

Classification of trilinear operations

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Abstract

We use the representation theory of the symmetric group to classify up to equivalence all multilinear operations over the field \mathbb{Q} of rational numbers. In the case $n = 3$, we obtain explicit representatives of the equivalence classes of trilinear operations

$$[a, b, c] = x_1 abc + x_2 acb + x_3 bac + x_4 bca + x_5 cab + x_6 cba \quad (x_i \in \mathbb{Q}).$$

From these results we obtain one-parameter families of deformations of the classical Lie, Jordan and anti-Jordan triple products and the corresponding varieties of triple systems. For one representative of each equivalence class, we use computational algebra to study the nonassociative polynomial identities satisfied by the operation in every totally associative ternary algebra. We obtain 19 new trilinear operations for which the corresponding varieties of triple systems are defined by identities of degrees 3 and 5. For 10 of these operations we classify their obvious identities in degree 3 and their minimal identities in degree 5. Our main goal is to give new examples of classes of triple systems.

Introduction

We use the representation theory of the symmetric group S_n to give a definition of equivalence for multilinear n -ary operations, and to obtain a canonical representative of each equivalence class. On the underlying vector space of a totally associative n -ary algebra, a multilinear n -ary operation defines a new structure of nonassociative n -ary algebra. In the familiar case $n = 2$, we recover the Lie bracket and the Jordan product and their basic polynomial identities: respectively, the anticommutative and Jacobi identities, and the commutative and Jordan identities. In the case $n = 3$, the representatives of the equivalence classes of trilinear operations provide one-parameter families of deformations of the classical Lie, Jordan and anti-Jordan triple products, a new fourth family of operations, and another six isolated operations. Our results clarify the position of the classical triple systems in the complete classification of triple systems arising from trilinear operations. All of our operations satisfy certain “obvious” identities in degree 3. For one representative of each equivalence class, we use computational algebra to determine whether the operation satisfies additional “minimal” identities of degree 5 which are not consequences of the obvious identities; this gives a total of 19 new trilinear operations. For 10 of these operations we explicitly determine identities which imply all the identities in degree ≤ 5 . These identities define new varieties of triple systems and open up a wide territory for further investigation; our identities should be useful in developing the structure theory of these new systems. Our methods make extensive use of computational algebra, especially linear algebra on large matrices, the representation theory of the symmetric group, and a new evolutionary algorithm which will be discussed in detail in a companion paper to be published elsewhere.

1 Preliminaries

1.1 General definitions

An n -ary algebra is a vector space A over a field \mathbb{F} together with a multilinear map

$$p: \overbrace{A \times \cdots \times A}^{n \text{ factors}} \rightarrow A.$$

We usually write $a_1 \cdots a_n$ instead of $p(a_1, \dots, a_n)$. The n -ary algebra A is **totally associative** if it satisfies the polynomial identities

$$a_1 \cdots (a_i \cdots a_{i+n-1}) \cdots a_{2n-1} = a_1 \cdots (a_j \cdots a_{j+n-1}) \cdots a_{2n-1} \text{ for } 1 \leq i < j \leq n;$$

that is, any product of $2n-1$ factors depends only on the order of the factors, not the placement of the parentheses.

A **multilinear operation** on an n -ary algebra is a linear combination of permutations of the original operation:

$$[a_1, a_2, \dots, a_n] = \sum_{\sigma \in S_n} x_\sigma a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} \quad (x_\sigma \in \mathbb{F}).$$

If we regard the arguments a_i as indeterminates then a multilinear operation can be identified with an element of the group algebra $\mathbb{F}S_n$ of the symmetric group S_n . We say that two multilinear operations $[a_1, a_2, \dots, a_n]_1$ and $[a_1, a_2, \dots, a_n]_2$ are **equivalent** if each can be written as a linear combination of permutations of the other:

$$[a_1, a_2, \dots, a_n]_1 = \sum_{\sigma \in S_n} y_\sigma [a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}]_2 \quad (y_\sigma \in \mathbb{F}),$$

$$[a_1, a_2, \dots, a_n]_2 = \sum_{\sigma \in S_n} z_\sigma [a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}]_1 \quad (z_\sigma \in \mathbb{F}).$$

We can restate this definition as follows:

Lemma 1. *Let O_1 and O_2 in $\mathbb{F}S_n$ be multilinear n -ary operations. Then O_1 and O_2 are equivalent if and only if $O_1 = YO_2$ and $O_2 = ZO_1$ for some $Y, Z \in \mathbb{F}S_n$. In other words, two operations are equivalent if and only if they generate the same left ideal in $\mathbb{F}S_n$. \square*

Every multilinear operation O satisfies certain **obvious identities** in its defining degree. These are precisely the multilinear identities of degree n satisfied by O :

Lemma 2. *If $O \in \mathbb{F}S_n$ is a multilinear n -ary operation then any nonzero element $I \in \mathbb{F}S_n$ for which $IO = 0$ (the zero element) is an identity of degree n satisfied by O . The set of all such identities is a left ideal in $\mathbb{F}S_n$. \square*

As an example, for $n = 2$ take $O = ab - ba$ (the Lie bracket) and $I = ab + ba$ (the anticommutative identity); then $IO = 0$. If we reverse I and O , we have $O = ab + ba$ (the Jordan product) and $I = ab - ba$ (the commutative identity); and again $IO = 0$.

A **minimal identity** for a multilinear n -ary operation O over a fixed field \mathbb{F} is a multilinear identity I of lowest degree $> n$ for which one (hence both) of the following equivalent conditions holds, but which is not a consequence of the obvious identities:

- O satisfies I in every totally associative n -ary algebra over \mathbb{F} .
- O satisfies I in the free totally associative n -ary algebra over \mathbb{F} on an infinite number of generators.

The free totally associative n -ary algebra over \mathbb{F} on a generating set S can be realized as the n -ary subalgebra generated by the monomials of degree n in the free (binary) associative algebra over \mathbb{F} on the set S . So it is not essential to use n -ary algebras in this definition.

From now on we assume that $\mathbb{F} = \mathbb{Q}$, the field of rational numbers.

1.2 The binary case

Total associativity for a binary operation reduces to the familiar identity $(ab)c = a(bc)$. Up to equivalence, there are four bilinear operations $[a, b] = x_1 ab + x_2 ba$ ($x_i \in \mathbb{Q}$), the first and last of which are trivial:

- the zero operation $[a, b] = 0$,
- the Lie bracket $[a, b] = ab - ba$,

- the Jordan product $[a, b] = ab + ba$ (we use the standard notation $a \circ b = ab + ba$),
- the original operation $[a, b] = ab$.

Lemma 3. *Every minimal identity for the Lie bracket is implied by the Jacobi identity*

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0.$$

Proof. This can be verified by hand. Construct the 6×3 matrix whose rows are labeled by the six associative monomials (permutations of a, b, c) and whose columns are labeled by the three inequivalent anticommutative monomials $[[a, b], c]$, $[[a, c], b]$, $[[b, c], a]$. In each column put the coefficients of the expansion of the corresponding anticommutative monomial as a linear combination of associative monomials. The matrix has rank 2, and its nullspace is spanned by the Jacobi identity. \square

Lemma 4. *Every minimal identity for the Jordan product is implied by the linearized form of the Jordan identity*

$$((a \circ a) \circ b) \circ a = (a \circ a) \circ (b \circ a).$$

Proof. Verifying this requires a computer algebra system. The “expansion matrix” in this case has size 24×15 , since in degree four there are $4!$ associative monomials and 15 commutative monomials. For details, see Bremner, Murakami and Shestakov [7] (Section 9: Computational methods, Example 2). \square

Thus the two most important varieties of nonassociative binary algebras are defined by the obvious and minimal identities for the non-trivial bilinear operations.

1.3 The ternary case

Total associativity for a ternary operation reduces to the identities

$$(abc)de = a(bcd)e = ab(cde).$$

A trilinear operation is an expression of the form

$$[a, b, c] = x_1 abc + x_2 acb + x_3 bac + x_4 bca + x_5 cab + x_6 cba \quad (x_i \in \mathbb{Q}).$$

Three of these operations have received attention for their connections with Lie and Jordan structures. Their obvious identities in degree 3 together with their minimal identities in degree 5 define the three most important varieties of triple systems. In each case the obvious identities can be verified by hand (for an example see Lemma 10), but verifying the minimal identities requires a computer algebra system (for an example see Theorem 11):

- The **Lie triple product** is the operation

$$[a, b, c] = abc - bac - cab + cba.$$

(This can be written in terms of a binary operation as the iterated Lie bracket $[[a, b], c]$.)

The obvious and minimal identities for this operation are

$$[a, a, b] = 0,$$

$$[a, b, c] + [b, c, a] + [c, a, b] = 0,$$

$$[a, b, [c, d, e]] = [[a, b, c], d, e] + [c, [a, b, d], e] + [c, d, [a, b, e]].$$

These identities define the variety of **Lie triple systems**; see Hodge and Parshall [14].

- The **Jordan triple product** is the operation

$$[a, b, c] = abc + cba.$$

(This can be written in terms of a binary operation as $\frac{1}{2}((a \circ b) \circ c + (c \circ b) \circ a - b \circ (a \circ c))$.) The obvious and minimal identities for this operation are

$$[a, b, c] = [c, b, a],$$

$$[a, b, [c, d, e]] = [[a, b, c], d, e] - [c, [b, a, d], e] + [c, d, [a, b, e]].$$

These identities define the variety of **Jordan triple systems**; see Zelmanov [18].

- The **anti-Jordan triple product** is the operation

$$[a, b, c] = abc - cba.$$

(This can be written in terms of a binary operation as the Jordan triple product on the odd subspace of a special Jordan superalgebra.) The obvious and minimal identities for this operation are

$$[a, b, a] = 0,$$

$$[a, b, [c, d, e]] = [[a, b, c], d, e] + [c, [b, a, d], e] + [c, d, [a, b, e]].$$

These identities define the variety of **anti-Jordan triple systems**; see Faulkner and Ferrar [12].

In this paper we will see that there are many other trilinear operations which satisfy obvious identities in degree 3 and minimal identities in degree 5.

2 Classification of multilinear operations

2.1 Structure of the group algebra

The classical structure theory for $\mathbb{Q}S_n$ was discovered by Young [17]; see also James and Kerber [15].

Theorem 5. *For any partition λ of n , let d_λ be the number of standard tableaux. Then $\mathbb{Q}S_n$ can be represented as the direct sum over all λ of full matrix algebras of size $d_\lambda \times d_\lambda$:*

$$\pi: \mathbb{Q}S_n \xrightarrow{\cong} \bigoplus_{\lambda} M_{d_\lambda}(\mathbb{Q}).$$

The mapping π is an isomorphism of associative algebras. □

In the natural representation of S_n , a basis E_{ij}^λ ($1 \leq i, j \leq d_\lambda$) of $M_{d_\lambda}(\mathbb{Q})$ can be constructed as follows. Enumerate the standard tableaux as $T_1^\lambda, \dots, T_{d_\lambda}^\lambda$. For T_i^λ let R_i^λ be the sum over the row permutations and let C_i^λ be the alternating sum over the column permutations. Let s_{ij}^λ be the permutation which interchanges T_i^λ and T_j^λ . Define $E_{ij}^\lambda \in \mathbb{Q}S_n$ by

$$E_{ii}^\lambda = \frac{d_\lambda}{n!} C_i^\lambda R_i^\lambda \text{ and } E_{ij}^\lambda = E_{ii}^\lambda s_{ij}^\lambda \text{ (} i \neq j \text{)}.$$

Then for partitions λ, μ we have

$$E_{ij}^\lambda E_{kl}^\mu = \delta_{\lambda\mu} \epsilon_{jk}^\lambda E_{il}^\lambda \text{ where } \epsilon_{jk}^\lambda \in \{-1, 0, 1\}.$$

Clifton [10] gives a modification of E_{ij}^λ which satisfies the usual matrix unit relations

$$E_{ij}^\lambda E_{kl}^\mu = \delta_{\lambda\mu} \delta_{jk} E_{il}^\lambda.$$

The diagonal elements E_{ii}^λ are idempotent elements of $\mathbb{Q}S_n$.

2.2 Equivalence classes of operations

The isomorphism of Theorem 5 allows us to study multilinear operations in terms of matrix algebra. Lemma 1 depends only on the structure of $\mathbb{Q}S_n$ as a left $\mathbb{Q}S_n$ -module. For each λ , the E_{ij}^λ for fixed j form a basis of a simple left module: column j of $M_{d_\lambda}(\mathbb{Q})$. We can therefore use arguments from linear algebra even though the E_{ij}^λ do not satisfy the usual matrix unit relations. (In fact, for $n \leq 4$ we have $\epsilon_{jk} = \delta_{jk}$, so this is only a concern for $n \geq 5$.) An operation $O \in \mathbb{Q}S_n$ corresponds to a sequence of matrices $\pi(O)$ in the direct sum of matrix algebras; for component λ of this sequence we write $\pi(O)_\lambda \in M_{d_\lambda}(\mathbb{Q})$. An invertible element $X \in \mathbb{Q}S_n$ is represented by a sequence of invertible matrices $\pi(X)_\lambda \in M_{d_\lambda}(\mathbb{Q})$.

Proposition 6. *If $O_1, O_2 \in \mathbb{Q}S_n$ satisfy $O_1 = YO_2$ and $O_2 = ZO_1$ for some $Y, Z \in \mathbb{Q}S_n$ then we may assume that Y is invertible and $Z = Y^{-1}$.*

Proof. Theorem 5 implies that the assumptions are equivalent to

$$\pi(O_1)_\lambda = \pi(Y)_\lambda \pi(O_2)_\lambda \text{ and } \pi(O_2)_\lambda = \pi(Z)_\lambda \pi(O_1)_\lambda \text{ for all } \lambda.$$

From this it follows that $\text{rowspan}(\pi(O_1)_\lambda) \subseteq \text{rowspan}(\pi(O_2)_\lambda)$ and conversely. Hence the rowspaces are equal, and so $\pi(O_1)_\lambda$ and $\pi(O_2)_\lambda$ have the same row canonical form. Therefore we may take $\pi(Y)_\lambda$ to be an invertible matrix and $\pi(Z)_\lambda = (\pi(Y)_\lambda)^{-1}$. Since this holds for all λ , the conclusion follows. \square

Corollary 7. *As representatives of the equivalence classes of multilinear n -ary operations, we may take the operations $O \in \mathbb{Q}S_n$ for which all the matrices $\pi(O)_\lambda \in M_{d_\lambda}(\mathbb{Q})$ are in row canonical form.*

Proof. Proposition 6 implies that the operations O_1 and O_2 are equivalent if and only if for every λ there is an invertible matrix $\pi(Y)_\lambda$ for which $\pi(Y)_\lambda \pi(O_2)_\lambda = \pi(O_1)_\lambda$, and this condition is equivalent to the statement that for every λ the matrices $\pi(O_1)_\lambda$ and $\pi(O_2)_\lambda$ have the same row canonical form. \square

Suppose that we have two operations O_1 and O_2 satisfying the condition that for every λ the matrix $\pi(O_1)_\lambda$ is the row canonical form of the matrix $\pi(O_2)_\lambda$. Then for every λ there is an invertible matrix R_λ such that $R_\lambda \pi(O_2)_\lambda = \pi(O_1)_\lambda$. To find an explicit element $Y \in \mathbb{Q}S_n$ for which $YO_2 = O_1$, we use the inverse of the isomorphism π from Theorem 5. Since $\pi(Y)_\lambda \pi(O_2)_\lambda = \pi(O_1)_\lambda$ for every λ , we have $\pi(Y)_\lambda = R_\lambda$ and so

$$Y = \sum_{\lambda} \pi^{-1}(R_\lambda).$$

For an example see Lemma 12.

Equivalent operations do not necessarily satisfy the same polynomial identities, although the equivalence gives a linear isomorphism between the spaces of obvious identities in the defining degree n .

Proposition 8. *Let O_1 and O_2 be equivalent multilinear n -ary operations. Then there exists an invertible $Y \in \mathbb{Q}S_n$ such that for any obvious identity I satisfied by O_1 , the element IY^{-1} represents an obvious identity satisfied by O_2 .*

Proof. Since O_1 and O_2 are equivalent, by definition there exists an invertible $Y \in \mathbb{Q}S_n$ for which $O_2 = YO_1$. If I represents an obvious identity satisfied by O_1 , then $IO_1 = 0$ by Lemma 2. Since $O_1 = Y^{-1}O_2$ we obtain $IY^{-1}O_2 = 0$, and so IY^{-1} represents an obvious identity satisfied by O_2 . \square

Given two equivalent multilinear n -ary operations O_1 and O_2 , an important open problem is to determine the relation between the identities of degree $\geq 2n-1$ satisfied by the two operations. (Since n -ary monomials have degrees congruent to 1 modulo $n-1$, the next possible degree is $2n-1$). In particular, is there a bijection between the minimal identities for O_1 and O_2 ? A natural setting for this problem is the theory of operads, especially its application to classes of algebraic operations; see Markl, Shnider and Stasheff [16].

3 Trilinear operations

For the rest of this paper $n = 3$. We assume throughout that the underlying ternary algebra A is totally associative.

3.1 Explicit decomposition of $\mathbb{Q}S_3$

Let the symmetric group S_3 act on the set $\{a, b, c\}$. Each permutation $\sigma \in S_3$ will be represented as the word $\sigma(a)\sigma(b)\sigma(c)$. We write the partitions of 3 in the short form 3, 21, 111, and so $d_3 = 1$, $d_{21} = 2$, $d_{111} = 1$. The group algebra decomposes as the direct sum

$$\mathbb{Q}S_3 = \mathbb{Q} \oplus M_2(\mathbb{Q}) \oplus \mathbb{Q}.$$

For $\lambda = 3$ there is one standard tableau:

$$\boxed{\begin{array}{|c|c|c|} \hline a & b & c \\ \hline \end{array}}$$

The corresponding idempotent is the symmetric sum

$$S = \frac{1}{6}(abc + acb + bac + bca + cab + cba). \quad (1)$$

For $\lambda = 21$ there are two standard tableaux:

$$\begin{array}{|c|c|} \hline a & b \\ \hline c & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline a & c \\ \hline b & \\ \hline \end{array}$$

The corresponding idempotents are

$$E_{11} = \frac{1}{3}(abc - cba)(abc + bac) = \frac{1}{3}(abc + bac - bca - cba), \quad (2)$$

$$E_{22} = \frac{1}{3}(abc - bac)(abc + cba) = \frac{1}{3}(abc - bac - cab + cba). \quad (3)$$

Multiplying on the right by the transposition which interchanges the tableaux gives

$$E_{12} = \frac{1}{3}(abc + bac - bca - cba)acb = \frac{1}{3}(acb - bac + bca - cab), \quad (4)$$

$$E_{21} = \frac{1}{3}(abc - bac - cab + cba)acb = \frac{1}{3}(acb - bca + cab - cba). \quad (5)$$

The four elements E_{ij} span the subalgebra $M_2(\mathbb{Q})$ of $\mathbb{Q}S_3$ and satisfy the matrix unit relations

$$E_{ij}E_{kl} = \delta_{jk}E_{il}.$$

For $\lambda = 111$ there is one standard tableau:

$$\begin{array}{|c|} \hline a \\ \hline b \\ \hline c \\ \hline \end{array}$$

The corresponding idempotent is the alternating sum

$$A = \frac{1}{6}(abc - acb - bac + bca + cab - cba). \quad (6)$$

The basis elements S, E_{ij} ($1 \leq i, j \leq 2$), A satisfy the orthogonality relations

$$SE_{ij} = E_{ij}S = 0, \quad AE_{ij} = E_{ij}A = 0, \quad AS = SA = 0.$$

3.2 Change of basis in $\mathbb{Q}S_3$

We have two bases of $\mathbb{Q}S_3$: the first (the permutation basis) consists of the words in lexicographical order,

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba; \quad (7)$$

the second (the idempotent basis) consists of the matrix units in standard order,

$$S, \quad E_{11}, \quad E_{12}, \quad E_{21}, \quad E_{22}, \quad A. \quad (8)$$

We write M for the change of basis matrix from the latter to the former. The columns of M are the coefficients of the matrix units in terms of the words:

$$M = \frac{1}{6} \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & 1 \\ 1 & 0 & 2 & 2 & 0 & -1 \\ 1 & 2 & -2 & 0 & -2 & -1 \\ 1 & -2 & 2 & -2 & 0 & 1 \\ 1 & 0 & -2 & 2 & -2 & 1 \\ 1 & -2 & 0 & -2 & 2 & -1 \end{pmatrix} \quad (9)$$

The columns of M^{-1} are the coefficients of the words in terms of the matrix units:

$$M^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix} \quad (10)$$

3.3 Equivalence classes of trilinear operations

By Corollary 7, for representatives of the equivalence classes of trilinear operations, we can choose ordered triples of matrices with rational entries in row canonical form:

$$\left[x, \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}, z \right]$$

For the first and last components the possible row canonical forms are 0 and 1. For the middle component the possible row canonical forms are

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & q \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (q \in \mathbb{Q})$$

We exclude the two trivial cases: the zero operation (the zero element of $\mathbb{Q}S_3$, the zero matrix in each component), and the original totally associative operation (the identity element of $\mathbb{Q}S_3$, the identity matrix in each component). This leaves ten cases: four families of operations with a parameter q (each including the case $q = \infty$) and six isolated operations. On the different question of trilinear identities for triple systems, a classification using similar techniques has been obtained by Faulkner [11, Theorem 6].

In the following sections we determine the identities of degree ≤ 5 for these ten (families of) operations; that is, for one representative of each equivalence class of trilinear operations. The obvious identities in degree 3 can easily be computed by hand using Lemma 2 (for an example see Lemma 10). However, finding the minimal identities in degree 5 requires the use of a computer algebra system, since we need to compute the row canonical form of a large matrix, the “expansion matrix” (for an example see Theorem 11).

We order the operations by increasing rank of the middle component in the matrix form. We use without further comment the fact that the polynomial identities satisfied by an operation remain unchanged for nonzero scalar multiples of the operation.

4 The symmetric, alternating, and cyclic sums

These operations have the zero matrix as the middle component. Each can be expressed in two forms corresponding to the two bases of $\mathbb{Q}S_3$:

$$\text{symmetric sum} \quad \frac{1}{6}(abc + acb + bac + bca + cab + cba) \quad \left[1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right] \quad (11)$$

$$\text{alternating sum} \quad \frac{1}{6}(abc - acb - bac + bca + cab - cba) \quad \left[0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1 \right] \quad (12)$$

$$\text{cyclic sum} \quad \frac{1}{3}(abc + bca + cab) \quad \left[1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1 \right] \quad (13)$$

Each of these operations satisfies obvious identities in degree 3:

$$\text{symmetric sum} \quad [\sigma(a), \sigma(b), \sigma(c)] = [a, b, c] \quad \text{for all } \sigma \in S_3, \quad (14)$$

$$\text{alternating sum} \quad [\sigma(a), \sigma(b), \sigma(c)] = \varepsilon(\sigma)[a, b, c] \quad \text{for all } \sigma \in S_3, \quad (15)$$

$$\text{cyclic sum} \quad [\sigma(a), \sigma(b), \sigma(c)] = [a, b, c] \quad \text{for all } \sigma \in A_3. \quad (16)$$

Here $\varepsilon: S_3 \rightarrow \{\pm 1\}$ is the sign homomorphism and $A_3 = \ker \varepsilon$ is the alternating group. The polynomial identities of degree ≤ 9 satisfied by these operations in every totally associative ternary algebra have been studied by Bremner and Hentzel [3]; in each case the minimal identities have degree 7.

In the next four sections we consider the operations which have a matrix of rank 1 as the middle component.

5 The Lie family of operations

In this section we consider the deformation of the classical Lie triple product. This operation and its matrix form are

$$\frac{1}{3}(abc - bac - cab + cba) \quad \left[0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right] \quad (17)$$

An equivalent form of operation (17) is obtained by taking the row canonical form of each matrix:

$$\frac{1}{3}(acb - bac + bca - cab) \quad \left[0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0 \right] \quad (18)$$

Both of these operations satisfy the Jacobi identity $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$. But operation (17) satisfies $[a, a, b] = 0$, whereas operation (18) satisfies $[a, b, a] = 0$. In the group algebra $\mathbb{Q}S_3$, we obtain operation (18) from operation (17) by multiplying by acb (the transposition which interchanges b and c). We replace the middle component of operation (18) with another matrix of rank 1, and obtain a deformation of the Lie triple product.

Definition 9. *The deformed Lie triple product is the trilinear operation*

$$\frac{1}{3} \left(abc + q acb + (1-q)bac + (q-1)bca - q cab - cba \right) \quad \left[0, \begin{pmatrix} 1 & q \\ 0 & 0 \end{pmatrix}, 0 \right] \quad (q \in \mathbb{Q})$$

Operation (18) corresponds to the limiting case $q \rightarrow \infty$ in the sense of projective geometry: the ordered pair $(1, q)$ in row 1 of the 2×2 matrix corresponds to the point $q \in \mathbb{Q}$, and the ordered pair $(0, 1)$ corresponds to the point $q = \infty$. In this section we determine the identities of degree ≤ 5 for this family of operations.

Lemma 10. *In every totally associative ternary algebra, the deformed Lie triple product satisfies the obvious identities*

$$[a, b, c] + [c, b, a] = 0, \quad (19)$$

$$[a, b, c] + [b, c, a] + [c, a, b] = 0. \quad (20)$$

These two identities imply all the obvious identities of degree 3 satisfied by the deformed Lie triple product.

Proof. We give two proofs to illustrate the techniques of this paper. The first proof is an application of Lemma 2. We write O for the deformed Lie triple product, and I for an obvious identity that it satisfies. We have the matrix forms

$$I = \left[x, \begin{pmatrix} 0 & y_{12} \\ 0 & y_{22} \end{pmatrix}, z \right] \quad O = \left[0, \begin{pmatrix} 1 & q \\ 0 & 0 \end{pmatrix}, 0 \right]$$

Here I represents all solutions to the matrix equation $IO = 0$; the subspace of $\mathbb{Q}S_3$ represented by I is a left ideal. Using the matrix M^{-1} from equation (10), we find the matrix forms of the anticommutative identity I_1 (19) and the ternary Jacobi identity I_2 (20):

$$I_1 = \left[2, \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}, 0 \right] \quad I_2 = \left[3, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 3 \right]$$

Both I_1 and I_2 have the form I , so $I_1O = I_2O = 0$. Furthermore, the identities I_1 and I_2 generate the left ideal represented by I : taking linear combinations of triples of matrices row-equivalent to I_1 and I_2 gives all the identities of the form I . We have shown that I_1 and I_2 are satisfied by O , and that they imply all the identities satisfied by O in degree 3.

$$\mathcal{X} = \frac{1}{3} \begin{pmatrix} 1 & q & 1-q & -q & q-1 & -1 \\ q & 1 & q-1 & -1 & 1-q & -q \\ 1-q & -q & 1 & q & -1 & q-1 \\ q-1 & -1 & q & 1 & -q & 1-q \\ -q & 1-q & -1 & q-1 & 1 & q \\ -1 & q-1 & -q & 1-q & q & 1 \end{pmatrix}$$

Table 1: The expansion matrix for the obvious identities of the deformed Lie triple product

The second proof uses a small version of the “expansion matrix” that we will need for the minimal identities of degree 5. Let \mathcal{X} be the 6×6 matrix whose rows are labeled by the associative monomials (the six permutations of a, b, c), whose columns are labeled by the nonassociative monomials (the six permutations of the arguments in the trilinear operation $[a, b, c]$), and whose entry \mathcal{X}_{ij} is the coefficient of the i -th associative monomial in

the expansion of the j -th nonassociative monomial. (Here expansion refers to an application of the definition of the deformed Lie triple product.) The matrix \mathcal{X} is displayed in Table 1. We compute the row canonical form of \mathcal{X} over the polynomial ring $\mathbb{Q}[q]$ (we can only divide by elements of \mathbb{Q}). The matrix has rank 2, and a nullspace of dimension 4. The nonzero rows of the row canonical form, and a matrix whose rows form a basis of the nullspace, are

$$\begin{pmatrix} 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The basis vectors of the nullspace represent the identities

$$\begin{aligned} -[a, b, c] + [a, c, b] + [b, a, c] &= 0, & [a, c, b] + [b, c, a] &= 0, \\ [a, b, c] - [a, c, b] + [c, a, b] &= 0, & [a, b, c] + [c, b, a] &= 0. \end{aligned}$$

The second and fourth are forms of the anticommutative identity (19), and using this we see that the first and third are forms of the ternary Jacobi identity (20). This shows that these two identities are satisfied by the deformed Lie triple product for all values of q , and that they imply all the obvious identities of degree 3. \square

Theorem 11. *In every totally associative ternary algebra, the deformed Lie triple product satisfies no new identities in degree 5, except in two cases:*

$$q = \infty: \quad [a, b, c] = \frac{1}{3}(acb - bac + bca - cab), \quad (21)$$

$$q = \frac{1}{2}: \quad [a, b, c] = \frac{1}{6}(2abc + acb + bac - bca - cab - 2cba). \quad (22)$$

For operation (21) we have the ternary derivation identity:

$$[a, b, [c, d, e]] = [[a, b, c], d, e] + [c, [a, b, d], e] + [c, d, [a, b, e]]. \quad (23)$$

For operation (22) multiplication by a “square” is a ternary derivation:

$$[[a, b, c], d, d] = [[a, d, d], b, c] + [a, [b, d, d], c] + [a, b, [c, d, d]]. \quad (24)$$

These identities (or the linearized form for $q = 1/2$) imply all the identities in degree 5 for operations (21, 22) which do not follow from the obvious identities in degree 3. For all other values of q the minimal identities have degree ≥ 7 .

Proof. We consider only the case of finite q ; the case $q = \infty$ is similar, but simpler, since the expansion matrix has scalar entries. In degree 5 there are three association types for a ternary operation:

$$[[a, b, c], d, e], \quad [a, [b, c, d], e], \quad [a, b, [c, d, e]]. \quad (25)$$

The linearization of identity (19) shows that any monomial in the third type can be rewritten in the first type:

$$[a, b, [c, d, e]] = -[[c, d, e], b, a]. \quad (26)$$

Combining this with identity (20) shows that any monomial in the second type can be rewritten in the first type:

$$[a, [b, c, d], e] = -[[b, c, d], e, a] - [e, a, [b, c, d]] = -[[b, c, d], e, a] + [[b, c, d], a, e]. \quad (27)$$

Therefore any identity of degree 5 for this operation can be written as a linear combination of monomials in the first association type. The same reasoning shows that we only need to consider the monomials in which the inner triple has the property that the lexicographically first letter occurs in the leftmost position:

$$[c, b, a] = -[a, b, c], \quad (28)$$

$$[b, a, c] = -[a, c, b] - [c, b, a] = -[a, c, b] + [a, b, c]. \quad (29)$$

This reduces the number of nonassociative monomials to $5!/3 = 40$.

1	q	q	q^2
$abcde$	$abcd$	$acbde$	$acbed$
$1-q$	$-(q-1)q$	$q-1$	$(q-1)q$
$bacde$	$bacd$	$bcade$	$bcaed$
$-q$	$-q^2$	-1	$-q$
$cabde$	$cabed$	$cbade$	$cbaed$
$1-q$	$-(q-1)q$	$(q-1)^2$	$-(q-1)^2$
$dabce$	$dacbe$	$dbace$	$dbcae$
$(q-1)q$	$q-1$	$q-1$	$(q-1)q$
$dcabe$	$dcbae$	$deabc$	$deacb$
$-(q-1)^2$	$(q-1)^2$	$-(q-1)q$	$1-q$
$debac$	$debca$	$decab$	$decba$
$-q$	$-q^2$	$(q-1)q$	$-(q-1)q$
$eabcd$	$eachd$	$ebacd$	$ebcad$
q^2	q	-1	$-q$
$ecabd$	$ecbad$	$edabc$	$edacb$
$q-1$	$1-q$	q	1
$edbca$	$edbca$	$edcab$	$edcba$

Table 2: Expansion of $[[a, b, c], d, e]$ using the deformed Lie triple product

We now construct the expansion matrix \mathcal{X} of size 120×40 . The columns are labeled by the nonassociative monomials and the rows are labeled by the associative monomials (both in lexicographical order). The columns represent the expansions of the nonassociative monomials obtained by interpreting the trilinear operation as the deformed Lie triple product: \mathcal{X}_{ij} is the coefficient of the i -th associative monomial in the expansion of the j -th nonassociative monomial. The expansion of (9 times) $[[a, b, c], d, e]$ appears in Table 2: the associative monomials are in lexicographical order and are preceded by their coefficients. Each column of \mathcal{X} contains 36 nonzero entries which are polynomials in q of degree ≤ 2 . The i -th row of \mathcal{X} represents the linear relation on the coefficients of a polynomial identity which expresses the condition that when the identity is expanded using the deformed Lie triple product, the coefficient of the i -th associative monomial must be 0. The nonzero elements of the nullspace of \mathcal{X} represent the identities satisfied by the deformed Lie triple product in every totally associative ternary algebra.

In order to determine how the nullspace of \mathcal{X} depends on the parameter q , we use Maple's `linalg[smith]` procedure to compute the Smith normal form of \mathcal{X} (Adkins and Weintraub [1], Section 5.3): we used this procedure rather than `LinearAlgebra[SmithForm]` since (for

some unknown reason) the earlier package seemed more efficient on this particular matrix. The diagonal entries of the Smith form of \mathcal{X} are

$$1 \text{ (24 times)}, \quad q - \frac{1}{2} \text{ (16 times)}.$$

Therefore the expansion matrix has full rank (zero nullspace, no identities) except when $q = 1/2$, in which case there is a 16-dimensional nullspace of identities.

We now fix $q = 1/2$ and compute the nullspace using Maple. One of the 16 basis vectors for the nullspace represents the identity

$$\begin{aligned} & [[a, c, b], d, e] + [[a, c, b], e, d] - [[a, d, e], c, d] - [[a, e, d], c, b] + [[b, d, e], c, a] \\ & + [[b, e, d], c, a] - [[c, d, e], a, b] + [[c, d, e], b, a] - [[c, e, d], a, b] + [[c, e, d], b, a] = 0. \end{aligned} \quad (30)$$

This identity alternates in a, b and is symmetric in d, e . If we set $d = e$, interchange b and c , and divide by 2, we obtain the identity

$$[[a, b, c], d, d] = [[a, d, d], b, c] + [[b, d, d], a, c] - [[b, d, d], c, a] - [[c, d, d], b, a],$$

which now alternates in a, c . Using equations (26) and (27) we see that this is equivalent to the identity in the statement of the Theorem. We now apply all permutations of a, b, c, d, e to identity (30), straighten the inner triples using equations (28) and (29), and store the results in a matrix of size 120×40 . We use Maple to compute the rank of this matrix, which is 16; so this identity generates the entire space of identities under the action of the symmetric group. \square

λ	d_λ	lifted and expansion	lifted	$q = \infty, 1/2$	
				all	new
5	1	3	3	3	0
41	4	12	11	11	0
32	5	15	13	14	1
311	6	18	16	17	1
221	5	15	13	14	1
2111	4	12	11	11	0
11111	1	3	3	3	0

Table 3: Ranks of representations for the Lie family

We verified these results in a different way using the representation theory of the symmetric group S_5 . The following description is rather brief; for more detailed discussions of these methods, see our earlier papers: Bremner and Hentzel [4, 5, 6], Bremner and Peresi [8], Hentzel and Peresi [13].

For each partition of 5 there is a corresponding irreducible representation of S_5 . The group algebra $\mathbb{Q}S_5$ decomposes as the direct sum of full matrix algebras according to Theorem 5. We consider four association types: the first represents the original totally associative operation,

and the other three are the nonassociative types (25). We form the direct sum of four copies of the group algebra $\mathbb{Q}S_5$, one copy for each association type. We lift the obvious identities in degree 3 to degree 5 in every possible way (replacing one variable by a triple, or embedding the identity in a triple) and compute the rank of these identities for each representation of S_5 : these identities lie entirely in the last three association types. We represent the expansions of the basic monomials (25) by storing the expansion in the associative type and the basic monomial in the appropriate nonassociative type: these identities relate the first type with the last three types. We then compute the rank of these expansions for each representation of S_5 ; identities which occur only in the last three types, and do not follow from the liftings of the obvious identities, will be new identities for the operation. After computing the row canonical forms of the resulting matrices, we can tell which representations of S_5 contain new identities for the operation. These results are summarized in Table 3: column 1 is the partition of 5, column 2 is the dimension of the corresponding irreducible representation of S_5 , column 3 is the rank of the combined lifted and expansion identities (this is $3d_\lambda$, the dimension of the space of nonassociative polynomials in the last three association types), column 4 is the rank of the lifted identities alone, column 5 is the rank (in the last three types only) of the combined lifted and expansion identities, and column 6 is the difference of columns 4 and 5. There is one new identity for each of the three partitions 32, 311, 221, giving a total dimension of $5 + 6 + 5 = 16$, which confirms our earlier results. For $q = \infty$ (equivalent to the classical case) the ranks are the same but the identities are different.

6 The Jordan family of operations

In this section we consider the deformation of the classical Jordan triple product. This operation and its matrix form are

$$abc + cba \quad \left[2, \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}, 0 \right] \quad (31)$$

An operation equivalent to the classical Jordan triple product is

$$\frac{1}{6}(abc + 3acb - bac + 3bca - cab + cba) \quad \left[1, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0 \right] \quad (32)$$

If we take the row canonical form of each matrix in (31) we obtain (32). Note that the obvious identity $[a, b, c] = [c, b, a]$ for operation (31) does not hold for operation (32).

Lemma 12. *Let O_1 be operation (32) and let O_2 be operation (31). Then for any nonzero $r \in \mathbb{Q}$, the group algebra element*

$$Y_r = \frac{1}{12} \left((1+2r)abc + (9-2r)acb - (7+2r)bac - (3-2r)bca + (5+2r)cab + (1-2r)cba \right),$$

is invertible and satisfies $Y_r O_2 = O_1$.

Proof. For each λ we need to find the inverse image under the isomorphism π from Theorem 5 of an invertible $d_\lambda \times d_\lambda$ matrix R_λ which satisfies $R_\lambda \pi(O_2)_\lambda = \pi(O_1)_\lambda$. The matrices R_λ

and their inverse images $\pi^{-1}(R_\lambda)$ are as follows:

$$\begin{aligned} R_3 &= \frac{1}{2} & R_{21} &= \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} & R_{111} &= r \in \mathbb{Q} - \{0\} \\ \pi^{-1}(R_3) &= \frac{1}{2}S & \pi^{-1}(R_{21}) &= -E_{11} + 2E_{21} + E_{22} & \pi^{-1}(R_{111}) &= rA \end{aligned}$$

With respect to the idempotent basis the element Y_r has coefficients $[1/2, -1, 0, 2, 1, r]$. Multiplying this (column) vector by the change-of-basis matrix M from equation (9) gives the coefficients of Y_r with respect to the permutation basis. \square

Definition 13. *The deformed Jordan triple product is the trilinear operation*

$$\frac{1}{3} \left(2q abc + (3-q)acb - q bac - q bca + (3-q)cab + 2q cba \right) \quad \left[2, \begin{pmatrix} -1 & -q \\ 2 & 2q \end{pmatrix}, 0 \right]$$

In this section we determine the identities of degree ≤ 5 for this family of operations.

Lemma 14. *In every totally associative ternary algebra, the deformed Jordan triple product satisfies the obvious identity*

$$[a, b, c] = [c, b, a]. \quad (33)$$

This identity implies all the obvious identities of degree 3 satisfied by the deformed Jordan triple product.

Proof. Similar to the proof of Lemma 10. \square

Theorem 15. *In every totally associative ternary algebra, the deformed Jordan triple product satisfies the following new identities in degree 5. In the case*

$$q = \infty: \quad [a, b, c] = abc + cba, \quad (34)$$

we have the identity

$$[a, b, [c, d, e]] = [[a, b, c], d, e] - [c, [b, a, d], e] + [c, d, [a, b, e]], \quad (35)$$

which does not follow from the obvious identity, and which implies all the identities of degree 5 satisfied by operation (34). The deformed Jordan triple product satisfies the following identity in degree 5 for all finite values of q (from now on we omit the commas in nonassociative monomials of degree 5):

$$\begin{aligned} & [[abc]de] + [[abc]ed] + [[abd]ce] + [[abd]ec] + [[abe]cd] + [[abe]dc] \\ & + [[cbd]ae] + [[cbd]ea] + [[cbe]ad] + [[cbe]da] + [[dbe]ac] + [[dbe]ca] \\ & - [a[bcd]e] - [a[bce]d] - [a[bdc]e] - [a[bde]c] - [a[bec]d] - [a[bed]c] \\ & - [c[adb]e] - [c[aeb]d] - [c[bad]e] - [c[bae]d] - [d[acb]e] - [d[bac]e] = 0. \end{aligned} \quad (36)$$

This identity does not follow from the obvious identity, and it implies all the identities of degree 5 for the deformed Jordan triple product that hold for all finite values of q . The

deformed Jordan triple product satisfies additional identities in degree 5 in the following three cases:

$$q = 0: \quad [a, b, c] = acb + cab, \quad (37)$$

$$q = \frac{1}{2}: \quad [a, b, c] = \frac{1}{6}(2abc + 5acb - bac - bca + 5cab + 2cba), \quad (38)$$

$$q = 1: \quad [a, b, c] = \frac{1}{3}(2abc + 2acb - bac - bca + 2cab + 2cba). \quad (39)$$

These operations satisfy the identities

$$q = 0: \quad [[acb]de] + [[ace]db] + [[bce]da] - [a[bd]c]e] - [a[cd]e]b] - [b[adc]e] = 0, \quad (40)$$

$$\begin{aligned} q = \frac{1}{2}: \quad & - [[acd]be] - [[ace]bd] + 2[[adc]be] + 2[[aec]bd] - [[cad]be] - [[cae]bd] \\ & + [[dbe]ac] + [[dbe]ca] - 2[a[dbe]c] + 2[d[abc]e] - [d[acb]e] - [d[bac]e] = 0, \end{aligned} \quad (41)$$

$$\begin{aligned} q = 1: \quad & 8[[abc]bb] + 4[[acb]bb] - 4[[bac]bb] + 4[[bbc]ab] - 8[[bbc]ba] \\ & - 3[[bcb]ab] - 4[a[bbc]b] + 2[a[bc]b] - 2[b[acb]b] + 3[b[bac]b] = 0. \end{aligned} \quad (42)$$

These identities (or the linearized form for $q = 1$) imply (respectively) all the identities in degree 5 for operations (37, 38, 39) which do not follow from the obvious identity.

Proof. We consider only the case of finite q ; the case $q = \infty$ is similar, but simpler, since the expansion matrix has scalar entries. The identity of Lemma 14 shows that any monomial in the third type can be rewritten in the first type:

$$[a, b, [c, d, e]] = [[c, d, e], b, a]. \quad (43)$$

Hence any identity of degree 5 for this operation can be written as a linear combination of monomials in the first two types. For the first type, we only need the monomials in which the first letter lexicographically precedes the third letter:

$$[[c, b, a], d, e] = [[a, b, c], d, e],$$

giving $5!/2 = 60$ distinct monomials. For the second type, we only need the monomials in which the first precedes the fifth and the second precedes the fourth:

$$[e, [d, c, b], a] = [e, [b, c, d], a] = [a, [d, c, b], e] = [a, [b, c, d], e],$$

giving $5!/4 = 30$ distinct monomials. The total number of nonassociative monomials is 90, and so the expansion matrix \mathcal{X} in this case has size 120×90 .

As before, we use the Maple procedure `linalg[smith]` to compute the Smith normal form of \mathcal{X} . The diagonal entries of the Smith form of \mathcal{X} are

$$1 \text{ (64 times)}, \quad q - \frac{1}{2} \text{ (6 times)}, \quad q \left(q - \frac{1}{2} \right) (q-1) \text{ (15 times)}, \quad 0 \text{ (5 times)}.$$

This shows that the expansion matrix never has full rank: there is a 5-dimensional space of identities which hold for all q . For $q = 0$ and $q = 1$ this extends to a 20-dimensional space. For $q = 1/2$ it extends further to a 26-dimensional space. In each of these cases we took the expansion matrix, set q equal to the appropriate value, computed the nullspace of the matrix, and found a basis vector which generated the entire space of identities. For the case $q = 1$, the identities produced by this method were very complicated, with many large and apparently random coefficients. To find the much simpler but equivalent identity (42) we used an evolutionary algorithm which is described in detail in the companion paper; see Bremner [2]. For the identity which holds for all q , we used the value $q = -1$; from the Smith form we know that the identities for any value of q except $q = 0, 1/2, 1$ are the same as for q indeterminate. \square

λ	lifted	$q = \infty$		general		$q = 0, 1$		$q = 1/2$	
		all	new	all	new	all	new	all	new
5	1	2	1	2	1	2	1	2	1
41	7	10	3	8	1	9	2	8	1
32	10	12	2	10	0	11	1	12	2
311	14	16	2	14	0	15	1	15	1
221	12	13	1	12	0	12	0	13	1
2111	11	11	0	11	0	11	0	11	0
11111	3	3	0	3	0	3	0	3	0

Table 4: Ranks of representations for the Jordan family

We verified these results using the representation theory of S_5 ; the results are displayed in Table 4. The column labeled “general” has the results for $q = -1$. For $q = 0, 1$ the ranks are the same, but the identities are different.

7 The anti-Jordan family of operations

In this section we consider the deformation of the classical anti-Jordan triple product. This operation, its matrix form, and its row canonical form, are

$$abc - cba \quad \left[0, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, 2 \right] \quad \left[0, \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}, 1 \right] \quad (44)$$

(The equivalent form is obtained simply by multiplying by $1/2$.) We therefore consider the following operation and its deformation:

$$\frac{1}{6}(abc + acb - 3bac + 3bca - cab - cba) \quad \left[0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 1 \right] \quad (45)$$

Definition 16. *The deformed anti-Jordan triple product is the trilinear operation*

$$\frac{1}{6} \left(3abc + (2q-1)acb + (1-2q)bac + (2q-1)bca + (1-2q)cab - 3cba \right) \quad \left[0, \begin{pmatrix} 1 & q \\ 0 & 0 \end{pmatrix}, 1 \right]$$

The anti-Jordan triple product is equivalent to the deformed operation for $q = 1/2$. In this section we determine the identities of degree ≤ 5 for this family of operations.

Lemma 17. *In every totally associative ternary algebra, the deformed anti-Jordan triple product satisfies the obvious identity*

$$[a, b, c] + [c, b, a] = 0. \quad (46)$$

This identity implies all the obvious identities of degree 3 satisfied by the deformed anti-Jordan triple product.

Proof. Similar to the proof of Lemma 10. □

Theorem 18. *In every totally associative ternary algebra, the deformed anti-Jordan triple product satisfies the following new identities in degree 5. In the case*

$$q = \infty: \quad [a, b, c] = \frac{1}{6}(abc + acb - 3bac + 3bca - cab - cba), \quad (47)$$

we have the identity

$$\begin{aligned} & [[abc]de] - [[acb]de] - [[ade]bc] + [[ade]cb] + 2[[bac]de] - [[bce]da] \\ & - 2[[bec]da] + [[cbe]da] + [a[bcd]e] + 2[a[bdc]e] - [a[cbd]e] - 2[b[ade]c] = 0, \end{aligned} \quad (48)$$

which does not follow from the obvious identity, and which implies all the identities of degree 5 satisfied by operation (47). (Identity (48) alternates in the variables b, c .) The deformed anti-Jordan triple product satisfies the following identity in degree 5 for all finite values of q :

$$\begin{aligned} & - [[adb]ce] + [[adb]ec] + [[adc]be] - [[adc]eb] - [[ade]bc] + [[ade]cb] \\ & - [[bdc]ae] + [[bdc]ea] + [[bde]ac] - [[bde]ca] - [[cde]ab] + [[cde]ba] \\ & + [a[bcd]e] - [a[bed]c] - [a[cbd]e] + [a[ced]b] - [a[dbe]c] + [a[dce]b] \\ & - [b[acd]e] + [b[aed]c] + [b[cad]e] + [b[dae]c] + [c[abd]e] - [c[bad]e] = 0. \end{aligned} \quad (49)$$

This identity does not follow from the obvious identity, and it implies all the identities of degree 5 for the deformed anti-Jordan triple product that hold for all finite values of q . The deformed anti-Jordan triple product also satisfies additional identities in the following three cases:

$$q = -1: \quad [a, b, c] = \frac{1}{2}(abc - acb + bac - bca + cab - cba), \quad (50)$$

$$q = \frac{1}{2}: \quad [a, b, c] = \frac{1}{2}(abc - cba), \quad (51)$$

$$q = 2: \quad [a, b, c] = \frac{1}{2}(abc + acb - bac + bca - cab - cba). \quad (52)$$

These operations satisfy the identities

$$\begin{aligned} q = -1: \quad & [[acb]de] - [[acb]ed] - [[acd]eb] + [[ace]db] - [[adb]ce] + [[adb]ec] \\ & [[adc]eb] - [[ade]cb] + [[aeb]cd] - [[aeb]dc] - [[aec]db] + [[aed]cb] \end{aligned}$$

$$\begin{aligned}
& - [[cad]be] + [[cad]eb] + [[cae]bd] - [[cae]db] - [[dae]bc] + [[dae]cb] \\
& - [c[abd]e] + [c[abe]d] + [c[adb]e] - [c[aeb]d] + [d[abc]e] - [d[acb]e] = 0, \quad (53)
\end{aligned}$$

$$q = \frac{1}{2}: \quad [a, b, [c, d, e]] = [[a, b, c], d, e] + [c, [b, a, d], e] + [c, d, [a, b, e]], \quad (54)$$

$$\begin{aligned}
q = 2: \quad & [[abc]de] - [[abd]ce] + [[acd]be] + [[ace]bd] - [[ace]db] - [[adc]be] \\
& - [[ade]bc] + [[ade]cb] - [[cad]be] - [[cae]bd] + [[cae]db] + [[cbd]ae] \\
& - [[cde]ab] + [[cde]ba] + [[dae]bc] - [[dae]cb] + [[dce]ab] - [[dce]ba] \\
& - [a[cbd]e] - [b[acd]e] + [b[adc]e] + [b[cad]e] + [c[abd]e] - [d[abc]e] = 0. \quad (55)
\end{aligned}$$

These identities imply (respectively) all the identities in degree 5 for operations (50, 51, 52) which do not follow from the obvious identity.

Proof. For $q = \infty$, the expansion matrix has size 120×90 (with scalar entries). It has rank 64, and hence its nullspace has dimension 26. One of the basis vectors of the nullspace corresponds to the identity of the Theorem. We compute the submodule generated by this identity, and find that it has dimension 26.

The rest of the proof is similar to the Jordan case. The diagonal entries of the Smith form of the expansion matrix \mathcal{X} are

$$1 \text{ (50 times)}, \quad q - \frac{1}{2} \text{ (20 times)}, \quad (q+1)(q-2) \left(q - \frac{1}{2} \right) \text{ (15 times)}, \quad 0 \text{ (5 times)}.$$

The expansion matrix never has full rank: there is a 5-dimensional space of identities which hold for all q . For $q = -1$ and $q = 2$ this extends to a 20-dimensional space. For $q = 1/2$ it extends further to a 40-dimensional space. In each of the non-classical cases ($q \neq 1/2$) we took the expansion matrix, set q equal to the appropriate value, computed the nullspace of the matrix, and found a basis vector which generated the entire space of identities. For the identity which holds for all q , we used the value $q = 1$; from the Smith form we know that the identities for any finite value of q except $q = -1, 1/2, 2$ are the same as for q indeterminate. \square

λ	lifted	$q = \infty$		general		$q = -1, 2$		$q = 1/2$	
		all	new	all	new	all	new	all	new
5	3	3	0	3	0	3	0	3	0
41	11	11	0	11	0	11	0	11	0
32	12	13	1	12	0	12	0	13	1
311	14	15	1	14	0	15	1	16	2
221	10	12	2	10	0	11	1	12	2
2111	7	8	1	8	1	9	2	10	3
11111	1	2	1	2	1	2	1	2	1

Table 5: Ranks of representations for the anti-Jordan family

We verified these results using the representation theory of S_5 ; the results are displayed in Table 5. The column labeled “general” has the results for $q = 1$. For $q = -1, 2$ the

ranks are the same, and the identities are also the same: in other words, the identities (53) and (55) imply each other. This provides an example of two inequivalent operations which satisfy the same minimal identities.

8 The fourth family of operations

This is the only parameterized family which appears not to be related to the classical Lie, Jordan and anti-Jordan triple systems. The basic operation, which we call the “fourth operation” for want of a better name, is

$$\frac{1}{3}(abc + acb - bac + 2bca) \quad \left[1, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 1\right] \quad (56)$$

To get this operation we add the cyclic sum (13) and the equivalent form (18) of the classical Lie triple product.

Definition 19. *The deformed fourth operation is the trilinear operation*

$$\frac{1}{3}\left(2abc + q acb + (1-q)bac + q bca + (1-q)cab - cba\right) \quad \left[1, \begin{pmatrix} 1 & q \\ 0 & 0 \end{pmatrix}, 1\right]$$

This family of operations (and the three in the next section) are significantly more obscure than the operations studied in the previous sections, especially in view of their lack of connection with triple systems that have appeared in the literature of nonassociative algebra. For this reason, our discussions in this section and the next will be relatively brief.

Lemma 20. *In every totally associative ternary algebra, the deformed fourth operation satisfies the obvious identity*

$$[a, c, b] - [b, a, c] + [b, c, a] - [c, a, b] = 0. \quad (57)$$

This identity implies all the obvious identities of degree 3 satisfied by the deformed fourth operation.

Proof. Similar to the proof of Lemma 10. □

Theorem 21. *In every totally associative ternary algebra, the deformed fourth operation satisfies the following new identities in degree 5. For the case of general finite q , and for the following special values of q , the deformed fourth operation satisfies additional identities which do not follow from the obvious identity:*

$$q = \infty: \quad [a, b, c] = \frac{1}{3}(abc + acb - bac + 2bca), \quad (58)$$

$$q = 0: \quad [a, b, c] = \frac{1}{3}(2abc + bac + cab - cba), \quad (59)$$

$$q = 1: \quad [a, b, c] = \frac{1}{3}(2abc + acb + bca - cba), \quad (60)$$

$$q = -1: \quad [a, b, c] = \frac{1}{3}(2abc - acb + 2bac - bca + 2cab - cba), \quad (61)$$

$$q = 2: \quad [a, b, c] = \frac{1}{3}(2abc + 2acb - bac + 2bca - cab - cba), \quad (62)$$

$$q = \frac{1}{2}: \quad [a, b, c] = \frac{1}{6}(4abc + acb + bac + bca + cab - 2cba). \quad (63)$$

For $q = \infty$ there is a 54-dimensional space of identities in degree 5. There is a 40-dimensional space of identities in degree 5 which hold for all finite q . For $q = 0, 1, -1, 2$ this extends to a 49-dimensional space. For $q = 1/2$ it extends further to a 54-dimensional space.

Proof. In the identity of Lemma 20, if we interchange a, b and rearrange the terms, we get

$$[a, b, c] = [a, c, b] + [b, c, a] - [c, b, a].$$

This shows that any monomial in the third association type is a linear combination of terms in the first and second types:

$$[a, b, [c, d, e]] = [a, [c, d, e], b] + [b, [c, d, e], a] - [[c, d, e], b, a].$$

We also get

$$[c, b, a] = -[a, b, c] + [a, c, b] + [b, c, a].$$

This shows that in the inner triple of any monomial of degree 5, we may assume that the first factor lexicographically precedes either the second factor or the third factor; in other words, the lexicographically greatest factor cannot be in the first position. We therefore have $(2/3)5! = 80$ distinct monomials in each of association types 1 and 2, for a total of 160. Hence for this operation the expansion matrix \mathcal{X} has size 120×160 . From this we see that there is a nullspace of identities of dimension ≥ 40 which hold for all q . The diagonal entries of the Smith form of \mathcal{X} are

$$1 \text{ (106 times)}, \quad q - \frac{1}{2} \text{ (5 times)}, \quad q(q-1)(q+1)(q-2) \left(q - \frac{1}{2} \right) \text{ (9 times)}.$$

In each of these cases we took the expansion matrix, set q equal to the appropriate value, computed the nullspace of the matrix, and found a basis vector which generated the entire space of identities. \square

We verified the dimensions obtained from the Smith form by using the representation theory of S_5 and obtained the results in Table 6. The column labeled “general” has the results for $q = 5$. For $q = 0, 1$ the ranks are the same, but the identities are different; and similarly for $q = -1, 2$ the ranks are the same, but the identities are different. In all these cases, the identities produced by this method were very complicated, with many large and apparently random coefficients. Since the evolutionary algorithm which we used to discover identity (42) for the Jordan family takes a very long time to run (weeks or even months), we plan to return to these identities in a future publication (Bremner and Peresi [9]).

9 The last three operations

In this section we consider the operations which have the identity matrix as the middle component. The proofs of the results in this section are similar to those in previous sections.

λ	lifted	$q = \infty$		general q		$q = 0, 1$		$q = -1, 2$		$q = 1/2$	
		all	new	all	new	all	new	all	new	all	new
5	1	2	1	2	1	2	1	2	1	2	1
41	6	9	3	8	2	9	3	8	2	8	2
32	9	11	2	10	1	11	2	10	1	11	2
311	10	12	2	12	2	12	2	12	2	12	2
221	9	11	2	10	1	10	1	11	2	11	2
2111	6	8	2	8	2	8	2	9	3	9	3
11111	1	2	1	2	1	2	1	2	1	2	1

Table 6: Ranks of representations for the fourth family

9.1 The cyclic commutator

The simplest of these operations is

$$\frac{1}{3}(2abc - bca - cab) \quad \left[0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right] \quad (64)$$

The cyclic commutator and its matrix form are

$$abc - bca \quad \left[0, \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}, 0 \right] \quad (65)$$

If we take the row canonical form of each matrix in (65) we obtain (64), so these two operations are equivalent.

Theorem 22. *In every totally associative ternary algebra, the cyclic commutator (65) satisfies the obvious identity*

$$[a, b, c] + [b, c, a] + [c, a, b] = 0. \quad (66)$$

This identity implies all the identities in degree 3 satisfied by operation (65). The cyclic commutator satisfies this minimal identity in degree 5:

$$[[bad]ec] - [[bec]ad] + [[dec]ba] + [a[dec]b] - [b[aec]d] = 0. \quad (67)$$

This identity is not a consequence of the obvious identity and it implies all the identities in degree 5 satisfied by operation (65). \square

9.2 The weakly commutative operation

The commutative identity $ab - ba = 0$ is the alternating sum over S_2 , and the anticommutative identity $ab + ba = 0$ is the symmetric sum over S_2 . If we generalize this to the ternary case, we obtain the weakly commutative and weakly anticommutative identities: the alternating sum and the symmetric sum (regarded as identities, not as operations). We use

λ	cyclic			weakly			weakly		
	commutator			commutative			anticommutative		
	lifted	all	new	lifted	all	new	lifted	all	new
5	3	3	0	0	2	2	3	3	0
41	8	9	1	1	8	7	8	9	1
32	7	11	4	2	10	8	5	10	5
311	10	14	4	6	13	7	6	13	7
221	7	11	4	5	10	5	2	10	8
2111	8	9	1	8	9	1	1	8	7
11111	3	3	0	3	3	0	0	2	2

Table 7: Ranks of representations for the last three operations

the adverb *weakly* to distinguish these identities from the *strongly* commutative and anti-commutative identities (14) and (15) satisfied by the symmetric sum and the alternating sum. The weakly commutative operation is

$$\frac{1}{6}(5abc + acb + bac - bca - cab + cba) \quad \left[1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right] \quad (68)$$

Theorem 23. *In every totally associative ternary algebra, every identity of degree 3 for operation (68) is a consequence of the obvious identity*

$$[a, b, c] - [a, c, b] - [b, a, c] + [b, c, a] + [c, a, b] - [c, b, a] = 0. \quad (69)$$

The weakly commutative operation satisfies a 141-dimensional space of identities in degree 5 which do not follow from the obvious identity. \square

9.3 The weakly anticommutative operation

The weakly anticommutative operation is

$$\frac{1}{6}(5abc - acb - bac - bca - cab - cba) \quad \left[0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right] \quad (70)$$

Theorem 24. *In every totally associative ternary algebra, every identity of degree 3 for operation (70) is a consequence of the obvious identity*

$$[a, b, c] + [a, c, b] + [b, a, c] + [b, c, a] + [c, a, b] + [c, b, a] = 0. \quad (71)$$

The weakly anticommutative operation satisfies a 141-dimensional space of identities in degree 5 which do not follow from the obvious identity. \square

The ranks of the irreducible representations for these last three operations are displayed in Table 7. We plan to return in a future publication to the problem of using an evolutionary algorithm to simplify the very complicated identities of degree 5 satisfied by the last two operations.

10 Summary of results

We have identified 22 trilinear operations which satisfy obvious identities in degree 3 and minimal identities in degree 5:

- the Lie family gives 2 operations:
 - the case $q = \infty$ (equivalent to the classical Lie triple product),
 - the case $q = 1/2$ of the deformation;
- the Jordan family gives 5 operations:
 - the classical Jordan triple product (the case $q = \infty$),
 - the general case of the deformation,
 - the cases $q = 0$, $q = 1$ and $q = 1/2$ of the deformation;
- the anti-Jordan family gives 5 operations:
 - the classical anti-Jordan triple product (equivalent to the case $q = 1/2$),
 - the general case of the deformation,
 - the cases $q = \infty$, $q = -1$ and $q = 2$ of the deformation;
- the fourth family gives 7 operations:
 - the general case of the deformation,
 - the cases $q = \infty$, $q = 0$, $q = 1$, $q = -1$, $q = 2$ and $q = 1/2$ of the deformation;
- the cyclic commutator;
- the weakly commutative operation;
- the weakly anticommutative operation.

Of these operations, only the classical Lie, Jordan and anti-Jordan triple products have been studied in previous research on nonassociative algebra; the remaining 19 operations are new. The obvious and minimal identities for these 19 operations define new varieties of triple systems; in the first three cases, these varieties seem to be closely related to Lie and Jordan structures.

11 Open problems

11.1 Identities for equivalent operations

In this paper we have considered only one representative of each equivalence class of trilinear operations, and determined whether each representative satisfies any identities in degree 5. A natural problem is to determine whether there is a bijection between the identities of degree ≥ 5 (in particular, the minimal identities) satisfied by equivalent operations. For identities of degree 3 we have Proposition 8, but for higher degrees the problem seems much harder.

11.2 Minimal identities of degree ≥ 7

The Lie family of operations has the special feature that it has no identities in degree 5 that hold for all q . The next step is to determine those values of q for which the minimal identities have degree 7. The obvious identities in degree 3 imply that we only need to consider the two association types in degree 7 which are represented by the monomials

$$[[[a, b, c], d, e], f, g], \quad [[a, b, c], [d, e, f], g].$$

The obvious identities further imply that there are only $7!/3 = 1680$ distinct monomials for the first type and $7!/9 = 560$ distinct monomials for the second type, giving a total of 2240. The expansion matrix will therefore have size 5040×2240 ; all the entries of this matrix are polynomials in q of degree ≤ 3 .

11.3 Quadrilinear operations

There are 5 partitions of 4, namely 4, 31, 22, 211, 1111; the corresponding irreducible representations of S_4 have dimensions $d_4 = 1$, $d_{31} = 3$, $d_{22} = 2$, $d_{211} = 3$, $d_{1111} = 1$. Therefore any quadrilinear operation can be represented by a list of matrices of the form

$$\left[v, \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}, \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}, z \right]$$

Since there are $n+1$ possible ranks for an $n \times n$ matrix, we see that (even with this coarse classification) there are $2 \cdot 4 \cdot 3 \cdot 4 \cdot 2 = 192$ different families of quadrilinear operations, and many of them have two or more parameters. One such operation is the Jordan tetrad $abcd + dcba$, which has the matrix form

$$\left[2, \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, 2 \right]$$

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