Lemma 4.2]; note that we have changed the sign of the ad-operator.

Lemma 2.5. *If A is a nonassociative algebra with $s, t \in N_{\mathrm{alt}}(A)$ and $x \in A$ then*

$$L_{sx} = L_s L_x + [R_s, L_x], \quad L_{xs} = L_x L_s + [L_x, R_s], \quad [L_s, L_t] = L_{[s,t]} - 2[R_s, L_t],$$
$$R_{sx} = R_x R_s + [R_x, L_s], \quad R_{xs} = R_s R_x + [L_s, R_x], \quad [R_s, R_t] = -R_{[s,t]} - 2[L_s, R_t],$$
$$[L_s, R_t] = [R_s, L_t].$$

The operator $D_{s,t}$ is a derivation, and we have

$$D_{s,t} = -\mathrm{ad}_{[s,t]} - 3[L_s, R_t], \qquad 2D_{s,t} = -\mathrm{ad}_{[s,t]} + [\mathrm{ad}_s, \mathrm{ad}_t].$$

Using $D_{s,t}$ we can rewrite the equations (2) and (3) as follows:

$$[ty, s] = [t, s]y + t[y, s] - D_{s,t}(y) - [y, [s,t]], \tag{5}$$

$$s(ty) = t(sy) + [s,t]y + \tfrac{2}{3} D_{s,t}(y) + \tfrac{2}{3}[y, [s,t]]. \tag{6}$$

3. Left multiplications on solvable Malcev algebras

We recall the classification of 5-dimensional solvable Malcev algebras from [6]. We omit the first case (the direct sum of the 4-dimensional solvable algebra and a 1-dimensional abelian algebra) and consider only the remaining five cases:

(7) $[b,c] = 2d,\ [a,b] = b,\quad [a,c] = c,\quad [a,d] = 2d - \tfrac{1}{2}e,\quad [a,e] = -e;$

(8) $[b,c] = 2d,\ [a,b] = b,\quad [a,c] = c,\quad [a,d] = -d,\quad [a,e] = -b - 2e;$

(9) $[b,c] = 2d,\ [a,b] = b,\quad [a,c] = c,\quad [a,d] = -d,\quad [a,e] = 2d - e;$

(10) $[b,c] = 2d,\ [a,b] = b + e,\quad [a,c] = c,\quad [a,d] = -d,\quad [a,e] = e;$

(11) $[b,c] = 2d,\ [a,b] = -b,\quad [a,c] = -c,\quad [a,d] = d,\quad [a,e] = \gamma e\ \ (\gamma \neq 0).$

(We have made a slight change of basis from [6] so that $[b,c] = 2d$ in every case.) Equation (11) coincides with equation (1); this is the one-parameter family on which we focus in this paper. But for the rest of this section we work more generally and let $M = \operatorname{span}\{a,b,c,d,e\}$ be one of the algebras in equations (7), (9), (10), (11). Then $L = \operatorname{span}\{b,c,d,e\}$ is a 4-dimensional nilpotent Lie algebra with $[b,c] = 2d$ and other products zero; furthermore, $[a,L] = L$.

Lemma 3.1. *If M is one of the algebras* (7), (9), (10), (11) *then in $U(M)$ we have*
$$D_{b,d} = D_{b,e} = D_{c,d} = D_{c,e} = D_{d,e} = 0.$$

Proof. We have $D_{s,t} = -\frac{1}{2}\mathrm{ad}_{[s,t]} + \frac{1}{2}[\mathrm{ad}_s, \mathrm{ad}_t]$ where $s,t \in M$. If $D_{s,t}$ is one of the above derivations then $[s,t] = 0$ and so $\mathrm{ad}_{[s,t]} = 0$; hence $D_{s,t} = \frac{1}{2}[\mathrm{ad}_s, \mathrm{ad}_t]$. From the multiplication table of M we have $\mathrm{ad}_d(M) \subseteq \operatorname{span}\{d,e\}$ and $\mathrm{ad}_e(M) \subseteq \operatorname{span}\{d,e\}$. Hence $\mathrm{ad}_s(\mathrm{ad}_d(M)) = 0$ and $\mathrm{ad}_s(\mathrm{ad}_e(M)) = 0$ for $s \in \{b,c,d,e\}$ (in fact $s \in L$). On the other hand, $\mathrm{ad}_d|_L = 0$ and $\mathrm{ad}_e|_L = 0$. It follows that $D_{s,t}(M) = \frac{1}{2}\mathrm{ad}_s\mathrm{ad}_t(M) - \frac{1}{2}\mathrm{ad}_t\mathrm{ad}_s(M) = 0$ for $t \in \{d,e\}$ and $s \in \{b,c,d,e\}$. Since $D_{s,t}$ is a derivation on $U(M)$, we have $D_{s,t}(U(M)) = 0$. □

Lemma 3.2. *In $U(M)$ we have $L_{s^m} = L_s^m$ for all $s \in M$.*

Proof. Lemma 2.5 implies that $L_{s^2} = L_s L_s + [R_s, L_s]$ for $s \in N_{\mathrm{alt}}(U(M))$. Since $s \in M \subseteq N_{\mathrm{alt}}(U(M))$ we have
$$[R_s, L_s](x) = R_s L_s(x) - L_s R_s(x) = (sx)s - s(xs) = (s,x,s) = 0.$$
Hence $L_{s^2} = L_s^2$. We now prove that $L_{s^m} = L_s^m$ for $m \geq 3$ by induction on m. By the inductive hypothesis and Lemma 2.5 we have
$$L_{s^{m+1}} = L_{ss^m} = L_s L_{s^m} + [R_s, L_{s^m}] = L_s L_s^m + [R_s, L_s^m] = L_s^{m+1},$$
since R_s commutes with L_s (and hence it commutes with L_s^m). □

Lemma 3.3. *In $U(M)$ the operators L_c, L_d, L_e are pairwise commutative, and*
$$L_{c^k d^l e^m} = L_c^k L_d^l L_e^m.$$

Proof. Lemma 2.5 implies that for any $x,y \in N_{\mathrm{alt}}(U(M))$ we have
$$[L_x, L_y] = L_{[x,y]} - 2[R_x, L_y], \quad D_{x,y} = -\mathrm{ad}_{[x,y]} - 3[L_x, R_y], \quad [L_x, R_y] = [R_x, L_y].$$
If $[x,y] = 0$ then $L_{[x,y]} = 0$ and $\mathrm{ad}_{[x,y]} = 0$, and hence
$$[L_x, L_y] = \tfrac{2}{3} D_{x,y}.$$
Therefore by Lemma 3.1 we have
$$[L_c, L_d] = [L_c, L_e] = [L_d, L_e] = 0.$$
By Lemma 3.2 we have
$$L_{c^k} = L_c^k, \quad L_{d^l} = L_d^l, \quad L_{e^m} = L_e^m.$$
If $[x,y] = 0$ then $D_{x,y} = -3[L_x, R_y] = -3[R_x, L_y]$, and so Lemma 3.1 implies
$$[L_c, R_d] = [R_c, L_d] = 0, \quad [L_c, R_e] = [R_c, L_e] = 0, \quad [L_d, R_e] = [R_d, L_e] = 0.$$
We show by induction on l that
$$L_{d^l e^m} = L_d^l L_e^m.$$
The basis $l = 0$ is Lemma 3.2. If $L_{d^{l-1} e^m} = L_d^{l-1} L_e^m$ for some $l \geq 1$ then
$$L_{d^l e^m} = L_{d(d^{l-1} e^m)} = L_d L_{d^{l-1} e^m} + [R_d, L_{d^{l-1} e^m}] = L_d^l L_e^m + [R_d, L_d^{l-1} L_e^m]$$

$$= L_d^l L_e^m + [R_d, L_d^{l-1}] L_e^m + L_d^{l-1} [R_d, L_e^m] = L_d^l L_e^m,$$

since R_d commutes with L_e^m and L_d^{l-1}. We show by induction on k that

$$L_{c^k d^l e^m} = L_c^k L_d^l L_e^m.$$

The basis $k = 0$ is the previous formula. If $L_{c^{k-1} d^l e^m} = L_c^{k-1} L_d^l L_e^m$ for $k \geq 1$ then

$$\begin{aligned} L_{c^k d^l e^m} &= L_{c(c^{k-1} d^l e^m)} = L_c L_{c^{k-1} d^l e^m} + [R_c, L_{c^{k-1} d^l e^m}] \\ &= L_c^k L_d^l L_e^m + [R_c, L_c^{k-1} L_d^l L_e^m] = L_c^k L_d^l L_e^m + [R_c, L_c^{k-1}] L_d^l L_e^m = L_c^k L_d^l L_e^m \end{aligned}$$

since R_c commutes with L_c^{k-1}, L_d^l and L_e^m. The proof is complete. □

Notation 3.4. We set $[\underbrace{X, \dots, X}_{q}, Y] = [\underbrace{X, \dots, [X, [X}_{q}, Y]] \cdots]$; if $q = 0$ we get Y.

Lemma 3.5. *If $s \in N_{\mathrm{alt}}(U(M))$ then*

$$L_{s^k x} = \sum_{q=0}^{k} \binom{k}{q} L_s^{k-q} [\underbrace{R_s, \dots, R_s}_{q}, L_x].$$

Proof. By induction on k; the basis $k = 0$ is trivial, and $k = 1$ is the first equation in Lemma 2.5. Assume that $k \geq 1$ and that

$$L_{s^k x} = \sum_{q=0}^{k} \binom{k}{q} L_s^{k-q} [\underbrace{R_s, \dots, R_s}_{q}, L_x].$$

Using Lemma 2.5, the fact that R_s and L_s commute (see the proof of Lemma 3.2) and Pascal's identity for binomial coefficients, we obtain

$$\begin{aligned} L_{s^{k+1} x} &= L_{s(s^k x)} = L_s L_{s^k x} + [R_s, L_{s^k x}] \\ &= \sum_{q=0}^{k} \binom{k}{q} L_s^{k+1-q} [\underbrace{R_s, \dots, R_s}_{q}, L_x] + \sum_{q=0}^{k} \binom{k}{q} L_s^{k-q} [\underbrace{R_s, \dots, R_s}_{q+1}, L_x] \\ &= L_s^{k+1} L_x + \sum_{q=1}^{k} \binom{k}{q} L_s^{k+1-q} [\underbrace{R_s, \dots, R_s}_{q}, L_x] \\ &\quad + \sum_{q=1}^{k} \binom{k}{q-1} L_s^{k+1-q} [\underbrace{R_s, \dots, R_s}_{q}, L_x] + [\underbrace{R_s, \dots, R_s}_{k+1}, L_x] \\ &= L_s^{k+1} L_x + \sum_{q=1}^{k} \binom{k+1}{q} L_s^{k+1-q} [\underbrace{R_s, \dots, R_s}_{q}, L_x] + [\underbrace{R_s, \dots, R_s}_{k+1}, L_x] \\ &= \sum_{q=0}^{k+1} \binom{k+1}{q} L_s^{k+1-q} [\underbrace{R_s, \dots, R_s}_{q}, L_x]. \end{aligned}$$

The proof is complete. □

Proposition 3.6. *In $U(M)$ we have*

$$L_{b^j c^k d^l e^m} = \sum_{\alpha=0}^{\min(j,k)} \alpha! \binom{j}{\alpha} \binom{k}{\alpha} L_b^{j-\alpha} L_c^{k-\alpha} L_d^l L_e^m D^\alpha, \qquad D = -\tfrac{1}{3}(2\,\mathrm{ad}_d + D_{b,c}).$$

Proof. Lemmas 3.3 and 3.5 give

$$L_{b^j c^k d^l e^m} = \sum_{\alpha=0}^{j} \binom{j}{\alpha} L_b^{j-\alpha} [\underbrace{R_b, \dots, R_b}_{\alpha}, L_c^k L_d^l L_e^m].$$

By Lemmas 2.5 and 3.1 we have

$$[L_d, R_b] = -\tfrac{1}{3}(\mathrm{ad}_{[d,b]} + D_{d,b}) = 0, \qquad [L_e, R_b] = -\tfrac{1}{3}(\mathrm{ad}_{[e,b]} + D_{e,b}) = 0.$$

Thus R_b commutes with L_d and L_e, and so

$$[\underbrace{R_b, \dots, R_b}_{\alpha}, L_c^k L_d^l L_e^m] = [\underbrace{R_b, \dots, R_b}_{\alpha}, L_c^k] L_d^l L_e^m.$$

We have

$$D_{b,c} = -\mathrm{ad}_{[b,c]} - 3[L_b, R_c] = -\mathrm{ad}_{[b,c]} - 3[R_b, L_c] = -2\,\mathrm{ad}_d - 3[R_b, L_c].$$

Hence $D = [R_b, L_c] = R_b L_c - L_c R_b$. We show that D commutes with R_b. For this we compute $[\mathrm{ad}_d, R_b]$ and $[D_{b,c}, R_b]$ using Lemma 2.5:

$$\begin{aligned}
[\mathrm{ad}_d, R_b] &= [\mathrm{ad}_d, \mathrm{ad}_b + L_b] = [\mathrm{ad}_d, \mathrm{ad}_b] + [\mathrm{ad}_d, L_b] \\
&= 2\,D_{d,b} + \mathrm{ad}_{[d,b]} + [R_d - L_d, L_b] = [R_d, L_b] - [L_d, L_b] \\
&= [R_d, L_b] - L_{[d,b]} + 2[R_d, L_b] = 3[R_d, L_b] = -D_{d,b} = 0, \\
[D_{b,c}, R_b](x) &= D_{b,c} R_b(x) - R_b D_{b,c}(x) = D_{b,c}(xb) - D_{b,c}(x)b \\
&= D_{b,c}(x)b + x D_{b,c}(b) - D_{b,c}(x)b = x D_{b,c}(b) = 0.
\end{aligned}$$

Thus $[D, R_b] = 0$. We show that $[R_b, L_c^k] = kDL_c^{k-1}$ by induction on k; the case $k = 1$ is clear. By the inductive hypothesis we have

$$\begin{aligned}
[R_b, L_c^{k+1}] &= R_b L_c^{k+1} - L_c^{k+1} R_b = (R_b L_c^k) L_c - L_c^{k+1} R_b \\
&= (L_c^k R_b + kDL_c^{k-1}) L_c - L_c^{k+1} R_b = L_c^k R_b L_c + kDL_c^k - L_c^{k+1} R_b \\
&= L_c^k (L_c R_b + D) + kDL_c^k - L_c^{k+1} R_b = (k+1) D L_c^k,
\end{aligned}$$

since D and L_c commute:

$$\begin{aligned}
[D, L_c](x) &= -\tfrac{1}{3}[D_{b,c} + 2\,\mathrm{ad}_d, L_c](x) = -\tfrac{1}{3}[D_{b,c}, L_c](x) - \tfrac{2}{3}[\mathrm{ad}_d, L_c](x) \\
&= -\tfrac{1}{3}\big(D_{b,c}(cx) - cD_{b,c}(x)\big) - \tfrac{2}{3}\big([R_d, L_c] - [L_d, L_c]\big)(x) \\
&= -\tfrac{1}{3} D_{b,c}(c) x = 0.
\end{aligned}$$

(Recall that $[R_d, L_c] = 0$ and $[L_d, L_c] = 0$ by the proof of Lemma 3.3.) Finally, we show by induction on α that

$$[\underbrace{R_b, \dots, R_b}_{\alpha}, L_c^k] = \alpha! \binom{k}{\alpha} D^\alpha L_c^{k-\alpha}.$$

This is clear for $\alpha = 0$ and we just proved it for $\alpha = 1$. Since $[R_b, D] = 0$ we have

$$\begin{aligned}
[\underbrace{R_b, \dots, R_b}_{\alpha+1}, L_c^k] &= [R_b, [\underbrace{R_b \dots, R_b}_{\alpha}, L_c^k]] = \alpha! \binom{k}{\alpha} [R_b, D^\alpha L_c^{k-\alpha}] \\
&= \alpha! \binom{k}{\alpha} D^\alpha [R_b, L_c^{k-\alpha}] = \alpha! \binom{k}{\alpha} (k-\alpha) D^{\alpha+1} L_c^{k-(\alpha+1)} \\
&= (\alpha+1)! \binom{k}{\alpha+1} D^{\alpha+1} L_c^{k-(\alpha+1)}.
\end{aligned}$$

The proof is complete. □

4. Representation of $\mathbb{M}$ by differential operators

We now return to the one-parameter family $\mathbb{M} = \mathbb{M}_\gamma$ of 5-dimensional solvable Malcev algebras with structure constants in equation (1).

Notation 4.1. We have these linear operators on the polynomial algebra $P(\mathbb{M})$:

- I is the identity;
- M_x is multiplication by $x \in \{a, b, c, d, e\}$;
- D_x is differentiation with respect to $x \in \{a, b, c, d, e\}$ (it is important to distinguish between this D_x and the previous $D_{s,t}$);
- S is the shift operator on the generator a: $S(a^i b^j c^k d^l e^m) = (a+1)^i b^j c^k d^l e^m$; more generally, we use exponential notation and write S^α $(\alpha \in F)$ for the shift-by-α operator $S_\alpha(a^i b^j c^k d^l e^m) = (a+\alpha)^i b^j c^k d^l e^m$.

Notation 4.2. We use the linear isomorphism $\phi\colon U(\mathbb{M}) \to P(\mathbb{M})$ to define operators on $P(\mathbb{M})$ expressing products and commutators in $U(\mathbb{M})$:

- L_x is left multiplication: $L_x(f) = \phi(x\phi^{-1}(f))$ for $x \in \mathbb{M}$, $f \in P(\mathbb{M})$;
- R_x is right multiplication: $R_x(f) = \phi(\phi^{-1}(f)x)$ for $x \in \mathbb{M}$, $f \in P(\mathbb{M})$;
- $\rho_x = R_x - L_x$ is the adjoint: $\rho_x(f) = \phi([\phi^{-1}(f), x])$ for $x \in \mathbb{M}$, $f \in P(\mathbb{M})$.

Lemma 4.3. *As operators on $P(\mathbb{M})$ we have*

$$L_a = M_a, \qquad \rho_a = M_b D_b + M_c D_c - M_d D_d + \gamma M_e D_e - 3 D_b D_c M_d,$$
$$R_a = M_a + M_b D_b + M_c D_c - M_d D_d + \gamma M_e D_e - 3 D_b D_c M_d.$$

Proof. The claim for L_a is trivial by our convention on basis monomials:

$$L_a(a^i b^j c^k d^l e^m) = a(a^i b^j c^k d^l e^m) = a^{i+1} b^j c^k d^l e^m.$$

Since by definition $R_a = L_a + \rho_a$, it remains only to prove the claim for ρ_a. We first show by induction on i that

$$[a^i b^j c^k d^l e^m, a] = a^i [b^j c^k d^l e^m, a].$$

For $i = 0$ the claim is trivial. For the inductive step, equation (5) gives

$$\begin{aligned}
&[a^{i+1} b^j c^k d^l e^m, a] = [a a^i b^j c^k d^l e^m, a] \\
&= [a, a] a^i b^j c^k d^l e^m + a[a^i b^j c^k d^l e^m, a] - D_{a,a}(a^i b^j c^k d^l e^m) - [a^i b^j c^k d^l e^m, [a, a]] \\
&= a[a^i b^j c^k d^l e^m, a].
\end{aligned}$$

By definition of M_x and D_x we have

$$\begin{aligned}
&(M_b D_b + M_c D_c - M_d D_d + \gamma M_e D_e - 3 D_b D_c M_d)(b^j c^k d^l e^m) \\
&= (j + k - l + m\gamma) b^j c^k d^l e^m - 3jk b^{j-1} c^{k-1} d^{l+1} e^m,
\end{aligned}$$

and so it now remains only to show by induction on j that

$$[b^j c^k d^l e^m, a] = (j + k - l + m\gamma) b^j c^k d^l e^m - 3jk b^{j-1} c^{k-1} d^{l+1} e^m.$$

For $j = 0$ we use the fact that a, c, d, e span a Lie subalgebra of $\mathbb{M}$ (a nilpotent Lie algebra, the split extension of the 1-dimensional Lie algebra with basis a by the module with basis c, d, e). Clearly ρ_a is a derivation of this Lie algebra, and we have $[x, [x, a]] = 0$ for $x = c, d, e$. Therefore

$$[c^k d^l e^m, a] = [c^k, a] d^l e^m + c^k [d^l, a] e^m + c^k d^l [e^m, a] = (k - l + m\gamma) c^k d^l e^m.$$

For the inductive step, equation (5) gives

$$\begin{aligned}
&[b^{j+1}c^kd^le^m, a] = [bb^jc^kd^le^m, a] \\
&= [b, a]b^jc^kd^le^m + b[b^jc^kd^le^m, a] - D_{a,b}(b^jc^kd^le^m) - [b^jc^kd^le^m, [a, b]] \\
&= b^{j+1}c^kd^le^m + b[b^jc^kd^le^m, a] - D_{a,b}(b^jc^kd^le^m) + [b^jc^kd^le^m, b].
\end{aligned}$$

Since b, c, d, e span a Lie subalgebra of $\mathbb{M}$, the structure constants give

$$[b^jc^kd^le^m, b] = -2kb^jc^{k-1}d^{l+1}e^m.$$

Furthermore, since $D_{a,b}(b) = 0$, $D_{a,b}(c) = d$, $D_{a,b}(d) = 0$, $D_{a,b}(e) = 0$, we obtain

$$\begin{aligned}
&D_{a,b}(b^jc^kd^le^m) \\
&= D_{a,b}(b^j)c^kd^le^m + b^jD_{a,b}(c^k)d^le^m + b^jc^kD_{a,b}(d^l)e^m + b^jc^kd^lD_{a,b}(e^m) \\
&= b^jD_{a,b}(c^k)d^le^m = kb^jc^{k-1}d^{l+1}e^m.
\end{aligned}$$

Combining these results gives

$$[b^{j+1}c^kd^le^m, a] = b^{j+1}c^kd^le^m + b[b^jc^kd^le^m, a] - 3kb^jc^{k-1}d^{l+1}e^m.$$

Now the inductive hypothesis gives

$$\begin{aligned}
&[b^{j+1}c^kd^le^m, a] = b^{j+1}c^kd^le^m + b[b^jc^kd^le^m, a] - 3kb^jc^{k-1}d^{l+1}e^m \\
&= b^{j+1}c^kd^le^m + b\big((j{+}k{-}l{+}m\gamma)b^jc^kd^le^m - 3jkb^{j-1}c^{k-1}d^{l+1}e^m\big) \\
&\quad - 3kb^jc^{k-1}d^{l+1}e^m \\
&= (j + 1 + k - l + m\gamma)b^{j+1}c^kd^le^m - 3(j + 1)kb^jc^{k-1}d^{l+1}e^m.
\end{aligned}$$

The proof is complete. □

Lemma 4.4. *As operators on $P(\mathbb{M})$ we have*

$$\begin{aligned}
L_b &= SM_b + (S^{-1} - S)D_cM_d, & \rho_b &= (I - S)M_b + (S - 2S^{-1} - I)D_cM_d, \\
R_b &= M_b - (S^{-1} + I)D_cM_d, & D_{a,b} &= (I - S)M_b + (S + S^{-1} - I)D_cM_d.
\end{aligned}$$

Proof. Since $R_b = L_b + \rho_b$, it suffices to prove the formulas for L_b, ρ_b and $D_{a,b}$. For this we set $y = a^ib^jc^kd^le^m$ and do simultaneous induction on i using the equations

$$\begin{aligned}
&b(ay) = a(by) + by - \tfrac{2}{3}D_{a,b}(y) + \tfrac{2}{3}[y, b], \qquad D_{a,b}(ay) = -by + aD_{a,b}(y), \\
&[ay, b] = -by + a[y, b] + D_{a,b}(y) - [y, b],
\end{aligned}$$

which follow from (5), (6) and Table 1. The basis of the induction consists of these equations from the proof of Lemma 4.3:

$$\begin{aligned}
&L_b(b^jc^kd^le^m) = b^{j+1}c^kd^le^m, \qquad D_{a,b}(b^jc^kd^le^m) = kb^jc^{k-1}d^{l+1}e^m, \\
&\rho_b(b^jc^kd^le^m) = -2kb^jc^{k-1}d^{l+1}e^m.
\end{aligned}$$

We now prove case $i+1$ of each equation separately, but in each case the inductive hypothesis is case i of all three equations. First, the formula for L_b:

$$\begin{aligned}
&b(a^{i+1}b^jc^kd^le^m) \\
&= a(b(a^ib^jc^kd^le^m)) + b(a^ib^jc^kd^le^m) - \tfrac{2}{3}D_{a,b}(a^ib^jc^kd^le^m) + \tfrac{2}{3}[a^ib^jc^kd^le^m, b] \\
&= (M_a + I)(SM_b + (S^{-1} - S)D_cM_d)(a^ib^jc^kd^le^m) \\
&\quad - \tfrac{2}{3}((I - S)M_b + (S + S^{-1} - I)D_cM_d)(a^ib^jc^kd^le^m) \\
&\quad + \tfrac{2}{3}((I - S)M_b + (S - 2S^{-1} - I)D_cM_d)(a^ib^jc^kd^le^m)
\end{aligned}$$

$$
\begin{aligned}
&= (M_a + I)(SM_b)(a^i b^j c^k d^l e^m) \\
&\quad + ((M_a{+}I)(S^{-1}{-}S) - \tfrac{2}{3}(S{+}S^{-1}{-}I) + \tfrac{2}{3}(S{-}2S^{-1}{-}I))(D_c M_d)(a^i b^j c^k d^l e^m) \\
&= (SM_a M_b)(a^i b^j c^k d^l e^m) + ((M_a - I)S^{-1} - (M_a + I)S)(D_c M_d)(a^i b^j c^k d^l e^m) \\
&= (SM_a M_b)(a^i b^j c^k d^l e^m) + (S^{-1} M_a - SM_a)(D_c M_d)(a^i b^j c^k d^l e^m) \\
&= (SM_b M_a)(a^i b^j c^k d^l e^m) + ((S^{-1} - S)M_a)(D_c M_d)(a^i b^j c^k d^l e^m) \\
&= (SM_b + (S^{-1} - S)D_c M_d)(a^{i+1} b^j c^k d^l e^m).
\end{aligned}
$$

Next, the formula for ρ_b:

$$
\begin{aligned}
&[a^{i+1} b^j c^k d^l e^m, b] \\
&= -b(a^i b^j c^k d^l e^m) + a[a^i b^j c^k d^l e^m, b] + D_{a,b}(a^i b^j c^k d^l e^m) - [a^i b^j c^k d^l e^m, b] \\
&= -(SM_b + (S^{-1} - S)D_c M_d)(a^i b^j c^k d^l e^m) \\
&\quad + (M_a - I)((I - S)M_b + (S - 2S^{-1} - I)D_c M_d)(a^i b^j c^k d^l e^m) \\
&\quad + ((I - S)M_b + (S + S^{-1} - I)D_c M_d)(a^i b^j c^k d^l e^m) \\
&= (-S + M_a(I - S))M_b(a^i b^j c^k d^l e^m) \\
&\quad + ((M_a + I)S - 2(M_a - I)S^{-1} - M_a)(D_c M_d)(a^i b^j c^k d^l e^m) \\
&= (M_a - (M_a + I)S)M_b(a^i b^j c^k d^l e^m) \\
&\quad + ((M_a + I)S - 2(M_a - I)S^{-1} - M_a)(D_c M_d)(a^i b^j c^k d^l e^m) \\
&= (M_a - SM_a)M_b(a^i b^j c^k d^l e^m) + (SM_a - 2S^{-1}M_a - M_a)(D_c M_d)(a^i b^j c^k d^l e^m) \\
&= (I - S)(M_a M_b)(a^i b^j c^k d^l e^m) + (SM_a - 2S^{-1}M_a - M_a)(D_c M_d)(a^i b^j c^k d^l e^m) \\
&= (I - S)(M_b M_a)(a^i b^j c^k d^l e^m) + (S - 2S^{-1} - I)(M_a D_c M_d)(a^i b^j c^k d^l e^m) \\
&= ((I - S)M_b + (S - 2S^{-1} - I)D_c M_d)(a^{i+1} b^j c^k d^l e^m).
\end{aligned}
$$

Finally, the formula for $D_{a,b}$:

$$
\begin{aligned}
&D_{a,b}(a^{i+1} b^j c^k d^l e^m) = -b(a^i b^j c^k d^l e^m) + aD_{a,b}(a^i b^j c^k d^l e^m) \\
&= -(SM_b + (S^{-1} - S)D_c M_d)(a^i b^j c^k d^l e^m) \\
&\quad + M_a((I - S)M_b + (S + S^{-1} - I)D_c M_d)(a^i b^j c^k d^l e^m) \\
&= (-S + M_a(I - S))M_b(a^i b^j c^k d^l e^m) \\
&\quad + ((I + M_a)S + (M_a - I)S^{-1} - M_a)(D_c M_d)(a^i b^j c^k d^l e^m) \\
&= (M_a - (I + M_a)S)M_b(a^i b^j c^k d^l e^m) \\
&\quad + ((I + M_a)S + (M_a - I)S^{-1} - M_a)(D_c M_d)(a^i b^j c^k d^l e^m) \\
&= (M_a - SM_a)M_b(a^i b^j c^k d^l e^m) + (SM_a + S^{-1}M_a - M_a)(D_c M_d)(a^i b^j c^k d^l e^m) \\
&= (I - S)(M_a M_b)(a^i b^j c^k d^l e^m) + (S + S^{-1} - I)(M_a D_c M_d)(a^i b^j c^k d^l e^m) \\
&= ((I - S)M_b + (S + S^{-1} - I)D_c M_d)(a^{i+1} b^j c^k d^l e^m).
\end{aligned}
$$

The proof is complete. □

Lemma 4.5. *As operators on $P(\mathbb{M})$ we have*

$$
L_c = SM_c - (S + S^{-1})D_b M_d, \qquad \rho_c = (I - S)M_c + (S + 2S^{-1} - I)D_b M_d,
$$

$$R_c = M_c + (S^{-1} - I)D_b M_d, \qquad D_{a,c} = (I - S)M_c + (S - S^{-1} - I)D_b M_d.$$

Proof. As before, it suffices to prove the formulas for L_c, ρ_c and $D_{a,c}$. The basis of the induction consists of the following equations which follow easily from the fact that b, c, d, e span a Lie subalgebra of $\mathbb{M}$:

$$c(b^j c^k d^l e^m) = b^j c^{k+1} d^l e^m - j b^{j-1} c^k d^{l+1} e^m, \qquad [b^j c^k d^l e^m, c] = 2j b^{j-1} c^k d^{l+1} e^m,$$
$$D_{a,c}(b^j c^k d^l e^m) = -j b^{j-1} c^k d^{l+1} e^m.$$

The strategy of the proof is the same as for Lemma 4.4. First, the formula for L_c:

$$\begin{aligned}
&c(a^{i+1} b^j c^k d^l e^m)\\
&= a(c(a^i b^j c^k d^l e^m)) + c(a^i b^j c^k d^l e^m) - \tfrac{2}{3} D_{a,c}(a^i b^j c^k d^l e^m) + \tfrac{2}{3}[a^i b^j c^k d^l e^m, c]\\
&= (M_a + I)(SM_c - (S + S^{-1})D_b M_d)(a^i b^j c^k d^l e^m)\\
&\quad - \tfrac{2}{3}((I - S)M_c + (S - S^{-1} - I)D_b M_d)(a^i b^j c^k d^l e^m)\\
&\quad + \tfrac{2}{3}((I - S)M_c + (S + 2S^{-1} - I)D_b M_d)(a^i b^j c^k d^l e^m)\\
&= ((M_a + I)(SM_c) - ((M_a + I)(S + S^{-1}) - 2S^{-1})(D_b M_d))(a^i b^j c^k d^l e^m)\\
&= ((M_a + I)(SM_c) - ((M_a + I)S + (M_a - I)S^{-1})(D_b M_d))(a^i b^j c^k d^l e^m)\\
&= ((SM_a)M_c - (SM_a + S^{-1}M_a)D_b M_d)(a^i b^j c^k d^l e^m)\\
&= ((SM_c)M_a - (S + S^{-1})M_a D_b M_d)(a^i b^j c^k d^l e^m)\\
&= (SM_c - (S + S^{-1})D_b M_d)(a^{i+1} b^j c^k d^l e^m).
\end{aligned}$$

Next, the formula for ρ_c:

$$\begin{aligned}
&[a^{i+1} b^j c^k d^l e^m, c]\\
&= -c(a^i b^j c^k d^l e^m) + a[a^i b^j c^k d^l e^m, c] + D_{a,c}(a^i b^j c^k d^l e^m) - [a^i b^j c^k d^l e^m, c]\\
&= (-SM_c + (S + S^{-1})D_b M_d)(a^i b^j c^k d^l e^m)\\
&\quad + (M_a - I)((I - S)M_c + (S + 2S^{-1} - I)D_b M_d)(a^i b^j c^k d^l e^m)\\
&\quad + ((I - S)M_c + (S - S^{-1} - I)D_b M_d)(a^i b^j c^k d^l e^m)\\
&= (-S + M_a(I - S))M_c(a^i b^j c^k d^l e^m)\\
&\quad + (S + S^{-1} + (M_a - I)(S{+}2S^{-1}{-}I) + (S{-}S^{-1}{-}I))(D_b M_d)(a^i b^j c^k d^l e^m)\\
&= (M_a - (M_a + I)S)M_c(a^i b^j c^k d^l e^m)\\
&\quad + ((M_a + I)S + 2(M_a - I)S^{-1} - M_a)(D_b M_d)(a^i b^j c^k d^l e^m)\\
&= (M_a - SM_a)M_c(a^i b^j c^k d^l e^m) + (SM_a + 2S^{-1}M_a - M_a)(D_b M_d)(a^i b^j c^k d^l e^m)\\
&= (I - S)(M_a M_c)(a^i b^j c^k d^l e^m) + (S + 2S^{-1} - I)(M_a D_b M_d)(a^i b^j c^k d^l e^m)\\
&= ((I - S)M_c + (S + 2S^{-1} - I)(D_b M_d))(a^{i+1} b^j c^k d^l e^m).
\end{aligned}$$

Finally, the formula for $D_{a,c}$:

$$\begin{aligned}
&D_{a,c}(a^{i+1} b^j c^k d^l e^m) = -c(a^i b^j c^k d^l e^m) + aD_{a,c}(a^i b^j c^k d^l e^m)\\
&= -(SM_c - (S + S^{-1})D_b M_d)(a^i b^j c^k d^l e^m)\\
&\quad + M_a((I - S)M_c + (S - S^{-1} - I)D_b M_d)(a^i b^j c^k d^l e^m)\\
&= ((-S{+}M_a(I{-}S))M_c + (S{+}S^{-1}{+}M_a S{-}M_a S^{-1}{-}M_a)(D_b M_d))(a^i b^j c^k d^l e^m)
\end{aligned}$$

$= (((I-S)M_a)M_c + (SM_a - S^{-1}M_a - M_a)D_bM_d)(a^ib^jc^kd^le^m)$
$= \big((I-S)M_c + (S - S^{-1} - I)D_bM_d\big)\,(a^{i+1}b^jc^kd^le^m).$

The proof is complete. ◻

Lemma 4.6. *As operators on $P(\mathbb{M})$ we have*

$$L_d = S^{-1}M_d, \qquad \rho_d = (I-S^{-1})M_d, \qquad R_d = M_d, \qquad D_{a,d} = (S^{-1}-I)M_d.$$

Proof. The basis of the induction consists of the equations

$$d(b^jc^kd^le^m) = b^jc^kd^{l+1}e^m, \qquad [b^jc^kd^le^m, d] = 0, \qquad D_{a,d}(b^jc^kd^le^m) = 0.$$

The rest of the proof is similar to that of Lemma 4.5. ◻

Lemma 4.7. *As operators on $P(\mathbb{M})$ we have*

$$L_e = S^{-\gamma}M_e, \qquad \rho_e = (I-S^{-\gamma})M_e, \qquad R_e = M_e, \qquad D_{a,e} = \gamma(S^{-\gamma}-I)M_e.$$

Proof. The basis of the induction consists of the three equations

$$D_{a,e}(b^jc^kd^le^m) = 0, \qquad e(b^jc^kd^le^m) = b^jc^kd^le^{m+1}, \qquad [b^jc^kd^le^m, e] = 0.$$

Recall that S^γ is defined by $S^\gamma(a^ib^jc^kd^le^m) = (a+\gamma)^ib^jc^kd^le^m$. We have

$e(a^{i+1}b^jc^kd^le^m)$
$= a(e(a^ib^jc^kd^le^m)) - \gamma e(a^ib^jc^kd^le^m) - \frac{2}{3}D_{a,e}(a^ib^jc^kd^le^m) - \frac{2}{3}\gamma[a^ib^jc^kd^le^m, e]$
$= ((M_a - \gamma I)S^{-\gamma}M_e - \frac{2}{3}\gamma(S^{-\gamma}-I)M_e - \frac{2}{3}\gamma(I-S^{-\gamma})M_e)(a^ib^jc^kd^le^m)$
$= S^{-\gamma}M_aM_e(a^ib^jc^kd^le^m) = S^{-\gamma}M_e(a^{i+1}b^jc^kd^le^m),$

$[a^{i+1}b^jc^kd^le^m, e]$
$= \gamma e(a^ib^jc^kd^le^m) + a[a^ib^jc^kd^le^m, e] + D_{a,e}(a^ib^jc^kd^le^m) + \gamma[a^ib^jc^kd^le^m, e]$
$= (\gamma S^{-\gamma}M_e + M_a(I-S^{-\gamma})M_e + \gamma(S^{-\gamma}-I)M_e + \gamma(I-S^{-\gamma})M_e)(a^ib^jc^kd^le^m)$
$= (\gamma S^{-\gamma} + M_a - M_aS^{-\gamma})M_e(a^ib^jc^kd^le^m) = (-(M_a-\gamma I)S^{-\gamma} + M_a)M_e(a^ib^jc^kd^le^m)$
$= (-S^{-\gamma}+I)M_aM_e(a^ib^jc^kd^le^m) = (I-S^{-\gamma})M_e(a^{i+1}b^jc^kd^le^m),$

$D_{a,e}(a^{i+1}b^jc^kd^le^m) = -\gamma^2e(a^ib^jc^kd^le^m) + aD_{a,e}(a^ib^jc^kd^le^m)$
$= -\gamma^2S^{-\gamma}M_e(a^ib^jc^kd^le^m) + M_a\gamma(S^{-\gamma}-I)M_e(a^ib^jc^kd^le^m)$
$= \gamma(-\gamma S^{-\gamma}M_e + M_aS^{-\gamma}M_e - M_aM_e)(a^ib^jc^kd^le^m)$
$= \gamma((M_a-\gamma I)S^{-\gamma} - M_a)M_e(a^ib^jc^kd^le^m) = \gamma((S^{-\gamma}M_a - M_a)M_e)(a^ib^jc^kd^le^m)$
$= \gamma(S^{-\gamma}-I)M_aM_e(a^ib^jc^kd^le^m) = \gamma(S^{-\gamma}-I)M_e(a^{i+1}b^jc^kd^le^m).$

The proof is complete. ◻

5. The center of the universal enveloping algebra

Our next goal is to use the results of Section 4 to compute the center of $U(\mathbb{M})$. Let $Z(U)$ and $K(U)$ denote the center and commutative center of $U(\mathbb{M})$. Shestakov and Zhelyabin [9] proved that

$$Z(U) = K(U) = \{\, n \in U(\mathbb{M}) \mid [n,x] = 0 \,\text{for all}\, x \in \mathbb{M}\,\}.$$

$Z(U)$ is a characteristic subalgebra: it is stable under automorphisms of $U(\mathbb{M})$.

Theorem 5.1. *Let* $\mathbb{M} = \mathbb{M}_\gamma$ *belong to the one-parameter family* (1) *of solvable 5-dimensional Malcev algebras over a field* F*. Then*

$$Z(U) = \begin{cases} F[\, d^{\gamma m} e^m \,] & \text{if } \gamma = l/m \text{ with } l, m \in \mathbb{Z},\ (l,m) = 1,\ m > 0; \\ F & \text{if } \gamma \notin \mathbb{Q}. \end{cases}$$

Proof. We first show that $Z(U) \subseteq U(L)$ where $L = \operatorname{span}\{b, c, d, e\}$. Choose

$$n = \sum_{i=0}^{m} a^i s_i \in Z(U), \quad s_i \in U(L),$$

where m is minimal satisfying $n \notin U(L)$ and $s_m \neq 0$. If $m = 0$ then we are done. If $m \geq 1$ then we show by contradiction that $m = 1$. Assume $m > 1$ and consider the automorphism φ of $U(\mathbb{M})$ defined by

$$\varphi(a) = a + 1, \quad \varphi(b) = b, \quad \varphi(c) = c, \quad \varphi(d) = d, \quad \varphi(e) = e.$$

Since $Z(U)$ is a characteristic subalgebra, $\varphi(n) \in Z(U)$ and so $\varphi(n) - n \in Z(U)$. However,

$$\varphi(n) - n = \sum_{i=0}^{m} (a+1)^i s_i - \sum_{i=0}^{m} a^i s_i = \sum_{i=0}^{m} ((a+1)^i - a^i) s_i.$$

Hence $\varphi(n) - n$ is a nonzero element of $Z(U) \setminus U(L)$, but its degree in a is strictly less than m, contradicting the choice of n. Hence $m = 1$ and $n = s_0 + a s_1$. Since $n \in Z(U)$ we have $\operatorname{ad}_d(n) = [n, d] = 0$. Lemma 4.6 shows that $\operatorname{ad}_d = -D_{a,d}$ and so ad_d is a derivation of $U(\mathbb{M})$; also $\operatorname{ad}_d = (I - S^{-1}) M_d$ which is zero on $U(L)$. Hence

$$0 = \operatorname{ad}_d(n) = \operatorname{ad}_d(s_0 + a s_1) = \operatorname{ad}_d(s_0) + \operatorname{ad}_d(a) s_1 + a \operatorname{ad}_d(s_1) = d s_1.$$

Hence $s_1 = 0$ since $U(\mathbb{M})$ has no zero divisors by [8]. Therefore $Z(U) \subseteq U(L)$.

Clearly $n \in Z(U)$ if and only if $\operatorname{ad}_s(n) = 0$ for $s \in \{a, b, c, d, e\}$. Consider

$$n = \sum_i \alpha_i b^{j_i} c^{k_i} d^{l_i} e^{m_i} \in Z(U), \quad \alpha_i \neq 0.$$

By the formula for ρ_b from Lemma 4.4 we have

$$\begin{aligned} \operatorname{ad}_b(n) &= (I - S) M_b(n) + (S - 2S^{-1} - I) D_c M_d(n) = 0 - 2 D_c M_d(n) \\ &= \sum_i (-2\alpha_i k_i) b^{j_i} c^{k_i - 1} d^{l_i + 1} e^{m_i}; \end{aligned}$$

hence $k_i = 0$ for all i. By the formula for ρ_c from Lemma 4.5 we have

$$\begin{aligned} \operatorname{ad}_b(n) &= (I - S) M_c(n) + (S + 2S^{-1} - I) D_b M_d(n) = 0 + 2 D_b M_d(n) \\ &= \sum_i (2\alpha_i j_i) b^{j_i - 1} d^{l_i + 1} e^{m_i}; \end{aligned}$$

hence $j_i = 0$ for all i. By the formula for ρ_a from Lemma 4.3 we have

$$\operatorname{ad}_a(n) = \sum_i \alpha_i (-l_i + \gamma m_i) d^{l_i} e^{m_i};$$

hence $l_i = \gamma m_i$ for all i. It is clear that $\operatorname{ad}_d(d^{l_i} e^{m_i}) = \operatorname{ad}_e(d^{l_i} e^{m_i}) = 0$.

Hence if $\gamma \in \mathbb{Q}$ then $Z(U)$ is generated by $d^{\gamma m} e^m$ where m is the smallest positive integer for which $\gamma m \in \mathbb{Z}$, and if $\gamma \notin \mathbb{Q}$ then $Z(U) = F$. $\square$

6. The universal nonassociative enveloping algebra

In this section we compute structure constants for $U(\mathbb{M})$ where $\mathbb{M} = \mathbb{M}_\gamma$ belongs to the one-parameter family (1) of solvable 5-dimensional Malcev algebras.

Lemma 6.1. *In $U(\mathbb{M})$ we have*

$$L_{b^j c^k d^l e^m} = \sum_{\alpha=0}^{\min(j,k)} \sum_{\beta=0}^{\alpha} (-1)^{\alpha-\beta} \alpha! \binom{\alpha}{\beta} \binom{j}{\alpha} \binom{k}{\alpha} S^{-\beta} L_b^{j-\alpha} L_c^{k-\alpha} M_d^\alpha L_d^l L_e^m.$$

Proof. By Lemma 4.6 we have $\mathrm{ad}_d = -D_{a,d} = (I - S^{-1}) M_d$, so ad_d is a derivation of $U(\mathbb{M})$. Since $\mathrm{ad}_d(a) = d$ and $D_{b,c}(a) = d$, we have $(D_{b,c} - \mathrm{ad}_d)(a) = 0$. Since $D_{b,c} - \mathrm{ad}_d$ is a derivation of $U(\mathbb{M})$, we have $(D_{b,c} - \mathrm{ad}_d)(a^k) = 0$. It follows from Table 1 and Lemma 4.6 that $\mathrm{ad}_d(x) = 0$ and $D_{b,c}(x) = 0$ for any $x \in U(L)$ where $L = \mathrm{span}\{b, c, d, e\}$. Hence

$$(D_{b,c} - \mathrm{ad}_d)(a^k x) = (D_{b,c} - \mathrm{ad}_d)(a^k) x + a^k (D_{b,c} - \mathrm{ad}_d)(x) = 0.$$

Therefore $\mathrm{ad}_d = D_{b,c}$ on $U(\mathbb{M})$. This implies that the operator D from the proof of Proposition 3.6 satisfies $D = -\mathrm{ad}_d = (S^{-1} - I) M_d$. Therefore

$$D^\alpha = (S^{-1} - I)^\alpha M_d^\alpha = \sum_{\beta=0}^{\alpha} (-1)^{\alpha-\beta} \binom{\alpha}{\beta} S^{-\beta} M_d^\alpha.$$

Using this in Proposition 3.6 gives the stated formula for $L_{b^j c^k d^l e^m}$. □

Remark 6.2. If we set $m = 0$ in Lemma 6.1 then we obtain the formula for $L_{b^j c^k d^l}$ in Lemma 4.2 of [1]. The following Lemma 6.3 generalizes Lemma 4.3 of [1].

Lemma 6.3. *In $U(\mathbb{M})$ we have*

$$\begin{aligned}
&[R_a, L_a^s S^t L_b^u D_b^v D_c^w L_c^x M_d^y L_d^z L_e^m] = -(t+v+w+y) L_a^s S^t L_b^u D_b^v D_c^w L_c^x M_d^y L_d^z L_e^m \\
&- u L_a^s S^{t-1} L_b^{u-1} D_b^v D_c^{w+1} L_c^x M_d^{y+1} L_d^z L_e^m + x L_a^s S^{t-1} L_b^u D_b^{v+1} D_c^w L_c^{x-1} M_d^{y+1} L_d^z L_e^m.
\end{aligned}$$

Proof. For any $x, y \in M$, by Lemma 2.5 we have

$$D_{x,y} = -\mathrm{ad}_{[x,y]} - 3[R_x, L_y], \quad \text{hence} \quad -3[R_x, L_y] = \mathrm{ad}_{[x,y]} + D_{x,y}.$$

Using this and Lemma 4.4 we show that $[R_a, L_b] = -S^{-1} D_c M_d$:

$$\begin{aligned}
&-3[R_a, L_b] = \mathrm{ad}_{[a,b]} + D_{a,b} = -\mathrm{ad}_b + D_{a,b} \\
&= -(I - S) M_b - (S - 2S^{-1} - I) D_c M_d + (I - S) M_b + (S + S^{-1} - I) D_c M_d \\
&= 3 S^{-1} D_c M_d.
\end{aligned}$$

Similarly, using Lemma 4.5 we show that $[R_a, L_c] = S^{-1} D_b M_d$:

$$\begin{aligned}
&-3[R_a, L_c] = \mathrm{ad}_{[a,c]} + D_{a,c} = -\mathrm{ad}_c + D_{a,c} \\
&= -(I - S) M_c - (S + 2S^{-1} - I) D_b M_d + (I - S) M_c + (S - S^{-1} - I) D_b M_d \\
&= -3 S^{-1} D_b M_d.
\end{aligned}$$

Using Lemmas 4.6 and 4.7 we see that $[R_a, L_d] = 0$ and $[R_a, L_e] = 0$:

$$\begin{aligned}
-3[R_a, L_d] &= \mathrm{ad}_{[a,d]} + D_{a,d} = \mathrm{ad}_d + D_{a,d} = 0, \\
-3[R_a, L_e] &= \mathrm{ad}_{[a,e]} + D_{a,e} = \gamma \mathrm{ad}_e + D_{a,e} = 0.
\end{aligned}$$

Similarly, using obvious commutation relations (Lemma 3.2 of [1]), we have

$$\begin{aligned}
[R_a, D_b] &= [M_b D_b, D_b] - 3[D_b D_c M_d, D_b] = [M_b, D_b] D_b - 0 = -D_b,\\
[R_a, D_c] &= [M_c D_c, D_c] - 3[D_b D_c M_d, D_c] = [M_c, D_c] D_c - 0 = -D_c,\\
[R_a, M_d] &= -[M_d D_d, M_d] - 3[D_b D_c, M_d] M_d = -M_d [D_d, M_d] = -M_d,\\
[R_a, S] &= [M_a, S] = -S.
\end{aligned}$$

We now apply these formulas to the derivation rule:

$$\begin{aligned}
&[R_a, L_a^s S^t L_b^u D_b^v D_c^w L_c^x M_d^y L_d^z L_e^m] = L_a^s [R_a, S^t] L_b^u D_b^v D_c^w L_c^x M_d^y L_d^z L_e^m\\
&+ L_a^s S^t [R_a, L_b^u] D_b^v D_c^w L_c^x M_d^y L_d^z L_e^m + L_a^s S^t L_b^u [R_a, D_b^v] D_c^w L_c^x M_d^y L_d^z L_e^m\\
&+ L_a^s S^t L_b^u D_b^v [R_a, D_c^w] L_c^x M_d^y L_d^z L_e^m + L_a^s S^t L_b^u D_b^v D_c^w [R_a, L_c^x] M_d^y L_d^z L_e^m\\
&+ L_a^s S^t L_b^u D_b^v D_c^w L_c^x [R_a, M_d^y] L_d^z L_e^m.
\end{aligned}$$

Expanding the right side and collecting terms gives the stated result. □

Lemma 6.4. *In $U(\mathbb{M})$ the operator $L_{a^i b^j c^k d^l e^m}$ equals*

$$\sum_{\alpha=0}^{\min(j,k)} \sum_{\beta=0}^{\alpha} \sum_{\kappa=0}^{i} \sum_{\delta=0}^{i-\kappa} \sum_{\epsilon=0}^{i-\kappa-\delta} (-1)^{i+\alpha-\beta-\kappa-\delta} \alpha! \delta! \epsilon! \binom{\alpha}{\beta} \binom{j}{\alpha, \epsilon} \binom{k}{\alpha, \delta} \times$$
$$X_i(\kappa, \delta, \epsilon) L_a^\kappa S^{-\beta-\delta-\epsilon} L_b^{j-\alpha-\epsilon} D_b^\delta D_c^\epsilon L_c^{k-\alpha-\delta} M_d^{\alpha+\delta+\epsilon} L_d^l L_e^m,$$

where $X_i(\kappa, \delta, \epsilon)$ is polynomial in $\alpha - \beta$ satisfying $X_0(0,0,0) = 1$,

$$X_{i+1}(\kappa, \delta, \epsilon) = (\alpha-\beta+\delta+\epsilon) X_i(\kappa, \delta, \epsilon) + X_i(\kappa-1, \delta, \epsilon) + X_i(\kappa, \delta-1, \epsilon) + X_i(\kappa, \delta, \epsilon-1),$$

and $X_i(\kappa, \delta, \epsilon) = 0$ unless $0 \le \kappa \le i$, $0 \le \delta \le i-\kappa$, $0 \le \epsilon \le i-\kappa-\delta$.

Proof. Induction on i. The basis is Lemma 6.1 and the inductive step is Lemma 6.3. The rest of the proof is a step-by-step repetition of that of Lemma 4.4 of [1]. □

Lemma 6.5. *We have*

$$X_i(\kappa, \delta, \epsilon) = \binom{\delta+\epsilon}{\epsilon} \sum_{\zeta=0}^{i-\kappa-\delta-\epsilon} \binom{i}{\kappa, \zeta} \left\{ {i-\kappa-\zeta \atop \delta+\epsilon} \right\} (\alpha-\beta)^\zeta,$$

where the Stirling numbers of the second kind are defined by

$$\left\{ {r \atop s} \right\} = \frac{1}{s!} \sum_{t=0}^{s} (-1)^{s-t} \binom{s}{t} t^r.$$

Proof. See [1], Definition 4.5 and Lemma 4.6. □

The formulas for L_b and L_c in Lemmas 4.4 and 4.5 are the same as for the 4-dimensional solvable Malcev algebra [1]. Thus Lemma 5.1 of [1] holds in our case:

$$L_b^u = \sum_{\eta=0}^{u} \sum_{\theta=0}^{u-\eta} (-1)^{u-\eta-\theta} \binom{u}{\eta, \theta} S^{u-2\theta} M_b^\eta M_d^{u-\eta} D_c^{u-\eta}, \tag{12}$$

$$L_c^x = \sum_{\lambda=0}^{x} \sum_{\mu=0}^{x-\lambda} (-1)^{x-\lambda} \binom{x}{\lambda, \mu} S^{x-2\mu} M_c^\lambda M_d^{x-\lambda} D_b^{x-\lambda}. \tag{13}$$

Therefore Lemma 6.4 can be rewritten in terms of M_x, D_x and S as follows.

Lemma 6.6. *In $U(\mathbb{M})$ the operator $L_{a^i b^j c^k d^l e^m}$ equals*

$$\sum_{\alpha=0}^{\min(j,k)} \sum_{\beta=0}^{\alpha} \sum_{\kappa=0}^{i} \sum_{\delta=0}^{i-\kappa} \sum_{\epsilon=0}^{i-\kappa-\delta} \sum_{\zeta=0}^{i-\kappa-\delta-\epsilon} \sum_{\eta=0}^{j-\alpha-\epsilon} \sum_{\theta=0}^{j-\alpha-\epsilon-\eta} \sum_{\lambda=0}^{k-\alpha-\delta} \sum_{\mu=0}^{k-\alpha-\delta-\lambda}$$
$$(-1)^{i+j+k+\alpha-\beta-\kappa-\epsilon-\eta-\theta-\lambda} (\alpha-\beta)^\zeta \alpha! \binom{\alpha}{\beta} (\delta+\epsilon)! \binom{i}{\kappa,\zeta} \times$$
$$\left\{ {i-\kappa-\zeta \atop \delta+\epsilon} \right\} \binom{j}{\alpha,\epsilon,\eta,\theta} \binom{k}{\alpha,\delta,\lambda,\mu} \times$$
$$M_a^\kappa S^{j+k-l-2\alpha-\beta-2\delta-2\epsilon-2\theta-2\mu-\gamma m} M_b^\eta D_b^{k-\alpha-\lambda} D_c^{j-\alpha-\eta} M_c^\lambda M_d^{j+k+l-\alpha-\eta-\lambda} M_e^m.$$

Theorem 6.7. *In $U(\mathbb{M})$ the product $(a^i b^j c^k d^l e^m)(a^r b^n c^p d^q e^s)$ equals*

$$\sum_{\alpha=0}^{\min(j,k)} \sum_{\beta=0}^{\alpha} \sum_{\kappa=0}^{i} \sum_{\delta=0}^{i-\kappa} \sum_{\epsilon=0}^{i-\kappa-\delta} \sum_{\zeta=0}^{i-\kappa-\delta-\epsilon} \sum_{\eta=0}^{j-\alpha-\epsilon} \sum_{\theta=0}^{j-\alpha-\epsilon-\eta} \sum_{\lambda=0}^{k-\alpha-\delta} \sum_{\mu=0}^{k-\alpha-\delta-\lambda} \sum_{\nu=0}^{r}$$
$$(-1)^{i+j+k+\alpha-\beta-\kappa-\epsilon-\eta-\theta-\lambda} (\alpha-\beta)^\zeta \alpha! \binom{\alpha}{\beta} (\delta+\epsilon)! \omega^\nu \binom{i}{\kappa,\zeta} \times$$
$$\left\{ {i-\kappa-\zeta \atop \delta+\epsilon} \right\} \binom{j}{\alpha,\epsilon,\eta,\theta} \binom{k}{\alpha,\delta,\lambda,\mu} \binom{r}{\nu} \begin{bmatrix} n \\ k-\alpha-\lambda \end{bmatrix} \begin{bmatrix} p+\lambda \\ j-\alpha-\eta \end{bmatrix} \times$$
$$a^{r+\kappa-\nu} b^{-k+n+\alpha+\eta+\lambda} c^{-j+p+\alpha+\eta+\lambda} d^{j+k+l+q-\alpha-\eta-\lambda} e^{m+s},$$

where $\omega = j+k-l-2\alpha-\beta-2\delta-2\epsilon-2\theta-2\mu-m\gamma$. (For $(\alpha-\beta)^\zeta$ we set $0^0 = 1$.)

Proof. Apply $L_{a^i b^j c^k d^l e^m}$ to $a^r b^n c^p d^q e^s$. □

7. The universal alternative enveloping algebra

By Lemma 6.3 of [1] we have

$$(c, ab, ab) = -bd, \quad (b, ac, ac) = cd, \quad (a, bc, bc) = 2d^2.$$

Using Theorem 6.7 (of the present paper) we calculate

$$(ac + be, ac + be, a) = (ac, be, a) + (be, ac, a) = de.$$

Let J be the ideal of $U(\mathbb{M})$ generated by $\{bd, cd, d^2, de\}$. We consider $U(\mathbb{M})/J$, and our goal is to prove that this is the universal alternative enveloping algebra of $\mathbb{M}$. Since (the cosets of) the elements d^2, cd, bd, de are zero in $U(\mathbb{M})/J$ we can reduce each basis monomial of $U(\mathbb{M})$ modulo J to either $a^i d$ (type 1) or $a^r b^n c^p e^s$ (type 2).

Lemma 7.1. *In $U(\mathbb{M})/J$ we have*

$$a^i d \cdot a^r d = 0,$$
$$a^i d \cdot a^r b^n c^p e^s = \delta_{0n} \delta_{0p} \delta_{0s} a^i (a-1)^r d,$$
$$a^i b^j c^k e^m \cdot a^r d = \delta_{j0} \delta_{k0} \delta_{m0} a^{i+r} d,$$
$$a^i b^j c^k e^m \cdot a^r b^n c^p e^s = a^i (a+j+k-\gamma m)^r b^{j+n} c^{k+p} e^{m+s} + \delta_{m,0} \delta_{s,0} \delta_{j+n,1} \delta_{k+p,1} T^{ir}_{jk},$$

where

$$T^{ir}_{jk} = \begin{cases} 0 & \text{if } (j,k) = (0,0), \\ (a-1)^{i+r} d - a^i (a+1)^r d & \text{if } (j,k) = (1,0), \\ -(a-1)^{i+r} d - a^i (a+1)^r d & \text{if } (j,k) = (0,1), \\ a^i (a-1)^r d - a^i (a+2)^r d & \text{if } (j,k) = (1,1). \end{cases}$$

Proof. The proof of the first equation, and of the last in the case $s = m = 0$, is given in Lemma 6.5 of [1]. For the second equation, we compute $L_{a^i d}$ using Lemma 6.4. Since $j = k = 0$ and $m = 0$ we have that $\alpha = \beta = 0$. Therefore $L_{a^i d}$ equals

$$\sum_{\kappa=0}^{i}\sum_{\delta=0}^{i-\kappa}\sum_{\epsilon=0}^{i-\kappa-\delta}(-1)^{i-\kappa-\delta}\delta!\epsilon!\binom{0}{0,\varepsilon}\binom{0}{0,\delta}X_i(\kappa,\delta,\epsilon)L_a^\kappa S^{-\delta-\epsilon}L_b^{-\epsilon}D_b^\delta D_c^\epsilon L_c^{-\delta}M_d^{\delta+\epsilon}L_d.$$

Clearly $\epsilon = \delta = 0$, and since $X_i(\kappa, 0, 0) = 0$ unless $\kappa = i$ by Lemma 6.5, we get $L_{a^i d} = L_a^i L_d$. Therefore in $U(\mathbb{M})/J$ we have

$$L_{a^i d}(a^r b^n c^p e^s) = M_a^i S^{-1} M_d(a^r b^n c^p e^s) = \delta_{s0}\delta_{n0}\delta_{p0}a^i(a{-}1)^r d.$$

For the third equation, we consider $L_{a^i b^j c^k e^m}$. Since the exponent of d is ≤ 1 in monomials of both types 1 and 2, we have $\alpha + \delta + \epsilon = 0$ in Lemma 6.4. Hence $\alpha = \delta = \epsilon = 0$ and so $\beta = 0$. Therefore,

$$L_{a^i b^j c^k e^m} = \sum_{\kappa=0}^{i}(-1)^{i-\kappa}X_i(\kappa,0,0)L_a^\kappa S^0 L_b^j L_c^k = L_a^i L_b^j L_c^k.$$

Using equations (12) and (13) we obtain the stated result. It remains only to consider the fourth equation in the case $s + m \neq 0$. Since the exponent of e in Theorem 6.7 is non-zero, the exponent of d must be zero. Hence $j{+}k{-}\alpha{-}\eta{-}\lambda = 0$ and so $\alpha + \eta + \lambda = j + k$. But then, since $\alpha + \eta \leq j$, $\lambda \leq k$, $\eta \leq j$, $\alpha + \lambda \leq k$, we see that $\lambda = k$, $\eta = j$, $\alpha = \beta = 0$, $\delta = \epsilon = \theta = \mu = \zeta = 0$. The sum collapses to

$$\sum_{\nu=0}^{r}(j{+}k{-}\gamma m)^\nu\binom{r}{\nu}a^{i+r-\nu}b^{j+n}c^{k+p}e^{m+s} = a^i(a{+}j{+}k{-}\gamma m)^r b^{j+n}c^{k+p}e^{m+s}.$$

The proof is complete. □

Theorem 7.2. *The universal alternative enveloping algebra* $A(\mathbb{M})$ *has the basis* $\{\, a^i d,\ a^i b^j c^k e^m \mid i, j, k \geq 0 \,\}$ *and the structure constants of Lemma 7.1.*

Proof. We compute all possible associators of basis monomials of $U(\mathbb{M})/J$ of types 1 and 2. Since the product of a monomial of type 1 with any monomial is a linear combination of monomials of type 1, and the product of two monomials of type 1 is zero, it is clear that every associator with two monomials of type 1 vanishes. Next consider an associator with one monomial of type 1 and two of type 2, for example

$$(a^i d, a^r b^n c^p e^s, a^l b^k c^t e^q) = (a^i d \cdot a^r b^n c^p e^s)a^l b^k c^t e^q - a^i d(a^r b^n c^p e^s \cdot a^l b^k c^t e^q).$$

If $s + q \neq 0$, then a straightforward application of Lemma 7.1 shows that the result is zero. If $s + q = 0$, then Theorem 6.6 of [1] shows that $(a^i d, a^r b^n c^p, a^l b^k c^t) = 0$. The other two cases are similar. We finally consider three monomials of type 2:

$$(a^i b^j c^k e^m, a^r b^n c^p e^s, a^l b^q c^t e^y) = A - B + C - D,$$

where

$$\begin{aligned} A - B &= a^i(a{+}j{+}k{-}\gamma m)^r b^{j+n}c^{k+p}e^{m+s}\cdot a^l b^q c^t e^y \\ &\quad - a^i b^j c^k e^m \cdot a^r(a{+}n{+}p{-}\gamma s)^l b^{n+q}c^{p+t}e^{s+y} \\ &= \delta_{j+n+q,1}\delta_{k+p+t,1}\left(\begin{array}{l} \sum_{\nu=0}^{r}\binom{r}{\nu}(j{+}k{-}\gamma m)^\nu\delta_{m+s,0}\delta_{y,0}T^{i+r-\nu,l}_{j+n,p+k} \\ \quad -\sum_{\mu=0}^{l}\binom{l}{\mu}(n{+}p{-}\gamma s)^\mu\delta_{m,0}\delta_{s+y,0}T^{i,r+l-\mu}_{jk}) \end{array}\right), \end{aligned}$$

$$C - D = \delta_{m,0}\delta_{s,0}\delta_{j+n,1}\delta_{k+p,1}T^{ir}_{jk}a^l b^q c^t e^y - \delta_{s0}\delta_{y0}\delta_{n+q,1}\delta_{p+t,1}a^i b^j c^k e^m T^{rl}_{np}.$$

If $y \neq 0$ then $s + y \neq 0$ and so $A - B = 0$; moreover, the second term in $C - D$ vanishes, and the second equation of Lemma 7.1 shows that the first term in $C - D$ also vanishes, so $C - D = 0$. Similar arguments apply if $m \neq 0$ or $s \neq 0$. Finally, if $s + m + y = 0$, then we know the value of the corresponding associator from Theorem 6.6 of [1], and hence the alternativity property is clear. □

Corollary 7.3. *Every Malcev algebra $\mathbb{M}_\gamma$ in the one-parameter family of solvable 5-dimensional Malcev algebras is special; that is, $\mathbb{M}_\gamma$ is isomorphic to a subalgebra of the commutator algebra of an alternative algebra.*

We can construct much smaller (but still infinite-dimensional) alternative enveloping algebras for the Malcev algebras $\mathbb{M}_\gamma$ as follows. We consider the algebra $\mathbb{A}_\gamma$ with basis $\{ a^r, b, c, d, e \mid r \in \mathbb{Z}, r \geq 1 \}$; the nonzero products of basis elements are $a^r \cdot a^s = a^{r+s}$ together with

$$(14) \qquad a^r \cdot d = d, \quad b \cdot a^r = b, \quad b \cdot c = d, \quad c \cdot a^r = c, \quad c \cdot b = -d, \quad e \cdot a^r = (-\gamma)^r e.$$

It is easy to check that $\mathbb{M}_\gamma$ is isomorphic to the subalgebra of $\mathbb{A}_\gamma^-$ with basis $\{a, b, c, d, e\}$. It remains to verify that $\mathbb{A}_\gamma$ is alternative; equivalently, that the associator (x, y, z) is a skew-symmetric function of $x, y, z \in \mathbb{A}_\gamma$. Since the associator is trilinear it suffices to check basis elements. We write $\mathbb{A}'_\gamma = \text{span}\{a^r, b, c, d\}$ and $\mathbb{I}_\gamma = \text{span}\{a^t - a^s\}$ for all $s, t > 0$. It follows from (14) that $\mathbb{A}'_\gamma$ is a subalgebra of $\mathbb{A}_\gamma$, that $\mathbb{I}_\gamma$ is an ideal in $\mathbb{A}'_\gamma$, and that $\mathbb{A}'_\gamma/\mathbb{I}_\gamma$ is isomorphic to the 4-dimensional alternative algebra of [1, Table 3]. Consider arbitrary elements $x+\mathbb{I}_\gamma$, $y+\mathbb{I}_\gamma$, $z+\mathbb{I}_\gamma$ in $\mathbb{A}'_\gamma/\mathbb{I}_\gamma$. Since the quotient algebra is alternative, we have

$$(x, y, z) + \mathbb{I}_\gamma = (x + \mathbb{I}_\gamma, y + \mathbb{I}_\gamma, z + \mathbb{I}_\gamma) = (y + \mathbb{I}_\gamma, x + \mathbb{I}_\gamma, z + \mathbb{I}_\gamma) = (y, x, z) + \mathbb{I}_\gamma.$$

Hence $(x, y, z) - (y, x, z) \in \mathbb{I}_\gamma$. But (14) implies that associators take values in $\mathbb{L} = \text{span}\{b, c, d, e\}$. Since $\mathbb{I}_\gamma \cap \mathbb{L} = \{0\}$, we obtain $(x, y, z) = (y, x, z)$. A similar argument shows that $(x, y, z) = (x, z, y)$, and so $\mathbb{A}'_\gamma$ is alternative.

It remains to consider the cases in which at least one of x, y, z is e. First, we show that $(e, x, y) = (x, e, y) = (x, y, e) = 0$ for any $x, y \in \mathbb{L}$: for this, it is easy to see that $\mathbb{L}^2 = \text{span}\{d\}$ and $\mathbb{L}^3 = \{0\}$. Second, suppose that $x = a^s$ and $y \in \text{span}\{b, c, d\}$, then $(e, a^s, y) = (e \cdot a^s)y - e(a^s \cdot y) = 0$ by (14); the other five permutations (e, a^s, y) are similar. Finally, for the remaining cases direct calculation easily shows that

$$(e, a^r, e) = (a^r, e, e) = (e, e, a^r) = (e, a^r, a^s) = (a^r, e, a^s) = (a^r, a^s, e) = 0.$$

Thus $\mathbb{A}_\gamma$ is alternative.

8. Conclusion

It follows from results of Elduque [4] that, unlike the 4-dimensional solvable Malcev algebra considered in [1], the Malcev algebras $\mathbb{M} = \mathbb{M}_\gamma$ in the one-parameter family of 5-dimensional solvable algebras are not isomorphic to subalgebras of the 7-dimensional simple Malcev algebra. This raises the question of the existence of a finite-dimensional alternative algebra $\mathbb{A}_\gamma$ for which the commutator algebra $\mathbb{A}_\gamma^-$ contains a subalgebra isomorphic to $\mathbb{M}_\gamma$. If such an algebra $\mathbb{A}_\gamma$ does not exist for some γ, then we have an example of a finite-dimensional special Malcev algebra which does not have a finite-dimensional enveloping alternative algebra. A related result is the generalization of the Ado theorem in [8]; this guarantees the existence of a finite-dimensional nonassociative enveloping algebra for any finite-dimensional Malcev algebra, but this enveloping algebra is not necessarily alternative. This leads to an important open problem: If $\mathbb{M}$ is a special finite-dimensional Malcev algebra, then does $\mathbb{M}$ necessarily have a finite-dimensional alternative envelope?

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