

Invariant Nonassociative Algebra Structures on Irreducible Representations of Simple Lie Algebras

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An irreducible representation of a simple Lie algebra can be a direct summand of its own tensor square. In this case, the representation admits a nonassociative algebra structure which is invariant in the sense that the Lie algebra acts as derivations. We study this situation for the Lie algebra $sl(2)$.

1. INTRODUCTION

In Section 2 we review the basic representation theory of $sl(2)$. We illustrate our methods on the familiar adjoint representation in Section 3. To go further, we need an explicit version of the classical Clebsch-Gordan Theorem, which is proved in Section 4. This gives highest weight vectors in the tensor square of an irreducible representation. The representation with highest weight n (and dimension $n + 1$) occurs as a summand of its symmetric square when $n \equiv 0 \pmod{4}$ and as a summand of its exterior square when $n \equiv 2 \pmod{4}$. In the latter case we obtain a series of anticommutative algebras beginning with $sl(2)$ itself ($n = 2$, dimension 3, the adjoint representation). The next algebra in the sequence is the simple non-Lie Malcev algebra ($n = 6$, dimension 7) which is discussed in Section 5.

In Section 6 we compute the structure constants for the algebra arising in the case $n = 10$ (dimension 11); this new anticommutative algebra is the focus of the present paper. In Section 7 we review our computational methods, which involve linear algebra on large matrices and the representation theory of the symmetric group. In Section 8 we describe a computer search for polynomial identities satisfied by the new 11-dimensional algebra. We show that all its identities in degree 6 or less are trivial consequences of anticommutativity. We show that it satisfies nontrivial identities in degree 7, classify them, and provide explicit examples. In Section 9 we consider unital extensions of our anticommutative algebras, and

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relate this to the work of Dixmier on nonassociative algebras defined by transvection of binary forms in classical invariant theory.

In Section 10 we go beyond $sl(2)$ and use the software system LiE to determine all fundamental representations of simple Lie algebras of rank less than or equal to 8 which occur as summands of their own exterior squares. This demonstrates the existence of a large number of new anticommutative algebras, with simple Lie algebras in their derivation algebras, which seem worthy of further study. In closing we provide an interesting new characterization of the Lie algebra E_8 .

2. REPRESENTATIONS OF THE LIE ALGEBRA $sl(2)$

We first recall some standard facts about $sl(2)$ and its representations. All vector spaces and tensor products are over \mathbb{F} , an algebraically closed field of characteristic zero. Our basic reference is [Humphreys 72], especially Section II.7.

2.1 The Lie Algebra $sl(2)$

As an abstract Lie algebra, $sl(2)$ has basis $\{E, H, F\}$ and commutation relations

$$[H, E] = 2E, [H, F] = -2F, \text{ and } [E, F] = H. \quad (2-1)$$

All other relations between basis elements follow from anticommutativity:

$$\begin{aligned} [E, H] &= -2E, & [F, H] &= 2F, & [F, E] &= -H, \\ [E, E] &= 0, & [H, H] &= 0, & [F, F] &= 0. \end{aligned}$$

Since the Lie bracket is bilinear, these relations determine the product $[X, Y]$ for all $X, Y \in sl(2)$. These relations are satisfied by the commutator $[X, Y] = XY - YX$ of the 2×2 matrices

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2-2)$$

These three matrices form a basis of the vector space of all 2×2 matrices of trace 0.

2.2 The Irreducible Representation $V(n)$

For any nonnegative integer n , there is an irreducible representation of $sl(2)$ containing a nonzero vector v_n (called the highest weight vector) satisfying the conditions

$$H.v_n = nv_n \quad \text{and} \quad E.v_n = 0. \quad (2-3)$$

This representation is unique up to isomorphism of $sl(2)$ -modules. It is denoted $V(n)$ and is called the representation with highest weight n . Its dimension is $n + 1$; a basis of $V(n)$ consists of the $n + 1$ vectors v_{n-2i} where, by definition,

$$v_{n-2i} = \frac{1}{i!} F^i.v_n, \quad 0 \leq i \leq n. \quad (2-4)$$

The action of $sl(2)$ on $V(n)$ is then as follows:

$$E.v_{n-2i} = (n - i + 1)v_{n-2i+2}, \quad (2-5a)$$

$$H.v_{n-2i} = (n - 2i)v_{n-2i}, \quad (2-5b)$$

$$F.v_{n-2i} = (i + 1)v_{n-2i-2}. \quad (2-5c)$$

The basis vectors v_{n-2i} are called weight vectors since they are eigenvectors for H . It is easy to check that the linear mapping $r: sl(2) \rightarrow \text{End } V(n)$ defined by these relations satisfies the defining property for representations of Lie algebras:

$$r([X, Y]) = r(X)r(Y) - r(Y)r(X).$$

2.3 Invariant Bilinear Forms

Up to a scalar multiple there is a unique $sl(2)$ -module homomorphism

$$V(n) \otimes V(n) \rightarrow V(0).$$

Since $V(0)$ is one-dimensional, we can identify $V(0)$ with the field \mathbb{F} , and so this homomorphism can be expressed as a bilinear form (x, y) satisfying the $sl(2)$ -invariance property

$$(D.x, y) + (x, D.y) = 0 \text{ for any } D \in sl(2) \text{ and } x, y \in V(n).$$

The next proposition gives the precise formula for this bilinear form in terms of the weight vectors.

Proposition 2.1. *Any $sl(2)$ -invariant bilinear form on $V(n)$ is a scalar multiple of*

$$(v_{n-2i}, v_{n-2j}) = \delta_{i+j, n} (-1)^i \binom{n}{i}, \quad 0 \leq i, j \leq n.$$

This form is symmetric if n is even and alternating if n is odd.

Proof: We first consider the action of H : we have

$$(H.x, y) + (x, H.y) = 0 \text{ for any } x, y \in V(n).$$

Setting $x = v_{n-2i}$ and $y = v_{n-2j}$ gives

$$(H.v_{n-2i}, v_{n-2j}) + (v_{n-2i}, H.v_{n-2j}) = 0.$$

Using the formula for the action of H on weight vectors and collecting terms gives

$$(2n - 2(i + j))(v_{n-2i}, v_{n-2j}) = 0.$$

Therefore, $(v_{n-2i}, v_{n-2j}) = 0$ unless $i + j = n$; or equivalently $(n - 2i) + (n - 2j) = 0$. Now assume that $i + j = n$, so that $n - 2j = 2i - n$. We need to determine

$$(v_{n-2i}, v_{2i-n}).$$

We consider the action of E on the pairing of v_{n-2i} and v_{n-2j} :

$$(E.v_{n-2i}, v_{n-2j}) + (v_{n-2i}, E.v_{n-2j}) = 0.$$

Using the formula for the action of E on weight vectors, we obtain

$$(n-i+1)(v_{n-2i+2}, v_{n-2j}) + (n-j+1)(v_{n-2i}, v_{n-2j+2}) = 0.$$

Both terms will be zero unless $2n - 2(i + j) + 2 = 0$; that is, $i + j = n + 1$. In this case we get

$$(n - i + 1)(v_{n-2(i-1)}, v_{2(i-1)-n}) + i(v_{n-2i}, v_{2i-n}) = 0.$$

This gives the recurrence relation

$$(v_{n-2i}, v_{2i-n}) = -\frac{n-i+1}{i}(v_{n-2(i-1)}, v_{2(i-1)-n})$$

for $i \geq 1$. If we write $f(i) = (v_{n-2i}, v_{2i-n})$, then we can write this relation more succinctly as

$$f(i) = -\frac{n-i+1}{i}f(i-1) \text{ for } i \geq 1.$$

From this we obtain

$$\begin{aligned} f(1) &= -nf(0), \\ f(2) &= \frac{n(n-1)}{2}f(0), \\ f(3) &= -\frac{n(n-1)(n-2)}{6}f(0), \quad \dots \end{aligned}$$

The general solution is therefore

$$f(i) = (-1)^i \binom{n}{i} f(0) \text{ for } 0 \leq i \leq n,$$

which is easily proved by induction on i . Taking $f(0) = 1$, this gives the formula stated in Proposition 1.1. Finally, we can verify the symmetric or alternating property as follows:

$$\begin{aligned} (v_{n-2j}, v_{n-2i}) &= \delta_{j+i,n} (-1)^j \binom{n}{j} = \delta_{i+j,n} (-1)^{n-i} \binom{n}{n-i} \\ &= (-1)^n \delta_{i+j,n} (-1)^i \binom{n}{i} \\ &= (-1)^n (v_{n-2i}, v_{n-2j}). \end{aligned}$$

This completes the proof. \square

2.4 The Clebsch-Gordan Theorem

The Clebsch-Gordan Theorem shows how the tensor product of two irreducible representations of $sl(2)$ can be expressed as a direct sum of other irreducible representations. See [Humphreys 72, Exercise 22.7].

Theorem 2.2. *We have the isomorphism*

$$V(n) \otimes V(m) \cong \bigoplus_{i=0}^m V(n+m-2i),$$

for any nonnegative integers $n \geq m$. In the special case $n = m$, we obtain

$$V(n) \otimes V(n) \cong \bigoplus_{i=0}^n V(2n-2i).$$

The examples of particular interest to us in this paper will be

$$\begin{aligned} V(2) \otimes V(2) &\cong V(4) \oplus V(2) \oplus V(0), \\ V(6) \otimes V(6) &\cong V(12) \oplus V(10) \oplus V(8) \oplus V(6) \\ &\quad \oplus V(4) \oplus V(2) \oplus V(0), \\ V(10) \otimes V(10) &\cong V(20) \oplus V(18) \oplus V(16) \oplus V(14) \\ &\quad \oplus V(12) \oplus V(10) \oplus V(8) \oplus V(6) \\ &\quad \oplus V(4) \oplus V(2) \oplus V(0). \end{aligned}$$

Recall the linear transposition map T on $V(n) \otimes V(n)$ defined by $T(v \otimes v') = v' \otimes v$. Using T , we define the symmetric and exterior squares of $V(n)$:

$$\begin{aligned} S^2V(n) &= \{ t \in V(n) \otimes V(n) \mid T(t) = t \}, \\ \Lambda^2V(n) &= \{ t \in V(n) \otimes V(n) \mid T(t) = -t \}. \end{aligned}$$

It is easy to verify that

$$V(n) \otimes V(n) = S^2V(n) \oplus \Lambda^2V(n).$$

In our three examples, we have

$$\begin{aligned} S^2V(2) &\cong V(4) \oplus V(0), \\ \Lambda^2V(2) &\cong V(2), \\ S^2V(6) &\cong V(12) \oplus V(8) \oplus V(4) \oplus V(0), \\ \Lambda^2V(6) &\cong V(10) \oplus V(6) \oplus V(2), \\ S^2V(10) &\cong V(20) \oplus V(16) \oplus V(12) \oplus V(8) \\ &\quad \oplus V(4) \oplus V(0), \\ \Lambda^2V(10) &\cong V(18) \oplus V(14) \oplus V(10) \oplus V(6) \oplus V(2). \end{aligned}$$

In Section 4 we will prove the Clebsch-Gordan Theorem in the case $m = n$ and give explicit formulas for the highest weight vectors of the summands of $V(n) \otimes V(n)$.

2.5 Action of $sl(2)$ on Polynomials

Following [Humphreys 72, Exercise 7.4], we let $\{X, Y\}$ be a basis for the vector space \mathbb{F}^2 , on which $sl(2)$ acts by the natural representation: the 2×2 matrices given in Equations (2-2). This means that we have

$$\begin{aligned} E.X &= 0, & H.X &= X, & F.X &= Y, \\ E.Y &= X, & H.Y &= -Y, & F.Y &= 0, \end{aligned}$$

and the action extends linearly to all of $sl(2)$ and all of \mathbb{F}^2 . We write $\mathbb{F}[X, Y]$ for the polynomial ring in the variables X and Y with coefficients from \mathbb{F} . Since X and Y generate $\mathbb{F}[X, Y]$, we can extend the action of $sl(2)$ to all of $\mathbb{F}[X, Y]$ by the derivation rule:

$$D.pq = (D.p)q + p(D.q)$$

for any $D \in sl(2)$ and any $p, q \in \mathbb{F}[X, Y]$. This makes $\mathbb{F}[X, Y]$ into a representation of $sl(2)$. The subspace of $\mathbb{F}[X, Y]$ consisting of the homogeneous polynomials of degree n has basis

$$\{X^n, X^{n-1}Y, \dots, XY^{n-1}, Y^n\},$$

which is invariant under the action of $sl(2)$ and forms a representation of $sl(2)$ isomorphic to $V(n)$. The action of $sl(2)$ on the polynomial ring can be succinctly expressed in terms of the differential operators

$$E = X \frac{\partial}{\partial Y}, \quad H = X \frac{\partial}{\partial X} - Y \frac{\partial}{\partial Y}, \quad F = Y \frac{\partial}{\partial X}.$$

Applying these operators to the basis monomials of the polynomial ring, we obtain

$$\begin{aligned} E.X^{n-i}Y^i &= iX^{n-i+1}Y^{i-1}, \\ H.X^{n-i}Y^i &= (n-2i)X^{n-i}Y^i, \\ F.X^{n-i}Y^i &= (n-i)X^{n-i-1}Y^{i+1}. \end{aligned}$$

The exact correspondence between the monomials and the abstract basis vectors of $V(n)$ is given by

$$v_{n-2i} = \binom{n}{i} X^{n-i} Y^i.$$

Using this basis of the space of homogeneous polynomials of degree n , we get

$$\begin{aligned} E. \binom{n}{i} X^{n-i} Y^i &= i \binom{n}{i} X^{n-(i-1)} Y^{i-1} \\ &= (n-i+1) \binom{n}{i-1} X^{n-(i-1)} Y^{i-1}, \\ H. \binom{n}{i} X^{n-i} Y^i &= (n-2i) \binom{n}{i} X^{n-i} Y^i, \end{aligned}$$

$$\begin{aligned} F. \binom{n}{i} X^{n-i} Y^i &= (n-i) \binom{n}{i} X^{n-(i+1)} Y^{i+1} \\ &= (i+1) \binom{n}{i+1} X^{n-(i+1)} Y^{i+1}. \end{aligned}$$

Expressing the same relations in terms of the weight vector basis of $V(n)$ we get Equations (2-5).

3. THE ADJOINT REPRESENTATION ($n = 2$)

We illustrate the results of Section 2 by recovering the three-dimensional representation $V(2)$ of $sl(2)$. It has basis $\{v_2, v_0, v_{-2}\}$ on which the Lie algebra acts as follows:

$$\begin{aligned} E.v_2 &= 0, & E.v_0 &= 2v_2, & E.v_{-2} &= v_0, \\ H.v_2 &= 2v_2, & H.v_0 &= 0, & H.v_{-2} &= -2v_{-2}, \\ F.v_2 &= v_0, & F.v_0 &= 2v_{-2}, & F.v_{-2} &= 0. \end{aligned}$$

By the Clebsch-Gordan Theorem we know that

$$V(2) \otimes V(2) \cong V(4) \oplus V(2) \oplus V(0).$$

We determine a basis for each of the three summands on the right side of this isomorphism.

3.1 The Summand $V(4)$

Recall that for any Lie algebra L , and any two L -modules V and W , the action of $D \in L$ on $V \otimes W$ is given by

$$D.(v \otimes w) = D.v \otimes w + v \otimes D.w.$$

It is easy to check that

$$x_4 = v_2 \otimes v_2$$

is a highest weight vector of weight 4 in $V(2) \otimes V(2)$. Applying F repeatedly, using the action of $sl(2)$ on $V(4)$, we obtain a basis for the summand of $V(2) \otimes V(2)$ isomorphic to $V(4)$:

$$\begin{aligned} x_2 &= F.x_4 = F.v_2 \otimes v_2 + v_2 \otimes F.v_2 = v_0 \otimes v_2 + v_2 \otimes v_0 \\ &= v_2 \otimes v_0 + v_0 \otimes v_2, \\ x_0 &= \frac{1}{2} F.x_2 \\ &= \frac{1}{2} (F.v_2 \otimes v_0 + v_2 \otimes F.v_0 + F.v_0 \otimes v_2 + v_0 \otimes F.v_2) \\ &= \frac{1}{2} (v_0 \otimes v_0 + v_2 \otimes 2v_{-2} + 2v_{-2} \otimes v_2 + v_0 \otimes v_0) \\ &= v_2 \otimes v_{-2} + v_0 \otimes v_0 + v_{-2} \otimes v_2, \end{aligned}$$

$$\begin{aligned}
 x_{-2} &= \frac{1}{3}F.x_0 \\
 &= \frac{1}{3}(F.v_2 \otimes v_{-2} + v_2 \otimes F.v_{-2} \\
 &\quad + F.v_0 \otimes v_0 + v_0 \otimes F.v_0 + F.v_{-2} \otimes v_2 \\
 &\quad + v_{-2} \otimes F.v_2) \\
 &= \frac{1}{3}(v_0 \otimes v_{-2} + v_2 \otimes 0 + 2v_{-2} \otimes v_0 + v_0 \otimes 2v_{-2} \\
 &\quad + 0 \otimes v_2 + v_{-2} \otimes v_0) \\
 &= v_0 \otimes v_{-2} + v_{-2} \otimes v_0, \\
 x_{-4} &= \frac{1}{4}F.x_{-2} \\
 &= \frac{1}{4}(F.v_0 \otimes v_{-2} + v_0 \otimes F.v_{-2} + F.v_{-2} \otimes v_0 \\
 &\quad + v_{-2} \otimes F.v_0) \\
 &= \frac{1}{4}(2v_{-2} \otimes v_{-2} + v_0 \otimes 0 + 0 \otimes v_0 + v_{-2} \otimes 2v_{-2}) \\
 &= v_{-2} \otimes v_{-2}.
 \end{aligned}$$

3.2 The Summand $V(2)$

We next find a highest weight vector y_2 of weight 2 in $V(2) \otimes V(2)$, and then we apply F twice to obtain vectors y_0 and y_{-2} ; together these three vectors form a basis of a subspace of $V(2) \otimes V(2)$ that is isomorphic to $V(2)$ as a representation of $sl(2)$. Any vector of weight 2 in $V(2) \otimes V(2)$ must have the form

$$y_2 = a v_2 \otimes v_0 + b v_0 \otimes v_2 \text{ for some } a, b \in \mathbb{F}.$$

Applying E to y_2 , we obtain

$$\begin{aligned}
 E.y_2 &= E.(a v_2 \otimes v_0 + b v_0 \otimes v_2) \\
 &= a(E.v_2 \otimes v_0 + v_2 \otimes E.v_0) \\
 &\quad + b(E.v_0 \otimes v_2 + v_0 \otimes E.v_2) \\
 &= a(0 + v_2 \otimes 2v_2) + b(2v_2 \otimes v_2 + 0) \\
 &= 2(a + b)v_2 \otimes v_2.
 \end{aligned}$$

For y_2 to be a highest weight vector we must have $E.y_2 = 0$, and therefore $a + b = 0$. Up to a nonzero scalar multiple we can take

$$y_2 = v_2 \otimes v_0 - v_0 \otimes v_2.$$

We have $F.y_2 = y_0$; therefore,

$$y_0 = F.y_2 = 2(v_2 \otimes v_{-2} - v_{-2} \otimes v_2).$$

Since $F.y_0 = 2y_{-2}$, we get

$$y_{-2} = \frac{1}{2}F.y_0 = F.\left(\frac{1}{2}y_0\right) = v_0 \otimes v_{-2} - v_{-2} \otimes v_0.$$

3.3 The Summand $V(0)$

Next and last we find a highest weight vector z_0 of weight 0 in $V(2) \otimes V(2)$. We have

$$z_0 = a v_2 \otimes v_{-2} + b v_0 \otimes v_0 + c v_{-2} \otimes v_2.$$

Applying E gives

$$\begin{aligned}
 E.z_0 &= a(E.v_2 \otimes v_{-2} + v_2 \otimes E.v_{-2}) \\
 &\quad + b(E.v_0 \otimes v_0 + v_0 \otimes E.v_0) \\
 &\quad + c(E.v_{-2} \otimes v_2 + v_{-2} \otimes E.v_2) \\
 &= a(0 \otimes v_{-2} + v_2 \otimes v_0) + b(2v_2 \otimes v_0 + v_0 \otimes 2v_2) \\
 &\quad + c(v_0 \otimes v_2 + v_{-2} \otimes 0) \\
 &= (a + 2b)v_2 \otimes v_0 + (2b + c)v_0 \otimes v_2.
 \end{aligned}$$

Since this must be 0, any highest weight vector of weight 0 must be a scalar multiple of

$$z_0 = v_2 \otimes v_{-2} - \frac{1}{2}v_0 \otimes v_0 + v_{-2} \otimes v_2.$$

3.4 The Nonassociative Product on $V(2)$

To determine the projection

$$V(2) \otimes V(2) \rightarrow V(2) = \Lambda^2 V(2),$$

we need to express each simple tensor $v_p \otimes v_q$ with $p, q \in \{2, 0, -2\}$ as a linear combination of the weight vectors of weight $p + q$ in the irreducible summands of $V(2) \otimes V(2)$. We consider two different ordered bases of $V(2) \otimes V(2)$. We call the first the ‘‘tensor basis’’:

$$\begin{aligned}
 &v_2 \otimes v_2, v_2 \otimes v_0, v_2 \otimes v_{-2}, v_0 \otimes v_2, v_0 \otimes v_0, v_0 \otimes v_{-2}, \\
 &v_{-2} \otimes v_2, v_{-2} \otimes v_0, v_{-2} \otimes v_{-2};
 \end{aligned}$$

we call the second the ‘‘module basis’’:

$$x_4, x_2, x_0, x_{-2}, x_{-4}, y_2, y_0, y_{-2}, z_0.$$

We use the module basis to label the columns of a 9×9 matrix A , and we use the tensor basis to label the rows; then, we set entry i, j of A equal to the coefficient of the i th tensor basis vector in the j th module basis vector:

$$A = \begin{pmatrix}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 1 \\
 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
 \end{pmatrix}.$$

The inverse matrix shows how to express the tensor basis vectors as linear combinations of the module basis vectors:

$$A^{-1} = \frac{1}{12} \begin{pmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 8 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \\ 0 & 6 & 0 & -6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & -6 & 0 \\ 0 & 0 & 4 & 0 & -8 & 0 & 4 & 0 & 0 \end{pmatrix}$$

From rows 6–8 of the inverse matrix, we can read off the projection P from $V(2) \otimes V(2)$ onto the summand isomorphic to $V(2)$ with basis y_2, y_0, y_{-2} ; this gives the multiplication table for a three-dimensional anticommutative algebra:

$$\begin{aligned} P(v_2 \otimes v_2) &= 0, & P(v_2 \otimes v_0) &= \frac{1}{2}y_2, & P(v_2 \otimes v_{-2}) &= \frac{1}{4}y_0, \\ P(v_0 \otimes v_2) &= -\frac{1}{2}y_2, & P(v_0 \otimes v_0) &= 0, & P(v_0 \otimes v_{-2}) &= \frac{1}{2}y_{-2}, \\ P(v_{-2} \otimes v_2) &= -\frac{1}{4}y_0, & P(v_{-2} \otimes v_0) &= -\frac{1}{2}y_{-2}, & P(v_{-2} \otimes v_{-2}) &= 0. \end{aligned}$$

We now identify v_p with y_p for $p \in \{2, 0, -2\}$ by the module isomorphism h , which sends v_p to y_p and extends linearly to all of $V(2)$. Then, $h^{-1} \circ P$ is a linear map from $V(2) \otimes V(2)$ to $V(2)$, which we can regard as a multiplication on $V(2)$:

	v_2	v_0	v_{-2}
v_2	0	$\frac{1}{2}$	$\frac{1}{4}$
v_0	$-\frac{1}{2}$	0	$\frac{1}{2}$
v_{-2}	$-\frac{1}{4}$	$-\frac{1}{2}$	0.

Since $v_p v_q = c_{pq} v_{p+q}$ for some scalar c_{pq} , we have included only c_{pq} in this table. The map

$$E \mapsto 4v_2, \quad H \mapsto -4v_0, \quad F \mapsto -4v_{-2}$$

induces an isomorphism of Lie algebras.

4. AN EXPLICIT VERSION OF THE CLEBSCH-GORDAN THEOREM

In this section we work out a general formula for the highest weight vector of weight n in the tensor product $V(n) \otimes V(n)$. Then, we generalize this and find an explicit formula for all the highest weight vectors in $V(n) \otimes V(n)$. From this we recover the Clebsch-Gordan Theorem in this special case, together with the additional result on

the structure of the symmetric and exterior squares. Recall that $V(n)$ has dimension $n + 1$ and basis

$$v_n, v_{n-2}, \dots, v_{-n+2}, v_{-n}.$$

The action of the $sl(2)$ basis elements E, H, F on $V(n)$ is given by Equations (2–5). In order for $V(n)$ to occur as a summand of $V(n) \otimes V(n)$ we must assume that n is even.

Theorem 4.1. *Let n be an even nonnegative integer. Then every highest weight vector of weight n in $V(n) \otimes V(n)$ is a nonzero scalar multiple of*

$$w_n = \sum_{i=0}^{n/2} (-1)^i \frac{\binom{\frac{n}{2}+i}{i}}{\binom{n}{i}} v_{n-2i} \otimes v_{2i}.$$

Proof: Any vector of weight n in $V(n) \otimes V(n)$ must have the form

$$w_n = \sum_{i=0}^{n/2} a_i v_{n-2i} \otimes v_{2i}.$$

For this to be a highest weight vector, we must have $E.w_n = 0$. We have

$$\begin{aligned} E.w_n &= \sum_{i=0}^{n/2} a_i (E.v_{n-2i} \otimes v_{2i} + v_{n-2i} \otimes E.v_{2i}) \\ &= \sum_{i=0}^{n/2} a_i \left((n-i+1)v_{n-2i+2} \otimes v_{2i} \right. \\ &\quad \left. + v_{n-2i} \otimes \left(\frac{n}{2} + i + 1 \right) v_{2i+2} \right) \\ &= \sum_{i=0}^{n/2} (n-i+1)a_i v_{n-2(i-1)} \otimes v_{2i} \\ &\quad + \sum_{i=0}^{n/2} \left(\frac{n}{2} + i + 1 \right) a_i v_{n-2i} \otimes v_{2(i+1)} \\ &= \sum_{i=1}^{n/2} (n-i+1)a_i v_{n-2(i-1)} \otimes v_{2i} \\ &\quad + \sum_{i=0}^{n/2-1} \left(\frac{n}{2} + i + 1 \right) a_i v_{n-2i} \otimes v_{2(i+1)} \\ &= \sum_{i=0}^{n/2-1} (n-i)a_{i+1} v_{n-2i} \otimes v_{2(i+1)} \\ &\quad + \sum_{i=0}^{n/2-1} \left(\frac{n}{2} + i + 1 \right) a_i v_{n-2i} \otimes v_{2(i+1)} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=0}^{n/2-1} ((n-i)a_{i+1} \\
 &\quad + \left(\frac{n}{2} + i + 1\right) a_i) v_{n-2i} \otimes v_{2(i+1)}
 \end{aligned}$$

Since the simple tensors are linearly independent, every coefficient must be zero, and so

$$a_{i+1} = -\frac{\frac{n}{2} + i + 1}{n - i} a_i, \quad 0 \leq i \leq \frac{n}{2} - 1.$$

Since we may choose $a_0 = 1$ without loss of generality, we get

$$a_i = (-1)^i \frac{\frac{n}{2} + i}{n - (i-1)} \cdots \frac{\frac{n}{2} + 2}{n-1} \cdot \frac{\frac{n}{2} + 1}{n}.$$

By induction on i , this simplifies to the compact formula in the statement of Theorem 4.1. \square

We now generalize this computation to establish the decomposition of $V(n) \otimes V(n)$ into a direct sum of irreducible representations; we then identify the symmetric and exterior squares.

Theorem 4.2. *Let n be an even nonnegative integer. Then $V(n) \otimes V(n)$ contains a highest weight vector of weight m if and only if $m = 2n - 2k$ where k is an integer and $0 \leq k \leq n$. Every such highest weight vector is a nonzero scalar multiple of*

$$w_m = \sum_{i=0}^k (-1)^i \frac{\binom{n-k+i}{i}}{\binom{n}{i}} v_{n-2i} \otimes v_{n-2(k-i)}.$$

From this it follows that

$$V(n) \otimes V(n) \cong \bigoplus_{k=0}^n V(2n - 2k),$$

and further follows that we have

$$\begin{aligned}
 S^2 V(n) &\cong \bigoplus_{k=0, \text{ even}}^n V(2n - 2k), \\
 \Lambda^2 V(n) &\cong \bigoplus_{k=1, \text{ odd}}^{n-1} V(2n - 2k).
 \end{aligned}$$

Proof: Any vector in $V(n) \otimes V(n)$ has the form

$$\sum_{i=0}^n \sum_{j=0}^n a_{ij} v_{n-2i} \otimes v_{n-2j}.$$

Since any highest weight vector must be a weight vector, we first break up this sum into its weight components:

$$\begin{aligned}
 &\sum_{k=0}^{n-1} \left(\sum_{i=0}^k a_{i,k-i} v_{n-2i} \otimes v_{n-2(k-i)} \right) \\
 &\quad \text{(terms of positive weight)} \\
 &+ \sum_{i=0}^n a_{i,n-i} v_{n-2i} \otimes v_{2i-n} \\
 &\quad \text{(terms of weight zero)} \\
 &+ \sum_{k=n+1}^{2n} \left(\sum_{i=k-n}^n a_{i,k-i} v_{n-2i} \otimes v_{n-2(k-i)} \right) \\
 &\quad \text{(terms of negative weight)}
 \end{aligned}$$

Since a highest weight vector must have nonnegative weight, we can ignore the terms of negative weight and include the weight zero case with the positive weight cases:

$$\sum_{k=0}^n \left(\sum_{i=0}^k a_{i,k-i} v_{n-2i} \otimes v_{n-2(k-i)} \right).$$

The inner sum, call it w , is a weight vector of weight $2n - 2k$. For w to be a highest weight vector, we must have $E.w = 0$. The formulas for the action of $sl(2)$ on $V(n)$ give

$$\begin{aligned}
 E.v_{n-2i} &= (n - (i-1)) v_{n-2(i-1)}, \\
 E.v_{n-2(k-i)} &= (n - (k-i-1)) v_{n-2(k-i-1)}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 E.w &= \sum_{i=0}^k a_{i,k-i} (E.v_{n-2i} \otimes v_{n-2(k-i)} \\
 &\quad + v_{n-2i} \otimes E.v_{n-2(k-i)}) \\
 &= \sum_{i=0}^k a_{i,k-i} \left((n - (i-1)) v_{n-2(i-1)} \otimes v_{n-2(k-i)} \right. \\
 &\quad \left. + (n - (k-i-1)) v_{n-2i} \otimes v_{n-2(k-i-1)} \right) \\
 &= \sum_{i=0}^k a_{i,k-i} ((n - (i-1)) v_{n-2(i-1)} \otimes v_{n-2(k-i)} \\
 &\quad + \sum_{i=0}^k a_{i,k-i} ((n - (k-i-1)) v_{n-2i} \\
 &\quad \otimes v_{n-2(k-i-1)}) \\
 &= \sum_{i=0}^k a_{i,k-i} ((n - (i-1)) v_{n-2(i-1)} \otimes v_{n-2(k-i)}
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^{k+1} a_{i-1, k-(i-1)} ((n - (k - i)) v_{n-2(i-1)} \\
 & \quad \otimes v_{n-2(k-i)}) \\
 = & a_{0, k} (n + 1) v_{n+2} \otimes v_{n-2k} \\
 & + \sum_{i=1}^k ((n - (i - 1)) a_{i, k-i} \\
 & \quad + (n - (k - i)) a_{i-1, k-(i-1)}) \\
 & \quad \times v_{n-2(i-1)} \otimes v_{n-2(k-i)} \\
 & + a_{k, 0} (n + 1) v_{n-2k} \otimes v_{n+2}.
 \end{aligned}$$

Since $v_{n+2} = 0$ and the simple tensors are linearly independent, we get the relations

$$\begin{aligned}
 (n - (i - 1)) a_{i, k-i} + (n - (k - i)) a_{i-1, k-(i-1)} &= 0, \\
 1 \leq i \leq k.
 \end{aligned}$$

Therefore,

$$a_{i, k-i} = -\frac{n - (k - i)}{n - (i - 1)} a_{i-1, k-(i-1)}, \quad 1 \leq i \leq k.$$

Now induction on i shows that

$$a_{i, k-i} = (-1)^i \frac{\binom{n-k+i}{i}}{\binom{n}{i}}.$$

Thus, we have a unique (up to scalar multiples) highest weight vector in $V(n) \otimes V(n)$ for each weight $2n - 2k$ for $0 \leq k \leq n$. Since a highest weight vector of weight $2n - 2k$ generates a summand $V(2n - 2k)$ of dimension $2n - 2k + 1$, the dimension check

$$(n + 1)^2 = \sum_{k=0}^n (2n - 2k + 1)$$

shows that we have the direct sum decomposition as claimed in the statement of Theorem 4.2. Furthermore, the symmetry or antisymmetry of the coefficients of the highest weight vectors,

$$a_{k-i, i} = (-1)^k a_{i, k-i},$$

shows that they lie either in the symmetric or exterior square of $V(n)$ depending on whether k is even or odd. \square

5. THE SIMPLE NON-LIE MALCEV ALGEBRA ($n = 6$)

The second well-understood example of an anticommutative algebra that can be obtained from a representation of $sl(2)$ is the seven-dimensional simple non-Lie Malcev

algebra M : the vector space of pure imaginary octonions under the commutator product. The identity of lowest degree satisfied by M , which does not follow from anti-commutativity, was originally published in [Malcev 55]. It has degree 4 and is now called the Malcev identity:

$$[[x, y], [x, z]] = [[[x, y], z], x] + [[[y, z], x], x] + [[[z, x], x], y]. \tag{5-1}$$

The linearized version of this identity has eight terms. An equivalent identity which has only five terms is

$$\begin{aligned}
 [[w, y], [x, z]] &= [[[w, x], y], z] + [[[x, y], z], w] \\
 &+ [[[y, z], w], x] + [[[z, w], x], y].
 \end{aligned} \tag{5-2}$$

The variety of Malcev algebras is defined by anticommutativity and the Malcev identity (or one of its equivalents).

Since the product of distinct pure imaginary octonion basis elements is anticommutative, the multiplication table for the seven-dimensional simple non-Lie Malcev algebra can be obtained from the octonion multiplication table by replacing the diagonal entries by 0 and multiplying the other entries by 2.

Definition 5.1. The simple seven-dimensional non-Lie Malcev algebra is the anticommutative algebra with ‘‘octonion’’ basis I, J, K, L, M, N, P and multiplication table

$[\cdot]$	I	J	K	L	M	N	P
I	0	$2K$	$-2J$	$2M$	$-2L$	$-2P$	$2N$
J	$-2K$	0	$2I$	$2N$	$2P$	$-2L$	$-2M$
K	$2J$	$-2I$	0	$2P$	$-2N$	$2M$	$-2L$
L	$-2M$	$-2N$	$-2P$	0	$2I$	$2J$	$2K$
M	$2L$	$-2P$	$2N$	$-2I$	0	$-2K$	$2J$
N	$2P$	$2L$	$-2M$	$-2J$	$2K$	0	$-2I$
P	$-2N$	$2M$	$2L$	$-2K$	$-2J$	$2I$	0

We first determine the structure constants for the anticommutative algebra coming from $V(6)$, and then we show that this algebra is isomorphic to M .

Theorem 5.2. *The structure constants for the anticommutative algebra resulting from the projection*

$$V(6) \otimes V(6) \rightarrow V(6) \subset \Lambda^2 V(6)$$

are displayed in Table 1.

Since the product of v_p and v_q equals $c_{pq} v_{p+q}$ for some scalar c_{pq} , we only record the scalars c_{pq} in this table.

Proof: By the Clebsch-Gordan Theorem we know how $V(6) \otimes V(6)$ decomposes as a direct sum of irreducible representations:

\otimes	v_6	v_4	v_2	v_0	v_{-2}	v_{-4}	v_{-6}
v_6	0	0	0	1	2	2	1
v_4	0	0	-1	-1	0	1	1
v_2	0	1	0	-1	-1	0	1
v_0	-1	1	1	0	-1	-1	1
v_{-2}	-2	0	1	1	0	-2	0
v_{-4}	-2	-1	0	1	2	0	0
v_{-6}	-1	-1	-1	-1	0	0	0

TABLE 1.

$$V(6) \otimes V(6) \cong V(12) \oplus V(10) \oplus V(8) \oplus V(6) \oplus V(4) \oplus V(2) \oplus V(0).$$

We want to compute the projection $P: V(6) \otimes V(6) \rightarrow V(6)$; for this we follow the method used in the example of the adjoint representation. We use the explicit Clebsch-Gordan Theorem to determine a highest weight vector in each irreducible summand of the tensor product. We then apply F to determine a basis of weight vectors for each irreducible summand. From this we form the transition matrix from the module basis to the tensor basis. Inverting this matrix gives the transition matrix from the tensor basis to the module basis, and from this we obtain the explicit projection map from the tensor product onto the $V(6)$ summand. These computations were done by a Maple [Maple 04] program written by the authors.

For the summand $V(12)$, a highest weight vector is $v_6 \otimes v_6$, and the other weight vectors can be found by applying F following Equation (2-5c):

$$\begin{aligned} s_{12} &= v_6 \otimes v_6, \\ s_{10} &= v_6 \otimes v_4 + v_4 \otimes v_6, \\ s_8 &= v_6 \otimes v_2 + v_4 \otimes v_4 + v_2 \otimes v_6, \\ s_6 &= v_6 \otimes v_0 + v_4 \otimes v_2 + v_2 \otimes v_4 + v_0 \otimes v_6, \\ s_4 &= v_6 \otimes v_{-2} + v_4 \otimes v_0 + v_2 \otimes v_2 + v_0 \otimes v_4 \\ &\quad + v_{-2} \otimes v_6, \\ s_2 &= v_6 \otimes v_{-4} + v_4 \otimes v_{-2} + v_2 \otimes v_0 + v_0 \otimes v_2 \\ &\quad + v_{-2} \otimes v_4 + v_{-4} \otimes v_6, \\ s_0 &= v_6 \otimes v_{-6} + v_4 \otimes v_{-4} + v_2 \otimes v_{-2} + v_0 \otimes v_0 \\ &\quad + v_{-2} \otimes v_2 + v_{-4} \otimes v_4 + v_{-6} \otimes v_6, \\ s_{-2} &= v_4 \otimes v_{-6} + v_2 \otimes v_{-4} + v_0 \otimes v_{-2} + v_{-2} \otimes v_0 \\ &\quad + v_{-4} \otimes v_2 + v_{-6} \otimes v_4, \\ s_{-4} &= v_2 \otimes v_{-6} + v_0 \otimes v_{-4} + v_{-2} \otimes v_{-2} + v_{-4} \otimes v_0 \\ &\quad + v_{-6} \otimes v_2, \end{aligned}$$

$$\begin{aligned} s_{-6} &= v_0 \otimes v_{-6} + v_{-2} \otimes v_{-4} + v_{-4} \otimes v_{-2} + v_{-6} \otimes v_0, \\ s_{-8} &= v_{-2} \otimes v_{-6} + v_{-4} \otimes v_{-4} + v_{-6} \otimes v_{-2}, \\ s_{-10} &= v_{-4} \otimes v_{-6} + v_{-6} \otimes v_{-4}, \\ s_{-12} &= v_{-6} \otimes v_{-6}. \end{aligned}$$

For the summand $V(10)$ (and all the following summands), a highest weight vector is given by the explicit Clebsch-Gordan Theorem, and the other weight vectors are found by applying F :

$$\begin{aligned} t_{10} &= v_6 \otimes v_4 - v_4 \otimes v_6, \\ t_8 &= 2v_6 \otimes v_2 - 2v_2 \otimes v_6, \\ t_6 &= 3v_6 \otimes v_0 + v_4 \otimes v_2 - v_2 \otimes v_4 - 3v_0 \otimes v_6, \\ t_4 &= 4v_6 \otimes v_{-2} + 2v_4 \otimes v_0 - 2v_0 \otimes v_4 - 4v_{-2} \otimes v_6, \\ t_2 &= 5v_6 \otimes v_{-4} + 3v_4 \otimes v_{-2} + v_2 \otimes v_0 - v_0 \otimes v_2 \\ &\quad - 3v_{-2} \otimes v_4 - 5v_{-4} \otimes v_6, \\ t_0 &= 6v_6 \otimes v_{-6} + 4v_4 \otimes v_{-4} + 2v_2 \otimes v_{-2} - 2v_{-2} \otimes v_2 \\ &\quad - 4v_{-4} \otimes v_4 - 6v_{-6} \otimes v_6, \\ t_{-2} &= 5v_4 \otimes v_{-6} + 3v_2 \otimes v_{-4} + v_0 \otimes v_{-2} - v_{-2} \otimes v_0 \\ &\quad - 3v_{-4} \otimes v_2 - 5v_{-6} \otimes v_4, \\ t_{-4} &= 4v_2 \otimes v_{-6} + 2v_0 \otimes v_{-4} - 2v_{-4} \otimes v_0 - 4v_{-6} \otimes v_2, \\ t_{-6} &= 3v_0 \otimes v_{-6} + v_{-2} \otimes v_{-4} - v_{-4} \otimes v_{-2} - 3v_{-6} \otimes v_0, \\ t_{-8} &= 2v_{-2} \otimes v_{-6} - 2v_{-6} \otimes v_{-2}, \\ t_{-10} &= v_{-4} \otimes v_{-6} - v_{-6} \otimes v_{-4} \end{aligned}$$

For the summand $V(8)$, we obtain this basis:

$$\begin{aligned} u_8 &= v_6 \otimes v_2 - \frac{5}{6}v_4 \otimes v_4 + v_2 \otimes v_6, \\ u_6 &= 3v_6 \otimes v_0 - \frac{2}{3}v_4 \otimes v_2 - \frac{2}{3}v_2 \otimes v_4 + 3v_0 \otimes v_6, \\ u_4 &= 6v_6 \otimes v_{-2} + \frac{1}{2}v_4 \otimes v_0 - \frac{4}{3}v_2 \otimes v_2 + \frac{1}{2}v_0 \otimes v_4 \\ &\quad + 6v_{-2} \otimes v_6, \\ u_2 &= 10v_6 \otimes v_{-4} + \frac{8}{3}v_4 \otimes v_{-2} - v_2 \otimes v_0 - v_0 \otimes v_2 \\ &\quad + \frac{8}{3}v_{-2} \otimes v_4 + 10v_{-4} \otimes v_6, \\ u_0 &= 15v_6 \otimes v_{-6} + \frac{35}{6}v_4 \otimes v_{-4} + \frac{1}{3}v_2 \otimes v_{-2} - \frac{3}{2}v_0 \otimes v_0 \\ &\quad + \frac{1}{3}v_{-2} \otimes v_2 + \frac{35}{6}v_{-4} \otimes v_4 + 15v_{-6} \otimes v_6, \\ u_{-2} &= 10v_4 \otimes v_{-6} + \frac{8}{3}v_2 \otimes v_{-4} - v_0 \otimes v_{-2} - v_{-2} \otimes v_0 \\ &\quad + \frac{8}{3}v_{-4} \otimes v_2 + 10v_{-6} \otimes v_4, \\ u_{-4} &= 6v_2 \otimes v_{-6} + \frac{1}{2}v_0 \otimes v_{-4} - \frac{4}{3}v_{-2} \otimes v_{-2} + \frac{1}{2}v_{-4} \otimes v_0 \\ &\quad + 6v_{-6} \otimes v_2, \\ u_{-6} &= 3v_0 \otimes v_{-6} - \frac{2}{3}v_{-2} \otimes v_{-4} - \frac{2}{3}v_{-4} \otimes v_{-2} \\ &\quad + 3v_{-6} \otimes v_0, \\ u_{-8} &= v_{-2} \otimes v_{-6} - \frac{5}{6}v_{-4} \otimes v_{-4} + v_{-6} \otimes v_{-2}. \end{aligned}$$

For the summand $V(6)$ (this is the summand onto which we will project), we obtain this basis:

$$\begin{aligned} w_6 &= v_6 \otimes v_0 - \frac{2}{3}v_4 \otimes v_2 + \frac{2}{3}v_2 \otimes v_4 - v_0 \otimes v_6, \\ w_4 &= 4v_6 \otimes v_{-2} - v_4 \otimes v_0 + v_0 \otimes v_4 - 4v_{-2} \otimes v_6, \\ w_2 &= 10v_6 \otimes v_{-4} - v_2 \otimes v_0 + v_0 \otimes v_2 - 10v_{-4} \otimes v_6, \\ w_0 &= 20v_6 \otimes v_{-6} + \frac{10}{3}v_4 \otimes v_{-4} - \frac{4}{3}v_2 \otimes v_{-2} + \frac{4}{3}v_{-2} \otimes v_2 \\ &\quad - \frac{10}{3}v_{-4} \otimes v_4 - 20v_{-6} \otimes v_6, \\ w_{-2} &= 10v_4 \otimes v_{-6} - v_0 \otimes v_{-2} + v_{-2} \otimes v_0 - 10v_{-6} \otimes v_4, \\ w_{-4} &= 4v_2 \otimes v_{-6} - v_0 \otimes v_{-4} + v_{-4} \otimes v_0 - 4v_{-6} \otimes v_2, \\ w_{-6} &= v_0 \otimes v_{-6} - \frac{2}{3}v_{-2} \otimes v_{-4} + \frac{2}{3}v_{-4} \otimes v_{-2} - v_{-6} \otimes v_0. \end{aligned}$$

For the summand $V(4)$, we obtain this basis:

$$\begin{aligned} x_4 &= v_6 \otimes v_{-2} - \frac{1}{2}v_4 \otimes v_0 + \frac{2}{5}v_2 \otimes v_2 - \frac{1}{2}v_0 \otimes v_4 \\ &\quad + v_{-2} \otimes v_6, \\ x_2 &= 5v_6 \otimes v_{-4} - v_4 \otimes v_{-2} + \frac{1}{5}v_2 \otimes v_0 + \frac{1}{5}v_0 \otimes v_2 \\ &\quad - v_{-2} \otimes v_4 + 5v_{-4} \otimes v_6, \\ x_0 &= 15v_6 \otimes v_{-6} - \frac{3}{5}v_2 \otimes v_{-2} + \frac{3}{5}v_0 \otimes v_0 - \frac{3}{5}v_{-2} \otimes v_2 \\ &\quad + 15v_{-6} \otimes v_6, \\ x_{-2} &= 5v_4 \otimes v_{-6} - v_2 \otimes v_{-4} + \frac{1}{5}v_0 \otimes v_{-2} + \frac{1}{5}v_{-2} \otimes v_0 \\ &\quad - v_{-4} \otimes v_2 + 5v_{-6} \otimes v_4, \\ x_{-4} &= v_2 \otimes v_{-6} - \frac{1}{2}v_0 \otimes v_{-4} + \frac{2}{5}v_{-2} \otimes v_{-2} - \frac{1}{2}v_{-4} \otimes v_0 \\ &\quad + v_{-6} \otimes v_2. \end{aligned}$$

For the summand $V(2)$, we obtain this basis:

$$\begin{aligned} y_2 &= v_6 \otimes v_{-4} - \frac{1}{3}v_4 \otimes v_{-2} \\ &\quad + \frac{1}{5}v_2 \otimes v_0 - \frac{1}{5}v_0 \otimes v_2 + \frac{1}{3}v_{-2} \otimes v_4 - v_{-4} \otimes v_6, \\ y_0 &= 6v_6 \otimes v_{-6} - \frac{2}{3}v_4 \otimes v_{-4} + \frac{2}{15}v_2 \otimes v_{-2} \\ &\quad - \frac{2}{15}v_{-2} \otimes v_2 + \frac{2}{3}v_{-4} \otimes v_4 \\ &\quad - 6v_{-6} \otimes v_6, \\ y_{-2} &= v_4 \otimes v_{-6} - \frac{1}{3}v_2 \otimes v_{-4} + \frac{1}{5}v_0 \otimes v_{-2} - \frac{1}{5}v_{-2} \otimes v_0 \\ &\quad + \frac{1}{3}v_{-4} \otimes v_2 - v_{-6} \otimes v_4. \end{aligned}$$

For the summand $V(0)$, we obtain this basis:

$$\begin{aligned} z_0 &= v_6 \otimes v_{-6} - \frac{1}{6}v_4 \otimes v_{-4} + \frac{1}{15}v_2 \otimes v_{-2} - \frac{1}{20}v_0 \otimes v_0 \\ &\quad + \frac{1}{15}v_{-2} \otimes v_2 - \frac{1}{6}v_{-4} \otimes v_4 + v_{-6} \otimes v_6. \end{aligned}$$

We consider two distinct ordered bases of the 49-dimensional space $V(6) \otimes V(6)$. The first is the tensor basis, consisting of all

$$v_p \otimes v_q, \quad p, q \in \{6, 4, 2, 0, -2, -4, -6\},$$

ordered by the rule that $v_p \otimes v_q$ precedes $v_{p'} \otimes v_{q'}$ if and only if either $p > p'$, or $p = p'$ and $q > q'$. The second is the module basis, consisting of all

$$r_p, \quad r \in \{s, t, u, w, x, y, z\},$$

with appropriate weights p depending on r , ordered by the rule that r_p precedes $r_{p'}$ if and only if r precedes r' in the alphabet, or $r = r'$ and $p > p'$.

We now let A be the 49×49 matrix in which the entry i, j is the coefficient of the i th tensor basis vector in the expression for the j th module basis vector (in the linear combinations listed above). This is simply the transition matrix from the module basis to the tensor basis. The inverse matrix A^{-1} is the transition matrix from the tensor basis to the module basis, and so its columns describe the expressions of the tensor basis vectors as linear combinations of the module basis vectors. In particular, rows 34–40 of A^{-1} describe the projection $P: V(6) \otimes V(6) \rightarrow V(6)$. We present the results in the following table; the scalar c_{pq} in row v_p and column v_q means that $P(v_p \otimes v_q) = c_{pq}w_{p+q}$:

\otimes	v_6	v_4	v_2	v_0	v_{-2}	v_{-4}	v_{-6}
v_6	0	0	0	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{30}$	$\frac{1}{120}$
v_4	0	0	$-\frac{1}{2}$	$-\frac{1}{6}$	0	$\frac{1}{20}$	$\frac{1}{30}$
v_2	0	$\frac{1}{2}$	0	$-\frac{1}{6}$	$-\frac{1}{8}$	0	$\frac{1}{12}$
v_0	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$
v_{-2}	$-\frac{1}{12}$	0	$\frac{1}{8}$	$\frac{1}{6}$	0	$-\frac{1}{2}$	0
v_{-4}	$-\frac{1}{30}$	$-\frac{1}{20}$	0	$\frac{1}{6}$	$\frac{1}{2}$	0	0
v_{-6}	$-\frac{1}{120}$	$-\frac{1}{30}$	$-\frac{1}{12}$	$-\frac{1}{6}$	0	0	0

Now, we write h for the $sl(2)$ -module isomorphism defined by $h(v_p) = w_p$, which sends the original $V(6)$ with basis v_p ($p = 6, \dots, -6$) onto the $V(6)$ summand of $V(6) \otimes V(6)$ with basis w_p ($p = 6, \dots, -6$). The composition $h^{-1} \circ P$ provides $V(6)$ with the structure of an anticommutative algebra for which the structure constants appear in the last table:

$$[v_p, v_q] = c_{pq}v_{p+q}.$$

If we now introduce a scaled basis

$$v'_p = i!v_p, \quad i = \frac{1}{2}(6 - p),$$

then we obtain the structure constants in the statement of Theorem 5.2. \square

We will show that the anticommutative algebra of Theorem 5.2 is a simple non-Lie Malcev algebra by giving an explicit isomorphism between it and the simple

non-Lie Malcev algebra obtained from the pure imaginary octonions under the commutator product.

Theorem 5.3. *The anticommutative algebra obtained from the projection $V(6) \otimes V(6) \rightarrow V(6) \subset \Lambda^2 V(6)$ is isomorphic to the simple seven-dimensional non-Lie Malcev algebra. An explicit isomorphism is given by*

$$\begin{aligned} v_6 &= \frac{1}{16bc}(I - iM), & v_4 &= -\frac{1}{8b}(J + iN), & v_2 &= \frac{1}{8c}(K + iP) \\ v_0 &= -\frac{1}{2}iL, \\ v_{-2} &= c(K - iP), & v_{-4} &= b(J - iN), & v_{-6} &= 2bc(I + iM), \end{aligned}$$

where b, c are arbitrary nonzero scalars (and $i = \sqrt{-1}$).

Proof: Let

$$X = aI + bJ + cK + dL + eM + fN + gP$$

be a general element of the Malcev algebra in the octonion basis. Using the structure constants for this basis given in Definition 5.1, we see that the matrix representing left multiplication by X in the octonion basis is

$$\begin{pmatrix} 0 & -2c & 2b & -2e & 2d & 2g & -2f \\ 2c & 0 & -2a & -2f & -2g & 2d & 2e \\ -2b & 2a & 0 & -2g & 2f & -2e & 2d \\ 2e & 2f & 2g & 0 & -2a & -2b & -2c \\ -2d & 2g & -2f & 2a & 0 & 2c & -2b \\ -2g & -2d & 2e & 2b & -2c & 0 & 2a \\ 2f & -2e & -2d & 2c & 2b & -2a & 0 \end{pmatrix}$$

The characteristic polynomial of this matrix is

$$t(t^2 + 4(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2))^3.$$

We want to find an element X in the Malcev algebra (octonion basis) that behaves like v_0 in the $V(6)$ algebra in the sense that its left multiplication has the same eigenvalues. The $V(6)$ multiplication table from Theorem 5.2 shows that these eigenvalues are 0, 1 (3 times), and -1 (3 times). So we get a match of the characteristic polynomials if

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 = -\frac{1}{4}.$$

For simplicity we take

$$d = \frac{1}{2}i, \quad a = b = c = e = f = g = 0,$$

which gives the left multiplication matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is the matrix representing left multiplication by the new basis element

$$x_4 = \frac{1}{2}iL.$$

Any eigenvector for left multiplication by x_4 corresponding to the eigenvalue $\lambda = 1$ is a linear combination of

$$x_1 = I - iM, \quad x_2 = J - iN, \quad x_3 = K - iP.$$

Likewise, any eigenvector for left multiplication by x_4 corresponding to the eigenvalue $\lambda = -1$ is a linear combination of

$$x_5 = I + iM, \quad x_6 = J + iN, \quad x_7 = K + iP.$$

The multiplication table for these basis vectors is

[,]	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	0	$4x_7$	$-4x_6$	$-x_1$	$-8x_4$	0	0
x_2	$-4x_7$	0	$4x_5$	$-x_2$	0	$-8x_4$	0
x_3	$4x_6$	$-4x_5$	0	$-x_3$	0	0	$-8x_4$
x_4	x_1	x_2	x_3	0	$-x_5$	$-x_6$	$-x_7$
x_5	$8x_4$	0	0	x_5	0	$4x_3$	$-4x_2$
x_6	0	$8x_4$	0	x_6	$-4x_3$	0	$4x_1$
x_7	0	0	$8x_4$	x_7	$4x_2$	$-4x_1$	0

We now compare the locations of the zeroes in this table and in the table of Theorem 5.2. We call two distinct basis elements “related” if their product is zero; since the product is anticommutative, this term is well-defined. For the last table the related basis elements form the cycle

$$x_1, x_6, x_3, x_5, x_2, x_7, x_1;$$

for the table in Theorem 5.2, the related basis elements form the cycle

$$v_6, v_4, v_{-2}, v_{-6}, v_{-4}, v_2, v_6.$$

This suggests that we make the identifications

$$\begin{aligned} v_6 &= ax_1, & v_4 &= fx_6, & v_2 &= gx_7, & v_0 &= dx_4, \\ v_{-2} &= cx_3, & v_{-4} &= bx_2, & v_{-6} &= ex_5, \end{aligned}$$

for some suitable scalars a, b, c, d, e, f, g . This gives the new table, Table 2.

[,]	ax_1	fx_6	gx_7	dx_4	cx_3	bx_2	ex_5
ax_1	0	0	0	$-\frac{ad}{a}ax_1$	$-4\frac{ac}{f}fx_6$	$4\frac{ab}{g}gx_7$	$-8\frac{ae}{d}dx_4$
fx_6	0	0	$4\frac{fg}{a}ax_1$	$\frac{df}{f}fx_6$	0	$8\frac{bf}{d}dx_4$	$-4\frac{ef}{c}cx_3$
gx_7	0	$-4\frac{fg}{a}ax_1$	0	$\frac{dg}{g}gx_7$	$8\frac{cg}{d}dx_4$	0	$4\frac{eg}{b}bx_2$
dx_4	$\frac{ad}{a}ax_1$	$-\frac{df}{f}fx_6$	$-\frac{dg}{g}gx_7$	0	$\frac{cd}{c}cx_3$	$\frac{bd}{b}bx_2$	$-\frac{de}{e}ex_5$
cx_3	$4\frac{ac}{f}fx_6$	0	$-8\frac{cg}{d}dx_4$	$-\frac{cd}{c}cx_3$	0	$-4\frac{bc}{e}ex_5$	0
bx_2	$-4\frac{ab}{g}gx_7$	$-8\frac{bf}{d}dx_4$	0	$-\frac{bd}{b}bx_2$	$4\frac{bc}{e}ex_5$	0	0
ex_5	$8\frac{ae}{d}dx_4$	$4\frac{ef}{c}cx_3$	$-4\frac{eg}{b}bx_2$	$\frac{de}{e}ex_5$	0	0	0

TABLE 2.

\otimes	v_{10}	v_8	v_6	v_4	v_2	v_0	v_{-2}	v_{-4}	v_{-6}	v_{-8}	v_{-10}
v_{10}	0	0	0	0	0	3	9	14	14	9	3
v_8	0	0	0	0	-3	-6	-5	0	5	6	3
v_6	0	0	0	3	3	-1	-5	-5	-1	3	3
v_4	0	0	-3	0	4	4	0	-4	-4	0	3
v_2	0	3	-3	-4	0	4	4	0	-4	-3	3
v_0	-3	6	1	-4	-4	0	4	4	-1	-6	3
v_{-2}	-9	5	5	0	-4	-4	0	5	5	-9	0
v_{-4}	-14	0	5	4	0	-4	-5	0	14	0	0
v_{-6}	-14	-5	1	4	4	1	-5	-14	0	0	0
v_{-8}	-9	-6	-3	0	3	6	9	0	0	0	0
v_{-10}	-3	-3	-3	-3	-3	-3	0	0	0	0	0

TABLE 3.

We set the entries of Table 2 equal to the corresponding entries of the table of Theorem 5.2 and use Maple to solve the resulting system of nonlinear equations in seven unknowns. The result includes two free parameters b and c :

$$a = \frac{1}{16bc}, \quad b = \text{free}, \quad c = \text{free}, \quad d = -1,$$

$$e = 2bc, \quad f = -\frac{1}{8b}, \quad g = \frac{1}{8c}.$$

For any nonzero choices of b and c we get an isomorphism between the $V(6)$ algebra and the seven-dimensional simple non-Lie Malcev algebra. This completes the proof. \square

6. A NEW 11-DIMENSIONAL ANTICOMMUTATIVE ALGEBRA ($n = 10$)

In this section we study the module $V(10)$ and the projection

$$V(10) \otimes V(10) \rightarrow V(10) \subset \Lambda^2 V(10).$$

This gives the module $V(10)$ an $sl(2)$ -invariant anticommutative algebra structure which, to the best of our knowledge, has not been studied elsewhere.

Theorem 6.1. *The structure constants for the anticommutative algebra obtained from the projection*

$$V(10) \otimes V(10) \rightarrow V(10) \subset \Lambda^2 V(10)$$

are displayed in Table 3.

Since the product of v_p and v_q equals $c_{pq}v_{p+q}$ for some scalar c_{pq} , we only record the scalars c_{pq} .

Proof: The explicit form of the Clebsch-Gordan Theorem gives a formula for the highest weight vectors of the irreducible summands in the decomposition

$$V(10) \otimes V(10) \cong V(20) \oplus V(18) \oplus V(16) \oplus V(14) \\ \oplus V(12) \oplus V(10) \oplus V(8) \oplus V(6) \\ \oplus V(4) \oplus V(2) \oplus V(0).$$

Since the computational methods in this case are the same as those in the case of the adjoint representation $V(2)$ and the Malcev algebra $V(6)$, we do not give complete details. We present only the basis of weight vectors for the summand of $V(10) \otimes V(10)$ isomorphic to $V(10)$:

$$t_{10} = v_{10} \otimes v_0 - \frac{3}{5}v_8 \otimes v_2 + \frac{7}{15}v_6 \otimes v_4 - \frac{7}{15}v_4 \otimes v_6 \\ + \frac{3}{5}v_2 \otimes v_8 - v_0 \otimes v_{10}.$$

0	0	0	0	0	$\frac{3}{52}$	$\frac{3}{104}$	$\frac{1}{78}$	$\frac{1}{208}$	$\frac{1}{728}$	$\frac{1}{4368}$
0	0	0	0	$-\frac{15}{52}$	$-\frac{3}{26}$	$-\frac{5}{156}$	0	$\frac{5}{728}$	$\frac{5}{1092}$	$\frac{1}{728}$
0	0	0	$\frac{15}{26}$	$\frac{15}{104}$	$-\frac{1}{52}$	$-\frac{5}{104}$	$-\frac{5}{182}$	$-\frac{5}{1456}$	$\frac{5}{728}$	$\frac{1}{208}$
0	0	$-\frac{15}{26}$	0	$\frac{5}{39}$	$\frac{1}{13}$	0	$-\frac{10}{273}$	$-\frac{5}{182}$	0	$\frac{1}{78}$
0	$\frac{15}{52}$	$-\frac{15}{104}$	$-\frac{5}{39}$	0	$\frac{1}{13}$	$\frac{5}{78}$	0	$-\frac{5}{104}$	$-\frac{5}{156}$	$\frac{3}{104}$
$-\frac{3}{52}$	$\frac{3}{26}$	$\frac{1}{52}$	$-\frac{1}{13}$	$-\frac{1}{13}$	0	$\frac{1}{13}$	$\frac{1}{13}$	$-\frac{1}{52}$	$-\frac{3}{26}$	$\frac{3}{52}$
$-\frac{3}{104}$	$\frac{5}{156}$	$\frac{5}{104}$	0	$-\frac{5}{78}$	$-\frac{1}{13}$	0	$\frac{5}{39}$	$\frac{15}{104}$	$-\frac{15}{52}$	0
$-\frac{1}{78}$	0	$\frac{5}{182}$	$\frac{10}{273}$	0	$-\frac{1}{13}$	$-\frac{5}{39}$	0	$\frac{15}{26}$	0	0
$-\frac{1}{208}$	$-\frac{5}{728}$	$\frac{5}{1456}$	$\frac{5}{182}$	$\frac{5}{104}$	$\frac{1}{52}$	$-\frac{15}{104}$	$-\frac{15}{26}$	0	0	0
$-\frac{1}{728}$	$-\frac{5}{1092}$	$-\frac{5}{728}$	0	$\frac{5}{156}$	$\frac{3}{26}$	$\frac{15}{52}$	0	0	0	0
$-\frac{1}{4368}$	$-\frac{1}{728}$	$-\frac{1}{208}$	$-\frac{1}{78}$	$-\frac{3}{104}$	$-\frac{3}{52}$	0	0	0	0	0

TABLE 4.

$$\begin{aligned}
 t_8 &= 6v_{10} \otimes v_{-2} - 2v_8 \otimes v_0 + \frac{2}{3}v_6 \otimes v_2 - \frac{2}{3}v_2 \otimes v_6 \\
 &\quad + 2v_0 \otimes v_8 - 6v_{-2} \otimes v_{10}, \\
 t_6 &= 21v_{10} \otimes v_{-4} - 3v_8 \otimes v_{-2} - \frac{1}{3}v_6 \otimes v_0 + v_4 \otimes v_2 \\
 &\quad - v_2 \otimes v_4 + \frac{1}{3}v_0 \otimes v_6 + 3v_{-2} \otimes v_8 - 21v_{-4} \otimes v_{10}, \\
 t_4 &= 56v_{10} \otimes v_{-6} - \frac{8}{3}v_6 \otimes v_{-2} + \frac{4}{3}v_4 \otimes v_0 - \frac{4}{3}v_0 \otimes v_4 \\
 &\quad + \frac{8}{3}v_{-2} \otimes v_6 - 56v_{-6} \otimes v_{10}, \\
 t_2 &= 126v_{10} \otimes v_{-8} + 14v_8 \otimes v_{-6} - \frac{14}{3}v_6 \otimes v_{-4} \\
 &\quad + \frac{4}{3}v_2 \otimes v_0 - \frac{4}{3}v_0 \otimes v_2 + \frac{14}{3}v_{-4} \otimes v_6 - 14v_{-6} \otimes v_8 \\
 &\quad - 126v_{-8} \otimes v_{10}, \\
 t_0 &= 252v_{10} \otimes v_{-10} + \frac{252}{5}v_8 \otimes v_{-8} - \frac{28}{15}v_6 \otimes v_{-6} \\
 &\quad - \frac{14}{5}v_4 \otimes v_{-4} + \frac{8}{5}v_2 \otimes v_{-2} - \frac{8}{5}v_{-2} \otimes v_2 \\
 &\quad + \frac{14}{5}v_{-4} \otimes v_4 + \frac{28}{15}v_{-6} \otimes v_6 - \frac{252}{5}v_{-8} \otimes v_8 \\
 &\quad - 252v_{-10} \otimes v_{10}, \\
 t_{-2} &= 126v_8 \otimes v_{-10} + 14v_6 \otimes v_{-8} - \frac{14}{3}v_4 \otimes v_{-6} \\
 &\quad + \frac{4}{3}v_0 \otimes v_{-2} - \frac{4}{3}v_{-2} \otimes v_0 + \frac{14}{3}v_{-6} \otimes v_4 \\
 &\quad - 14v_{-8} \otimes v_6 - 126v_{-10} \otimes v_8, \\
 t_{-4} &= 56v_6 \otimes v_{-10} - \frac{8}{3}v_2 \otimes v_{-6} + \frac{4}{3}v_0 \otimes v_{-4} - \frac{4}{3}v_{-4} \otimes v_0 \\
 &\quad + \frac{8}{3}v_{-6} \otimes v_2 - 56v_{-10} \otimes v_6, \\
 t_{-6} &= 21v_4 \otimes v_{-10} - 3v_2 \otimes v_{-8} - \frac{1}{3}v_0 \otimes v_{-6} + v_{-2} \otimes v_{-4} \\
 &\quad - v_{-4} \otimes v_{-2} + \frac{1}{3}v_{-6} \otimes v_0 + 3v_{-8} \otimes v_2 \\
 &\quad - 21v_{-10} \otimes v_4, \\
 t_{-8} &= 6v_2 \otimes v_{-10} - 2v_0 \otimes v_{-8} + \frac{2}{3}v_{-2} \otimes v_{-6} \\
 &\quad - \frac{2}{3}v_{-6} \otimes v_{-2} + 2v_{-8} \otimes v_0 - 6v_{-10} \otimes v_2, \\
 t_{-10} &= v_0 \otimes v_{-10} - \frac{3}{5}v_{-2} \otimes v_{-8} + \frac{7}{15}v_{-4} \otimes v_{-6} \\
 &\quad - \frac{7}{15}v_{-6} \otimes v_{-4} + \frac{3}{5}v_{-8} \otimes v_{-2} - v_{-10} \otimes v_0.
 \end{aligned}$$

We put this information, together with the coefficients of the expressions for the other module basis vectors as linear combinations of the tensor basis vectors, into a ma-

trix A of size 121×121 . The inverse matrix A^{-1} shows how to express the tensor basis vectors as linear combinations of the module basis vectors and, in particular, gives the projection P from $V(10) \otimes V(10)$ to the $V(10)$ summand. Table 4 gives this projection, in the sense that the scalar c_{pq} in row v_p and column v_q satisfies the equation $P(v_p \otimes v_q) = c_{pq}t_{p+q}$; as above we use t to denote a vector in the $V(10)$ summand of the tensor product. In this table the rows and columns are indexed by the weights $p, q = 10, 8, \dots, -8, -10$. If we now introduce a scaled basis

$$v'_p = \frac{13}{30}i!v_p, \quad i = \frac{1}{2}(10 - p),$$

then we obtain the structure constants in the statement of Theorem 6.1. \square

7. COMPUTATIONAL METHODS

Let A be any algebra (not necessarily associative) over a field \mathbb{F} . That is, let A be a vector space over \mathbb{F} , together with a bilinear map $A \times A \rightarrow A$ (equivalently, a linear map $A \otimes A \rightarrow A$). We are interested in the polynomial identities satisfied by the algebra A . To simplify the discussion, we will assume initially that the base field \mathbb{F} has characteristic 0. This assumption implies that any polynomial identity over \mathbb{F} is equivalent to a family of homogeneous multilinear identities.

7.1 Associative Polynomials

We fix a positive integer n and a set of n indeterminates

$$X = \{a_1, a_2, \dots, a_n\}.$$

We let S_n be the symmetric group on $\{1, 2, \dots, n\}$. We will write elements of S_n as monomials of degree n in the

set X . That is, the permutation $\sigma \in S_n$ corresponds to the monomial

$$p_\sigma = a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}.$$

Here we apply σ to the subscripts not the positions, so we have to multiply permutations from right to left. This convention assumes that we are dealing with multilinear identities; we regard an identity with a variable repeated k times as shorthand for the symmetric sum of $k!$ terms over all permutations of k new variables in the positions of the original repeated variable.

An example will make this clear. Suppose we want to look at

$$f(x_1, x_2, x_3, x_4, x_5) = x_5 x_1 x_3 x_2 x_4 + x_1 x_5 x_3 x_2 x_4.$$

In terms of the action on subscripts, we would write this identity as

$$(1542)(3) + (1)(254)(3) \quad \text{or} \quad (1542) + (254).$$

The left ideal generated by this element of the group ring will include

$$\begin{aligned} (123)[(1542) + (254)] &= (1543) + (12543) \\ &= x_5 x_2 x_1 x_3 x_4 + x_2 x_5 x_1 x_3 x_4. \end{aligned}$$

This is just what we want: those portions of the monomials that do not move are substituted for the same element in all terms. The parts which switch are switched into new elements but the pattern of the movement will be the same. But, we have to multiply from right to left because the right-hand permutation hits the subscript first.

In terms of the action of permutations on positions, we would write the same identity as $I + (12)$. If $I + (12)$ acts on $x_5 x_1 x_3 x_2 x_4$, we get

$$x_5 x_1 x_3 x_2 x_4 + x_1 x_5 x_3 x_2 x_4.$$

Here, we must multiply from left to right because we put the permutation in first and then apply the identity:

$$(123)[I + (12)] = (123) + (1)(23),$$

and $(123) + (23)$ acting on $x_5 x_1 x_3 x_2 x_4$ is

$$x_3 x_5 x_1 x_2 x_4 + x_5 x_3 x_1 x_2 x_4.$$

This is also correct if we make the assumption that permutations are acting on positions.

The action on subscripts gives a left ideal when we multiply from right to left. The action on positions gives a left ideal when we multiply from left to right. The first

method (action on subscripts) is easier to program and corresponds directly to the representation theory given in standard references such as [James and Kerber 81].

Let P_n be the linear span of all $n!$ monomials p_σ as σ ranges over S_n . We regard P_n as a module over S_n under the action induced by $\pi p_\sigma = p_{\pi\sigma}$. If A is an associative algebra, then the elements of P_n correspond to the homogeneous multilinear identities of degree n that could be satisfied by A . More precisely, let $f \in P_n$ be a homogeneous multilinear associative polynomial of degree n . We say that f is an identity for A if it vanishes identically on A , that is, if

$$f(x_1, x_2, \dots, x_n) = 0 \text{ for every } x_1, x_2, \dots, x_n \in A.$$

An identity can be identified with the submodule that it generates: an algebra A satisfies the identity f if and only if it satisfies all the identities in the submodule generated by f .

The group ring $\mathbb{F}S_n$ decomposes as the direct sum of full matrix rings of size $d_\lambda \times d_\lambda$, where d_λ is the dimension of the irreducible $\mathbb{F}S_n$ -module corresponding to λ as λ ranges over all partitions of n . The module P_n decomposes into the same direct sum. Any submodule is a direct sum of irreducible modules. Since we can perform computations one representation at a time, it is possible to study an identity by breaking the problem down into smaller pieces, each of which corresponds to a partition of the degree n .

7.2 Nonassociative Polynomials

If A is a nonassociative algebra then we also have to keep track of the possible association types that may occur in a monomial of degree n . The number of distinct association types (that is, the number of distinct ways to parenthesize n factors) is the Catalan number

$$t_n = \frac{1}{n} \binom{2n-2}{n-1}.$$

Here is a short table:

n	1	2	3	4	5	6	7	8
t_n	1	1	2	5	14	42	132	429

The total number of homogeneous multilinear monomials of degree n for a nonassociative algebra A is the Catalan number t_n (the number of association types) times the factorial $n!$ (the number of associative words, or unparenthesized monomials):

$$\frac{1}{n} \binom{2n-2}{n-1} n! = \frac{(2n-2)!}{(n-1)!}.$$

This is the dimension of the vector space Q_n of all possible multilinear homogeneous nonassociative identities of degree n . We can think of Q_n as the direct sum of t_n copies of P_n , one copy for each association type.

7.3 Random Vectors

Suppose we want to find the simplest identity satisfied by a nonassociative algebra A of dimension s . We represent elements of A as s -tuples with respect to some convenient basis, and we assume that we have an explicit procedure for computing the product in A with respect to this basis. (Such a procedure can be obtained from the structure constants for the algebra A .)

For each degree n and each partition $n = n_1 + \dots + n_k$ we list all the corresponding monomials involving k variables x_1, \dots, x_k with x_i occurring n_i times in each monomial. (We are no longer assuming multilinearity.) Let c be the number of these monomials; we have

$$c = \frac{1}{n} \binom{2n-2}{n-1} \binom{n}{n_1, \dots, n_k} = \frac{(2n-2)!}{(n-1)!} \cdot \frac{1}{n_1!n_2! \dots n_k!}.$$

We set up a matrix M with $c + s$ rows and c columns, initialized to zero.

We now begin the following iterative procedure: we generate k random elements of A . Each column of M is labeled by one of the c monomials, and each monomial involves k variables. We set the k variables equal to the k random elements of A and evaluate each of the c monomials using the product in A . For each column of M , we obtain another element of A which we view as an $s \times 1$ column vector. In column j of M , in rows $c + 1$ to $c + s$ (at the bottom of the matrix), we put the $s \times 1$ column vector obtained by evaluating the j th monomial on the k random elements of A . After the $s \times c$ bottom segment of M is filled in this way, each of the last s rows contains a linear relation which must be satisfied by the coefficients of any identity satisfied by A . To see this, let T_i ($1 \leq i \leq c$) be the monomials labeling the columns of the matrix M . Let

$$\sum_{i=1}^c a_i T_i$$

be the general linear combination of the monomials T_i where the coefficients a_i are indeterminates. When we evaluate the T_i on the k random elements of A , each T_i becomes an $s \times 1$ column vector:

$$T_i = (t_{i1}, t_{i2}, \dots, t_{is})^T.$$

Writing out the components of the general linear combination of the T_i , we get s linear relations that must be

satisfied by the a_i :

$$\sum_{i=1}^c t_{ij} a_i, \quad 1 \leq j \leq s.$$

These are the relations that occupy the last s rows of the matrix M . (Another way of viewing this process is to say that we are generating random counterexamples to the possible identities satisfied by A ; we thank Don Pigozzi for pointing this out.) We now compute the row canonical form of M . Since M has size $(c + s) \times c$, its rank must be $\leq c$, and so the bottom $s \times c$ submatrix will now be zero.

We repeat this fill-and-reduce process; in our experience, each iteration tends to increase the rank of M by $s = \dim A$. The process is continued until the rank of M stops increasing. We perform a few more iterations to be sure that M has reached full rank. If the nullspace of M at this point is nonzero, it contains candidates for non-trivial identities satisfied by A . We now test the candidates by seeing if they evaluate to zero on further choices of random arguments. Finally, we attempt to prove them directly.

7.4 The Symmetric Group Ring

Another technique we use to find identities is the representation theory of the symmetric group. The process of studying identities through group representations is indirect and complicated. It does, however, have two tremendous advantages. Because the process can be run separately on each representation of the symmetric group, the calculations can be broken up into smaller, more manageable portions. Also, the basic unit of the group algebra approach is the identity, rather than all substitutions in an identity. Since there are $n!$ possible substitutions, one can see that it is better to work with one object rather than $n!$ objects. Further details on this approach may be found in a previous publication of the authors [Bremner and Hentzel 04]).

To save space and time these computational methods were implemented over the field with p elements where p is a prime larger than the degree of the identities under consideration. This guarantees that the the group ring will be semisimple, and usually ensures that the results will be equivalent to the characteristic 0 case; that is, the dimension of a submodule of identities will be equal to the dimension of the corresponding submodule over \mathbb{Q} , and the basis which is computed will be formally the same in the two cases if the coefficients of the monomials are expressed as small integers.

$$\begin{pmatrix} 83 & 30 & 30 & 94 & 66 & 92 & 37 & 4 & 1 & 45 & 4 & 50 & 36 & 25 & 41 \\ 40 & 90 & 86 & 53 & 17 & 22 & 94 & 74 & 25 & 73 & 4 & 40 & 15 & 0 & 73 \\ 32 & 86 & 45 & 2 & 87 & 98 & 49 & 67 & 83 & 97 & 7 & 83 & 64 & 19 & 30 \\ 25 & 36 & 47 & 63 & 3 & 1 & 76 & 92 & 37 & 7 & 12 & 15 & 92 & 25 & 85 \\ 93 & 81 & 46 & 8 & 59 & 62 & 71 & 43 & 95 & 57 & 66 & 53 & 36 & 42 & 26 \\ 67 & 78 & 38 & 10 & 34 & 18 & 23 & 21 & 88 & 33 & 98 & 69 & 87 & 51 & 76 \\ 59 & 44 & 29 & 36 & 44 & 76 & 13 & 33 & 67 & 8 & 43 & 9 & 88 & 91 & 81 \end{pmatrix}$$

TABLE 5. Evaluation of the 15 monomials.

7.5 A Detailed Example: The Malcev Identity

We show how these computational methods can be used to discover the Malcev identity in degree 4 from the multiplication table for the seven-dimensional simple non-Lie Malcev algebra obtained from the representation $V(6)$ of $sl(2)$, which was presented in Theorem 5.2. It is important to have an integral matrix of structure constants, with no common factor in the entries, so that we can do these calculations in any characteristic.

There are five association types for a product of four variables:

$$((wx)y)z, (w(xy))z, (wx)(yz), w((xy)z), w(x(yz)).$$

For an anticommutative product, these five association types reduce to two:

$$((wx)y)z, (wx)(yz).$$

Using all 24 permutations of the four variables, and accounting for anticommutativity, we obtain a total of 15 inequivalent multilinear degree-4 monomials for an anticommutative product. There are 12 in the first association type and three more in the second:

$$\begin{aligned} &((wx)y)z, ((wx)z)y, ((wy)x)z, \\ &((wy)z)x, ((wz)x)y, ((wz)y)x, \\ &((xy)w)z, ((xy)z)w, ((xz)w)y, \\ &((xz)y)w, ((yz)w)x, ((yz)x)w, \\ &(wx)(yz), (wy)(xz), (wz)(xy). \end{aligned}$$

We now construct a matrix A of size 22×15 , initialized to zero. We think of A as consisting of a 15×15 square matrix on top of a 7×15 matrix.

We generate four pseudo-random vectors of length 7, which we call w, x, y, z . The components of these vectors can be regarded as either rational integers or elements of a finite field. In the former case, the components will be uniformly distributed single digits 0 through 9; in the latter case, the components will be uniformly distributed elements 0 through $p - 1$ in the field with p elements. We

then evaluate each of the 15 monomials using the $V(6)$ structure constants in Theorem 5.2. This produces 15 vectors of length 7, which we regard as column vectors and store in the bottom part of A . That is, the 7 components of the evaluation of monomial j are put in column j of A in rows 16 through 22.

We will use arithmetic modulo $p = 101$ and start with these 4 pseudo-random vectors generated by the Maple function $\mathbf{r} := \mathbf{rand}(101)$:

$$\begin{aligned} w &= (70, 76, 37, 82, 29, 56, 42), \\ x &= (47, 21, 41, 85, 35, 15, 97), \\ y &= (60, 39, 11, 14, 39, 61, 1), \\ z &= (21, 58, 99, 89, 51, 6, 32). \end{aligned}$$

Evaluating the 15 monomials on these 4 vectors gives the 7×15 matrix in Table 5.

At this point the 22×15 matrix A consists of that matrix as its bottom part and a 15×15 zero matrix as its top part. We now find the row canonical form of the matrix A . This completes the first iteration of the algorithm.

We now generate four more random vectors

$$\begin{aligned} w &= (27, 65, 24, 30, 90, 84, 42), \\ x &= (26, 51, 83, 96, 22, 23, 15), \\ y &= (24, 58, 63, 21, 36, 84, 67), \\ z &= (92, 60, 83, 2, 16, 75, 50), \end{aligned}$$

and fill in the last seven rows again as before. At this point, after the second fill but before the second row reduction, the matrix A is displayed in Table 6. The first seven rows of this matrix contain the row canonical form of the 7×15 matrix in Table 5. The row canonical form of the entire 22×15 matrix is given in Table 7. Here the last 12 rows have been omitted since they are zero.

Further iterations of the algorithm do not change the row canonical form of the matrix A . Since the row canonical form has rank 10, its nullspace has dimension 5. A basis for the nullspace consists of the rows of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 81 & 59 & 79 & 86 & 67 & 79 & 67 & 94 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 75 & 6 & 32 & 99 & 72 & 32 & 73 & 35 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 44 & 35 & 93 & 23 & 67 & 93 & 67 & 69 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 67 & 60 & 93 & 64 & 30 & 94 & 30 & 29 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 7 & 56 & 50 & 20 & 26 & 49 & 26 & 30 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 37 & 50 & 12 & 25 & 63 & 12 & 62 & 88 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 15 & 1 & 88 & 32 & 46 & 87 & 47 & 56 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 66 & 78 & 38 & 96 & 55 & 7 & 73 & 40 & 44 & 67 & 56 & 64 & 35 & 6 & 51 \\ 84 & 36 & 87 & 19 & 58 & 62 & 71 & 23 & 77 & 88 & 53 & 57 & 79 & 1 & 85 \\ 90 & 89 & 65 & 29 & 87 & 62 & 12 & 20 & 43 & 96 & 2 & 87 & 26 & 25 & 17 \\ 85 & 74 & 93 & 55 & 46 & 15 & 6 & 5 & 64 & 33 & 41 & 20 & 36 & 85 & 74 \\ 60 & 46 & 75 & 26 & 10 & 55 & 17 & 88 & 44 & 78 & 90 & 18 & 77 & 26 & 60 \\ 25 & 94 & 69 & 51 & 39 & 23 & 12 & 32 & 14 & 28 & 48 & 84 & 28 & 66 & 5 \\ 71 & 89 & 82 & 56 & 23 & 67 & 79 & 63 & 6 & 4 & 76 & 86 & 59 & 86 & 36 \end{pmatrix}$$

TABLE 6. First row reduction and second fill.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 100 & 0 & 0 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 100 & 0 & 100 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 100 & 0 & 1 & 100 & 0 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 100 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 100 & 0 & 1 & 100 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 100 \end{pmatrix}$$

TABLE 7. The final row canonical form.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \end{pmatrix}$$

TABLE 8. The matrix of identities.

in Table 8. Since $100 = -1$ modulo 101, we replace each 100 by -1 ; this allows us to regard the identities as polynomials over any field. Expressing the vectors as linear combinations of the original 15 monomials, we obtain the following five identities:

$$I_1 = ((wx)y)z - ((wz)x)y + ((xy)z)w + ((yz)w)x - (wy)(xz),$$

$$I_2 = ((wx)z)y - ((wz)y)x - ((xy)z)w + ((xz)w)y + ((xz)y)w - ((yz)w)x - (wx)(yz) - (wz)(xy),$$

$$I_3 = ((wy)x)z - ((wz)y)x - ((xy)z)w + ((xz)w)y - (wx)(yz),$$

$$I_4 = ((wy)z)x - ((wz)x)y + ((xy)z)w - ((xz)w)y + ((yz)w)x + ((yz)x)w - (wy)(xz) + (wz)(xy),$$

$$\begin{aligned}
 I_5 = & ((xy)w)z + ((xy)z)w - ((xz)w)y - ((xz)y)w \\
 & + ((yz)w)x + ((yz)x)w + (wx)(yz) - (wy)(xz) \\
 & + (wz)(xy).
 \end{aligned}$$

Identity I_1 is the same as the 5-term Malcev identity (5-2), after applying anticommutativity to one term. Identity I_2 is obtained by linearizing the 4-term Malcev identity (5-1) by replacing x by $w + x$ and then interchanging x and y . Identity I_3 is obtained from identity (5-2) by interchanging x and y . Identity I_4 is the linearized form of identity (5-1). The identity I_5 is a consequence of the Malcev identity, which is not equivalent to the Malcev identity: I_5 is the result of applying the element $\pi = 1 + (234) + (243)$ in the group ring of S_4 to the identity I_1 .

In the above calculations we have used characteristic $p = 101$. If we use $p = 2$, we obtain a six-dimensional space of identities. If we use $p = 3$, we obtain a nine-dimensional space of identities. For any other positive characteristic (that is, $p > 4$ where 4 is the degree of the identities), we obtain a five-dimensional space of identities. If we use characteristic 0 (rational arithmetic) and generate random single digits for the vector components, then the components in the evaluated monomials typically have three to five digits, and the matrix entries of the row canonical form typically have 20-digit numerators and denominators. This gives a relatively small but still impressive demonstration of “matrix entry blowup,” when using rational arithmetic to compute a row canonical form, and provides a convincing argument in favour of using modular arithmetic to save computer memory and CPU time.

8. IDENTITIES FOR THE 11-DIMENSIONAL ALGEBRA

In this section we classify all the identities of degree ≤ 7 for the 11-dimensional anticommutative algebra A obtained from the projection $V(10) \otimes V(10) \rightarrow V(10)$ described in Section 6 (Theorem 6.1).

Theorem 8.1. *Every identity of degree ≤ 6 satisfied by the algebra A is a consequence of anticommutativity.*

Proof: (By computer, following the methods and the example presented in Section 7.) There are six inequivalent anticommutative association types in degree 6:

$$\begin{aligned}
 & [[[[[a, b], c], d], e], f], & & [[[[[a, b], [c, d]], e], f], \\
 & [[[[[a, b], c], [d, e]], f], & & [[[[[a, b], c], d], [e, f]], \\
 & [[[[a, b], [c, d]], [e, f]], & & [[[[a, b], c], [[d, e], f]].
 \end{aligned}$$

There are altogether 945 multilinear anticommutative monomials; counting the number in each association type (dividing $6!$ by the number of antisymmetries which follow from anticommutativity), we have

$$\begin{aligned}
 & \frac{1}{2}6! + \frac{1}{8}6! + \frac{1}{4}6! + \frac{1}{4}6! + \frac{1}{16}6! + \frac{1}{8}6! \\
 & = 360 + 90 + 180 + 180 + 45 + 90 = 945.
 \end{aligned}$$

To determine the identities for the 11-dimensional algebra, we therefore need a matrix of size 956×945 : the upper part is a 945×945 square matrix, and the lower part is a 11×945 matrix into which we place the results of evaluating the anticommutative monomials. We repeatedly generate six random vectors of length 11 with components from the field with $p = 101$ elements. For each list of six random vectors, we evaluate all 945 monomials. For each monomial we put the result of the evaluation into the lower part of the matrix as a column vector in the column corresponding to the monomial. When the lower part of the matrix is filled in this way, we compute the row canonical form of the matrix. After each iteration of this process, the rank of the matrix increases by 11. After 86 iterations the matrix reaches full rank (945). This implies that the nullspace is zero and that there are no identities in degree 6, at least for characteristic $p = 101$.

Since an identity in degree < 6 would imply the existence of an identity of degree 6 (for example, replacing a variable by a product of variables, or by multiplying the identity by other variables), we have also shown that there are no identities of degree ≤ 6 in characteristic $p = 101$.

To complete the proof we need to argue that the nonexistence of identities in characteristic p for some p implies the nonexistence of identities in characteristic 0. We will show the contrapositive, that the existence of an identity in characteristic 0 implies the existence of a nonzero identity in characteristic p for every p . Let $c = 945$, let T_i for $1 \leq i \leq c$ be the multilinear anticommutative monomials of degree 6, and let a_i for $1 \leq i \leq c$ be rational numbers. Assume that

$$\sum_{i=1}^c a_i T_i$$

is an identity in characteristic 0 satisfied by the algebra A . Let m be the least common multiple of the denominators of the a_i for $1 \leq i \leq c$, and write $a'_i = ma_i$. Then, the a'_i for $1 \leq i \leq c$ are integers, and so

$$\sum_{i=1}^c a'_i T_i$$

is an identity for A with integer coefficients. Now let d be the greatest common divisor of the integers a'_i for $1 \leq i \leq c$, and write $a''_i = a'_i/d$. Then, the a''_i for $1 \leq i \leq c$ are integers with no common prime factor, which shows that for every prime p at least one of the a''_i remains nonzero when reduced modulo p . Let $a''_{i,p}$ for $1 \leq i \leq c$ be the residue class of a''_i modulo p ; then

$$\sum_{i=1}^c a''_{i,p} T_i$$

is a nonzero identity for A in characteristic p . This completes the proof. \square

We now consider identities of degree 7. Here are the 11 inequivalent anticommutative association types in degree 7:

- 1: [[[[[a, b, c, d, e, f, g], 2: [[[[[$a, b, [c, d], e, f, g$],
- 3: [[[[[$a, b, c, [d, e], f, g$], 4: [[[[[$a, b, c, d, [e, f], g$],
- 5: [[[[[$a, b, [c, d], [e, f], g$], 6: [[[[[$a, b, c, [[d, e, f], g$],
- 7: [[[[[$a, b, c, d, e, [f, g]$], 8: [[[[[$a, b, [c, d], e, [f, g]$],
- 9: [[[[[$a, b, c, [d, e], [f, g]$], 10: [[[[[$a, b, c, d, [[e, f], g]$],
- 11: [[[[[$a, b, [c, d], [[e, f], g]$].

Theorem 8.2. *The algebra A satisfies identities in degree 7 which are not consequences of anticommutativity. These identities exist only in the S_7 -representations labeled by the partitions 22111, 211111, and 1111111 (the last 3 representations). That is, the identities involve the variables $aabbcd$, $abcdef$, and $abcdefg$.*

Proof: (By computer.) To determine which partitions of 7 correspond to nontrivial identities we use the S_n -module methods described in Section 7. Table 9 shows the 15 partitions that label the distinct irreducible representations of the symmetric group S_7 . Column 2 gives a partition λ , and column 3 gives the dimension d_λ of the corresponding representation. Column 4 gives the product of 11 (the number of inequivalent anticommutative association types in degree 7) and the dimension of the representation; this is the dimension of the space of all possible identities in this representation in this degree. Column 5 gives the rank of the matrix of counterexamples, which was generated by the random procedure described in Section 7. Column 4 minus column 5 is the dimension of the space of identities satisfied by the 11-dimensional algebra, but this includes identities which are trivial consequences of anticommutativity. Column 6 gives the dimension of the space of identities which

i	$\lambda = \lambda_i$	d_λ	$11d_\lambda$	Rank	Skew	New
1	7	1	11	0	11	0
2	61	6	66	1	65	0
3	52	14	154	8	146	0
4	511	15	165	11	154	0
5	43	14	154	15	139	0
6	421	35	385	50	335	0
7	4111	20	220	35	185	0
8	331	21	231	41	190	0
9	322	21	231	49	182	0
10	3211	35	385	91	294	0
11	31111	15	165	48	117	0
12	2221	14	154	49	105	0
13	22111	14	154	55	98	1
14	211111	6	66	25	36	5
15	1111111	1	11	2	4	5

TABLE 9. Degree 7 identities for A .

are trivial consequences of anticommutativity. Column 7 gives column 4 minus the sum of columns 5 and 6: this number is always nonnegative, and if it is positive, it tells us there are new nontrivial identities satisfied by the 11-dimensional algebra in that representation. These new identities occur only in the last three representations. \square

We will now discuss the last 3 representations separately, starting with the last representation.

Theorem 8.3. *The space of multilinear identities of degree 7 (partition 1111111) for the algebra A has dimension 5. A basis for this space consists of these five identities:*

$$\begin{aligned}
 I_1 &= \sum_{alt} [[[[[ab]c]d][ef]]g] - \sum_{alt} [[[[[ab]c]d]e][fg]] \\
 &\quad + \sum_{alt} [[[[[ab]c]d][[ef]g]], \\
 I_2 &= \sum_{alt} [[[[[[ab]c]d]e]f]g] - 2 \sum_{alt} [[[[[ab]c]d][ef]]g] \\
 &\quad + \sum_{alt} [[[[[ab]c]d][[ef]g]], \\
 I_3 &= 2 \sum_{alt} [[[[[[ab]c][de]]f]g] - 3 \sum_{alt} [[[[[ab]c][[de]f]]g] \\
 &\quad - \sum_{alt} [[[[[ab]c][de]][fg]], \\
 I_4 &= 5 \sum_{alt} [[[[[ab]c][de]]f]g] + 11 \sum_{alt} [[[[[ab]c]d][ef]]g] \\
 &\quad + 5 \sum_{alt} [[[[[ab]c][de]][fg]],
 \end{aligned}$$

$$I_5 = 111 \sum_{alt} [[[[[ab]c]d]e]f]g] - 119 \sum_{alt} [[[[ab]c][[de]f]]g] - 185 \sum_{alt} [[[[ab]c]d][[ef]g]].$$

Here the sums are alternating sums over the seven variables a, b, c, d, e, f, g .

Proof: We wrote a Maple program to evaluate, in characteristic 0, the alternating sum over each of the 11 association types in degree 7 for any seven vectors in the algebra A . We ran this program over all $\binom{11}{7} = 330$ choices of seven distinct vectors from the 11 basis vectors v_{10}, \dots, v_{-10} . Remarkably, the 330 resulting vectors span a subspace of dimension 2. A basis for this subspace consists of the two rows of this matrix:

$$\begin{pmatrix} -75 & 0 & 44 & -60 & 0 & 0 & -105 & 0 & 88 & -45 & 0 \\ 42 & 0 & -299 & 105 & 0 & -222 & 273 & 0 & 68 & 168 & 0 \end{pmatrix}$$

In this matrix columns 2, 5, 8, and 11 are zero; it is easy to check that for the corresponding association types the alternating sum collapses to the zero polynomial as a result of anticommutativity. The seven nonzero columns are 1, 3, 4, 6, 7, 9, and 10. There are $\binom{7}{3} = 35$ subsets of three columns from among these seven columns. This gives 35 matrices of size 2×3 , for each of which we compute the row canonical form. We sort the resulting 35 reduced matrices, starting with the matrices with the simplest entries (in terms of the number of digits). We compute the one-dimensional nullspace of each of these 35 matrices, and obtain 35 identities, again ordered by the complexity of the coefficients. From among these 35 identities, we choose the simplest five that span the five-dimensional space of all identities. These five simplest identities are displayed above. Note that five is the number of independent new identities for this representation predicted by Table 9. \square

We now consider the representation labeled by partition 211111. The corresponding identities will involve the variables $abcdef$. Accounting for anticommutativity, there are altogether 87 monomials in these six variables with one repetition, in which the letters $bcdef$ occur in alphabetical order from left to right. We regard each of the 87 monomials as representing the alternating sum over the five variables $bcdef$; thus, each monomial actually represents a sum of 120 terms. The number of monomials in each of the 11 association types is given in this table:

association type	1	2	3	4	5	6	7	8	9	10	11
# of monomials	15	7	10	10	4	7	10	4	6	10	4

- 1: $[abcdef]$ (1), 2: $[abcdef]$ (1), 3: $[abcdæf]$ (1),
- 4: $[abcdeaf]$ (1), 5: $[abcdefa]$ (1), 6: $[bcaædf]$ (1),
- 7: $[bcdaæf]$ (1), 8: $[bcdaæf]$ (1), 9: $[bcdeæfa]$ (1),
- 10: $[bcdaæf]$ (1), 11: $[bcdaæf]$ (1), 12: $[bcdeæfa]$ (1),
- 13: $[bcdeæfa]$ (1), 14: $[bcdeæfa]$ (1), 15: $[bcdeæfa]$ (1),
- 16: $[abacdef]$ (2), 17: $[abcdæf]$ (2), 18: $[abcdeaf]$ (2),
- 19: $[abcdeæf]$ (2), 20: $[bcdeæfa]$ (2), 21: $[bcdeæfa]$ (2),
- 22: $[bcdeæfa]$ (2), 23: $[abacdef]$ (3), 24: $[abcdeaf]$ (3),
- 25: $[abcdeæf]$ (3), 26: $[abcdeæf]$ (3), 27: $[bcaædf]$ (3),
- 28: $[bcdaæf]$ (3), 29: $[bcdeæfa]$ (3), 30: $[bcdaæf]$ (3),
- 31: $[bcdeæfa]$ (3), 32: $[bcdeæfa]$ (3), 33: $[abacdef]$ (4),
- 34: $[abcdeæf]$ (4), 35: $[abcdæf]$ (4), 36: $[abcdeæf]$ (4),
- 37: $[bcaædf]$ (4), 38: $[bcdaæf]$ (4), 39: $[bcdeæfa]$ (4),
- 40: $[bcdaæf]$ (4), 41: $[bcdeæfa]$ (4), 42: $[bcdeæfa]$ (4),
- 43: $[abacdef]$ (5), 44: $[abcdæf]$ (5), 45: $[abcdeæf]$ (5),
- 46: $[bcdeæfa]$ (5), 47: $[abacdef]$ (6), 48: $[abcdeæf]$ (6),
- 49: $[abcdeæf]$ (6), 50: $[abcdeæf]$ (6), 51: $[bcdeæfa]$ (6),
- 52: $[bcdeæfa]$ (6), 53: $[bcdeæfa]$ (6), 54: $[abacdef]$ (7),
- 55: $[abcdeæf]$ (7), 56: $[abcdæf]$ (7), 57: $[abcdeæf]$ (7),
- 58: $[bcaædf]$ (7), 59: $[bcdaæf]$ (7), 60: $[bcdeæfa]$ (7),
- 61: $[bcdaæf]$ (7), 62: $[bcdaæf]$ (7), 63: $[bcdeæfa]$ (7),
- 64: $[abacdef]$ (8), 65: $[abcdæf]$ (8), 66: $[abcdeæf]$ (8),
- 67: $[bcdeæfa]$ (8), 68: $[abacdef]$ (9), 69: $[abcdeæf]$ (9),
- 70: $[abcdeæf]$ (9), 71: $[bcaædf]$ (9), 72: $[bcdeæfa]$ (9),
- 73: $[bcdaæf]$ (9), 74: $[abacdef]$ (10), 75: $[abcdeæf]$ (10),
- 76: $[abcdæf]$ (10), 77: $[abcdeæf]$ (10), 78: $[bcaædf]$ (10),
- 79: $[bcdaæf]$ (10), 80: $[bcdeæfa]$ (10), 81: $[bcdaæf]$ (10),
- 82: $[bcdeæfa]$ (10), 83: $[bcdeæfa]$ (10), 84: $[abacdef]$ (11),
- 85: $[abcdæf]$ (11), 86: $[abcdeæf]$ (11), 87: $[bcdeæfa]$ (11).

TABLE 10. The 87 monomials in representation 211111.

The complete list of all 87 monomials is given in Table 10, where we omit the brackets and give instead the permutation of the seven symbols and the association type (in parentheses). In what follows we will refer to these monomials by number.

Theorem 8.4. *The space of identities for the algebra A in variables $abcdef$ (partition 211111) has dimension 5. A basis for this space consists of these four identities where the monomials (alternating sums) are indicated by numbers in square brackets:*

$$J_1 = 4[24] + 2[27] + 2[30] - 2[31] - 2[50] + 1[53] - 2[69] - 1[71] - 2[73],$$

$$J_2 = -88[1] + 88[2] - 88[3] + 88[4] + 44[9] - 44[12] + 44[14] - 44[15] - 150[23] + 300[24] + 150[25] + 75[29] - 150[31] - 75[32] - 150[47] + 150[49] - 150[50] + 44[54] - 44[55] + 44[56] + 44[60] - 44[62]$$

$$\begin{aligned}
 &+ 44[63] + 75[68] - 150[69] + 75[72] - 150[73] + 44[74] && 3950368961407440 && [1] && - && 1913428424733144 && [2] \\
 &- 44[75] + 44[77] - 44[79] + 44[81] + 44[83], && + && 2773300851713040 && [3] && - && 4077469423453800 && [4] \\
 J_3 = &-176[2] + 176[3] - 88[6] + 88[7] - 88[11] + 88[12] && + && 456996975114944 && [5] && + && 1018470268337148 && [6] \\
 &+ 88[13] - 88[14] + 90[23] - 180[24] - 90[25] - 45[29] && - && 588534054847200 && [7] && - && 63550231023180 && [8] \\
 &+ 90[31] + 45[32] + 90[47] - 90[49] + 90[50] + 88[55] && - && 1746685993146248 && [9] && - && 429936213489948 && [10] \\
 &- 88[56] + 44[58] - 44[59] + 88[62] - 88[63] - 45[68] && + && 1082020499360328 && [11] && + && 728215724809100 && [12] \\
 &+ 90[69] - 45[72] + 90[73] + 88[75] + 44[78] - 88[81] && - && 652084285870380 && [13] && - && 1158151938299048 && [14] \\
 &- 44[82], && + && 1810236224169428 && [15] && + && 1471606883931976 && [16] \\
 J_4 = &-44[16] - 22[17] + 22[18] - 22[19] - 30[23] + 60[24] && + && 735803441965988 && [17] && - && 735803441965988 && [18] \\
 &+ 30[25] + 15[29] - 30[31] - 15[32] - 44[35] + 22[36] && + && 735803441965988 && [19] && + && 5861177270946330 && [23] \\
 &- 22[38] + 11[39] + 22[40] - 11[41] + 44[42] - 30[47] && - && 9579445919392388 && [24] && - && 5258835958813110 && [25] \\
 &+ 30[49] - 30[50] + 22[64] + 11[65] - 22[66] + 15[68] && + && 1149923171952060 && [26] && + && 1071454311250136 && [27] \\
 &- 30[69] + 15[72] - 30[73] + 22[84] + 22[85] + 11[86], && + && 301170656066610 && [28] && - && 2355627049497135 && [29] \\
 & && + && 469112999116916 && [30] && + && 3639799787744134 && [31] \\
 & && + && 2054456393430525 && [32] && + && 130877446257360 && [33] \\
 & && - && 75788640567948 && [34] && + && 1471606883931976 && [35] \\
 & && + && 130877446257360 && [36] && + && 27544402844706 && [37] \\
 & && + && 604925995708628 && [38] && - && 660014801398040 && [40] \\
 & && - && 27544402844706 && [41] && - && 604925995708628 && [42] \\
 & && + && 39381560807160 && [43] && + && 39381560807160 && [44] \\
 & && + && 19690780403580 && [45] && + && 5021454786062850 && [47] \\
 & && - && 5021454786062850 && [49] && + && 5021454786062850 && [50] \\
 & && - && 1202761859999040 && [56] && + && 127100462046360 && [57] \\
 & && - && 601380929999520 && [59] && + && 63550231023180 && [60] \\
 & && + && 601380929999520 && [61] && - && 63550231023180 && [62] \\
 & && - && 1139211628975860 && [63] && - && 604925995708628 && [64] \\
 & && - && 302462997854314 && [65] && + && 604925995708628 && [66] \\
 & && - && 1748892612436845 && [68] && + && 3950000474812714 && [69] \\
 & && - && 602341312133220 && [70] && + && 226107624969512 && [71] \\
 & && - && 2050063268503455 && [72] && + && 4552341786945934 && [73] \\
 & && - && 1018470268337148 && [74] && + && 55628434846908 && [75] \\
 & && - && 859872426979896 && [76] && - && 2221232128336188 && [77] \\
 & && - && 481420916745120 && [78] && + && 588534054847200 && [79] \\
 & && - && 601380929999520 && [80] && + && 374307778643040 && [81] \\
 & && + && 1082801846744640 && [82] && - && 1791295914846240 && [83] \\
 & && - && 660014801398040 && [84] && - && 660014801398040 && [85] \\
 & && - && 330007400699020 && [86] && && && && [86]
 \end{aligned}$$

together with a fifth identity that is displayed in Table 11.

Proof: (By computer.) We did the following computations first in characteristic $p = 101$ and then repeated them in characteristic 0. We wrote a Maple program to evaluate the alternating sum over each of the 87 monomials for any seven vectors from the algebra A . We created a matrix of size 98×87 and initialized it to zero. We generated six random vectors and assigned these to the variables $abcdef$. The results of evaluating the monomials (alternating sums) were stored in the bottom 11×87 submatrix. We then computed the row canonical form of the matrix. After each repetition of this fill and reduce process, the rank of the matrix increased by 11, until the seventh iteration, at which point the rank reached 73. Another 10 iterations did not increase the rank. The nullspace of the resulting matrix has dimension 14 and consists of the identities satisfied by A in representation 211111. Of these 14 identities, eight are trivial in the sense that they consist of a single monomial (alternating sum) which collapses to zero as a result of anticommutativity. These are monomials 20, 21, 22, 46, 48, 51, 67, and 87 in Table 10. Of the remaining six identities, two differ only by another trivial identity: the two-term identity which says that the sum of monomials (alternating sums) 52 and 53 is always 0 (this is another consequence of anticommutativity). This leaves five independent non-trivial identities (as predicted by Table 9). In characteristic 0, for each identity we cleared the denominators of the coefficients and then divided by the gcd of the coefficients. Four of the five resulting integral identities have coefficients with no more than three digits; these four are stated in Theorem 8.4. The fifth identity has coefficients

TABLE 11. The fifth identity in representation 211111.

with up to 16 digits, which contain large prime factors; it is displayed in Table 11. \square

Finally, we consider the representation labeled by partition 22111. Here the variables are $aabbcd$. There are altogether 460 monomials over all 11 association types, accounting for anticommutativity and requiring that cde occur in alphabetical order from left to right. We regard each monomial as representing the alternating sum over cde , so each monomial is actually a sum of six terms. Here is the number of monomials in each association type:

association type	1	2	3	4	5	6	7	8	9	10	11
# of monomials	110	30	55	55	14	31	55	14	27	55	14

1	7	66	14	100	94	35	87	5	0	94	96	0	7	72
44	29	57	64	37	93	36	86	73	7	21	97	100	8	12
66	7	22	64	6	30	2	78	5	89	7	67	49	29	64
64	30	86	8	7	12	66	7	97	100	8	6	30	22	64
78	2	93	36	86	73	7	21	29	64	67	49	30	64	5
89	7	7	8	86	39	48	14	1	17	61	36	90	98	83
51	68	61	36	1	17	98	90	39	48	14	68	51	83	4
2	95	95	2	4	74	89	79	27	12	22	29	13	72	88
53	48	31	53	17	70	48	70	48	84	84	59	77	66	31
53	17	66	77	59	4	21	22	97	80	79	70	30	23	71
78	45	56	63	53	58	35	67	82	72	67	77	67	100	54
48	19	34	66	72	67	67	82	67	77	63	53	58	1	47
34	19	48	69	51	82	84	36	32	50	36	84	82	33	68
0	17	42	0	84	59	53	24	48	77	79	0	0	84	59
67	48	77	70	66	5	17	17	96	26	53	53	22	70	66
48	77	96	17	17	84	59	67	75	53	53	96	79	31	72
35	35	95	29	35	35	31	6	37	64	65	18	36	83	18
0	0	36	0	65	65	83	36	18	57	21	99	44	80	2
23	11	23	90	78	86	15	15	48	82	67	0	53	19	36
84	67	82	53	67	0	84	36	82	67	50	47	14	51	54
87	48	94	53	7	44	57	51	4	86	7	2	8	95	7
64	30	64	89	49	64	44	94	95	7	2	8	30	64	37
4	86	7	64	49	57	89	7	48	17	36	65	0	36	17
36	48	0	48	53	64	79	37	22	13	88	90	53	48	48
0	24	53	77	99	22	2	79	66	23	78	78	53	33	82
67	45	54	78	68	67	82	56	53	47	23	51	15	50	86
66	50	87	33	51	14	68	7	0	94	0	51	64	37	15
18	80	93	50	94	33	64	23	99	94	44	71	37	87	94
33	93	50	37	99	23	15	18	80	57	37	71	94	14	87
65	3	11	33	36	11	3	87	68	54	47	22	95	79	6
90	53	48	84	0	17	17	59	84	42					

TABLE 12. The identity K in representation 22111 ($p = 101$).

For these computations the matrix will have size 471×460 . Because the matrix is so large, we did this representation in characteristic $p = 101$ only.

Theorem 8.5. *Over a field of characteristic $p = 101$, the algebra A satisfies a single identity in the variables $aabbcd$ (partition 22111). It has 443 nonzero coefficients, which are displayed in Table 12. The entry in row i and column j is the coefficient of monomial $15(i-1)+j$ in the lexicographically ordered list of 460 monomials.*

Proof: This proof is very similar to the computational proofs of the previous two theorems. The matrix has 460 columns, and the rank eventually stabilizes at 457. Of the three identities in the nullspace, two are trivial: one states that monomials 282 and 287 sum to 0, and the other states that monomials 291 and 295 sum to 0. The coefficients of the third identity K are presented in Table 12. \square

9. UNITAL EXTENSIONS

The description in Section 2 of the action of $sl(2)$ on $V(n)$ in terms of differential operators provides a connection with the notion from classical invariant theory of transvection of homogeneous polynomials, which was used in [Dixmier 84] to express nonassociative algebra structures closely related to ours in terms of partial differentiation of polynomials.

Let $f \in V(m)$ and $g \in V(n)$. Then, we can regard f and g as homogeneous polynomials in X and Y , where f has degree m and g has degree n . (In the terminology of classical invariant theory, f and g are binary forms of degrees m and n .) Let $f^{(i,j)}$ denote the derivative of f taken i times with respect to X and j times with respect to Y :

$$f^{(i,j)} = \left(\frac{\partial}{\partial X}\right)^i \left(\frac{\partial}{\partial Y}\right)^j f.$$

We define the k th transvectant of f and g as follows:

$$(f, g)_k = \frac{(m-k)! (n-k)!}{m! n!} \sum_{i=0}^k (-1)^i \binom{k}{i} f^{(k-i,i)} g^{(i,k-i)}.$$

It is clear that $(f, g)_k \in V(m+n-2k)$. This definition is from [Dixmier 84], but we have simplified the notation. In the special case $m = n$ we have $f, g \in V(n)$ and the transvectant

$$(f, g)_k = \left(\frac{(n-k)!}{n!} \right)^2 \sum_{i=0}^k (-1)^i \binom{k}{i} f^{(k-i,i)} g^{(i,k-i)},$$

which is nontrivial for $0 \leq k \leq n$. When n is even and $k = n/2$, we get a bilinear $sl(2)$ -invariant mapping from $V(n) \times V(n)$ to $V(n)$:

$$(f, g)_{n/2} = \left(\frac{\left(\frac{n}{2}\right)!}{n!} \right)^2 \sum_{i=0}^{n/2} (-1)^i \binom{n/2}{i} f^{(n/2-i,i)} g^{(i,n/2-i)}.$$

This defines a nonassociative algebra structure on the vector space $V(n)$, which contains $sl(2)$ in its derivation algebra; it is commutative when $n \equiv 0 \pmod{4}$ and anticommutative when $n \equiv 2 \pmod{4}$. Since $V(n)$ occurs only once as a direct summand of $V(n) \otimes V(n)$, this nonassociative algebra structure on $V(n)$ must coincide (up to a scalar multiple) with the projection $P: V(n) \otimes V(n) \rightarrow V(n)$.

In [Dixmier 84], the author considers unital nonassociative algebras that are obtained from these commutative or anticommutative algebras by forming the direct sum with a one-dimensional vector space spanned by a new unit element. Thus, we have ordered pairs (a, f) , where $a \in \mathbb{F}$ and $f \in V(n)$, and a product depending on arbitrary scalars $\lambda, \mu \in \mathbb{F}$, defined by

$$(a, f)(b, g) = (\lambda(f, g) + ab, \mu(f, g)_{n/2} + bf + ag), \tag{9-1}$$

where (f, g) is the bilinear form $V(n) \otimes V(n) \rightarrow V(0)$ of Section 2. Similarly, our nonassociative algebra structure on $V(n)$ can be extended to obtain a unital nonassociative algebra of dimension $n + 2$ using Dixmier's formula (9-1):

$$(a, f)(b, g) = (\lambda(f, g) + ab, \mu[f, g] + bf + ag).$$

Here, $a, b \in \mathbb{F}$, $f, g \in V(n)$, and $[f, g]$ is the $sl(2)$ -invariant algebra structure on $V(n)$.

Now assume that $n \equiv 2 \pmod{4}$ so that the algebra structure on $V(n)$ is anticommutative. Except for trivial choices of λ and μ , the unitaly extended algebras will not be anticommutative, and so we consider identities of

degree 3. For $n \geq 10$ we do not expect the unital algebras to be associative or even alternative.

We write $A(\lambda, \mu)$ for the 12-dimensional unital algebra obtained from the anticommutative algebra $V(10)$ by the above formula. From Proposition 2.1 the symmetric bilinear form on $V(10)$ is given by

$$(v_{10-2i}, v_{10-2j}) = \delta_{i+j,10} (-1)^i \binom{10}{i}.$$

In terms of the rescaled basis introduced at the end of the proof of Theorem 6.1, namely

$$v'_p = \frac{13}{30} i! v_p, \quad i = \frac{1}{2}(10 - p),$$

the symmetric bilinear form becomes

$$\begin{aligned} (v_{10-2i}, v_{10-2j}) &= \delta_{i+j,10} (-1)^i \left(\frac{13}{30} \right)^2 10! \\ &= \delta_{i+j,10} (-1)^i 681408. \end{aligned}$$

In Dixmier's formula (9-1) the constant 681408 can be absorbed into the parameter λ , and so (for our current purposes at least) we may make the simplifying assumption that the bilinear form is given by

$$(v_{10-2i}, v_{10-2j}) = \delta_{i+j,10} (-1)^i.$$

Using this, the bilinear multiplication for the unitaly extended algebra $A(\lambda, \mu)$ is defined on basis elements as follows:

$$\begin{aligned} (1, 0)(1, 0) &= (1, 0), \\ (1, 0)(0, v_{10-2j}) &= (0, v_{10-2j}), \\ (0, v_{10-2i})(1, 0) &= (0, v_{10-2i}), \\ (0, v_{10-2i})(0, v_{10-2j}) &= (\delta_{i+j,10} (-1)^i \lambda, \mu[v_{10-2i}, v_{10-2j}]), \end{aligned}$$

where the square brackets refer to the nonassociative structure of Theorem 5.2.

Definition 9.1. An algebra A (over a field \mathbb{F}) is called a noncommutative Jordan algebra if it satisfies the flexible identity

$$(x, y, x) = 0$$

and the Jordan identity

$$(x^2, y, x) = 0.$$

Both these identities are written in terms of the associator $(x, y, z) = (xy)z - x(yz)$. We say that A is quadratic if every element $x \in A$ is the root of some quadratic polynomial depending on x with coefficients in \mathbb{F} .

Theorem 9.2. *The nonassociative algebra $A(\lambda, \mu)$ is a noncommutative Jordan algebra for every $\lambda, \mu \in \mathbb{F}$.*

Proof: It is easy to check that $A(\lambda, \mu)$ is flexible and quadratic, and these together imply the Jordan identity. \square

10. OTHER SIMPLE LIE ALGEBRAS

10.1 Tensor Products

Let L be a simple (finite-dimensional) Lie algebra over an algebraically closed field \mathbb{F} of characteristic 0. Let V and W be two irreducible (finite-dimensional) representations of L . By Weyl's Theorem we know that any finite-dimensional representation of L is completely reducible; that is, it decomposes as the direct sum of irreducible representations. In particular, this holds for the tensor product $V \otimes W$. An important problem in representation theory is to find an explicit decomposition of $V \otimes W$ into a direct sum of irreducible representations. In the simplest case $L = sl(2)$, this problem is solved by the Clebsch-Gordan Theorem.

In the special case $V = W$, we can ask more specifically how many times V occurs as a direct summand of $V \otimes V$. If this multiplicity is nonzero, then there exist nonzero homomorphisms $P: V \otimes V \rightarrow V$ that give V a nonassociative algebra structure, which is L -invariant in the sense that L is contained in the derivation algebra. This structure is commutative when V occurs as a summand of $S^2(V)$ and anticommutative when V occurs as a summand of $\Lambda^2(V)$.

Up to isomorphism, the (finite-dimensional) simple Lie algebras, over an algebraically closed field \mathbb{F} of characteristic 0, are characterized by their Dynkin diagrams. For the numbering of the vertices of the Dynkin diagrams we follow [Humphreys 72].

10.2 Exterior Squares

A simple Lie algebra of rank ℓ has ℓ fundamental representations, which we will denote by Ω_i for $1 \leq i \leq \ell$. On the following pages we list, for each simple Lie algebra of rank $2 \leq \ell \leq 8$ and each fundamental representation, the multiplicity

$$\dim \text{Hom}_L(\Lambda^2 \Omega_i, \Omega_i)$$

of Ω_i as a direct summand of its exterior square. This multiplicity is the number of parameters that occur in the classification of L -invariant anticommutative algebra structures on Ω_i . To perform these calculations we used the software package LiE, which at the time of writing was available online at <http://young.sp2mi.univ-poitiers.fr/~marc/LiE/>.

For detailed information about LiE, see the articles [Cohen et al. 84] and [van Leeuwen 94], which are also available at the given URL.

10.3 Special Linear

For the special linear type A_ℓ none of the fundamental representations occurs as a summand of its own exterior square.

10.4 Orthogonal

For the orthogonal type B_ℓ we have the following results; all other multiplicities are zero. (Neither of the fundamental representations of B_2 occurs in its own exterior square.) The nonadjoint representations are starred:

$\dim B_3 = 21$	$\dim \Omega_2 = 21$	$\dim \text{Hom}_L(\Lambda^2 \Omega_2, \Omega_2) = 1$
$\dim B_4 = 36$	$\dim \Omega_2 = 36$	$\dim \text{Hom}_L(\Lambda^2 \Omega_2, \Omega_2) = 1$
	* $\dim \Omega_3 = 84$	$\dim \text{Hom}_L(\Lambda^2 \Omega_3, \Omega_3) = 1$
$\dim B_5 = 55$	$\dim \Omega_2 = 55$	$\dim \text{Hom}_L(\Lambda^2 \Omega_2, \Omega_2) = 1$
$\dim B_6 = 78$	$\dim \Omega_2 = 78$	$\dim \text{Hom}_L(\Lambda^2 \Omega_2, \Omega_2) = 1$
$\dim B_7 = 105$	$\dim \Omega_2 = 105$	$\dim \text{Hom}_L(\Lambda^2 \Omega_2, \Omega_2) = 1$
	* $\dim \Omega_5 = 3003$	$\dim \text{Hom}_L(\Lambda^2 \Omega_5, \Omega_5) = 1$
	* $\dim \Omega_6 = 5005$	$\dim \text{Hom}_L(\Lambda^2 \Omega_6, \Omega_6) = 1$
$\dim B_8 = 136$	$\dim \Omega_2 = 136$	$\dim \text{Hom}_L(\Lambda^2 \Omega_2, \Omega_2) = 1$
	* $\dim \Omega_6 = 12376$	$\dim \text{Hom}_L(\Lambda^2 \Omega_6, \Omega_6) = 1$
	* $\dim \Omega_7 = 19448$	$\dim \text{Hom}_L(\Lambda^2 \Omega_7, \Omega_7) = 1$

10.5 Symplectic

For the symplectic type C_ℓ none of the fundamental representations occurs as a summand of its own exterior square.

10.6 Orthogonal

For the orthogonal type D_ℓ we have the following results; all other multiplicities are zero. The nonadjoint representation is starred:

$\dim D_4 = 28$	$\dim \Omega_2 = 28$	$\dim \text{Hom}_L(\Lambda^2 \Omega_2, \Omega_2) = 1$
$\dim D_5 = 45$	$\dim \Omega_2 = 45$	$\dim \text{Hom}_L(\Lambda^2 \Omega_2, \Omega_2) = 1$
$\dim D_6 = 66$	$\dim \Omega_2 = 66$	$\dim \text{Hom}_L(\Lambda^2 \Omega_2, \Omega_2) = 1$
$\dim D_7 = 91$	$\dim \Omega_2 = 91$	$\dim \text{Hom}_L(\Lambda^2 \Omega_2, \Omega_2) = 1$
$\dim D_8 = 120$	$\dim \Omega_2 = 120$	$\dim \text{Hom}_L(\Lambda^2 \Omega_2, \Omega_2) = 1$
	* $\dim \Omega_6 = 8008$	$\dim \text{Hom}_L(\Lambda^2 \Omega_6, \Omega_6) = 2$

$\dim E_6 = 78$	$\dim \Omega_2 = 78$	$\dim \text{Hom}_L(\Lambda^2 \Omega_2, \Omega_2) = 1$
	* $\dim \Omega_4 = 2925$	$\dim \text{Hom}_L(\Lambda^2 \Omega_4, \Omega_4) = 2$
$\dim E_7 = 133$	$\dim \Omega_1 = 133$	$\dim \text{Hom}_L(\Lambda^2 \Omega_1, \Omega_1) = 1$
	* $\dim \Omega_3 = 8645$	$\dim \text{Hom}_L(\Lambda^2 \Omega_3, \Omega_3) = 2$
	* $\dim \Omega_4 = 365750$	$\dim \text{Hom}_L(\Lambda^2 \Omega_4, \Omega_4) = 1$
$\dim E_8 = 248$	* $\dim \Omega_3 = 6696000$	$\dim \text{Hom}_L(\Lambda^2 \Omega_3, \Omega_3) = 5$
	* $\dim \Omega_4 = 6899079264$	$\dim \text{Hom}_L(\Lambda^2 \Omega_4, \Omega_4) = 46$
	* $\dim \Omega_5 = 146325270$	$\dim \text{Hom}_L(\Lambda^2 \Omega_5, \Omega_5) = 6$
	* $\dim \Omega_6 = 2450240$	$\dim \text{Hom}_L(\Lambda^2 \Omega_6, \Omega_6) = 1$
	* $\dim \Omega_7 = 30380$	$\dim \text{Hom}_L(\Lambda^2 \Omega_7, \Omega_7) = 2$
	$\dim \Omega_8 = 248$	$\dim \text{Hom}_L(\Lambda^2 \Omega_8, \Omega_8) = 1$
$\dim F_4 = 52$	$\dim \Omega_1 = 52$	$\dim \text{Hom}_L(\Lambda^2 \Omega_1, \Omega_1) = 1$
	* $\dim \Omega_2 = 1274$	$\dim \text{Hom}_L(\Lambda^2 \Omega_2, \Omega_2) = 2$
	* $\dim \Omega_3 = 273$	$\dim \text{Hom}_L(\Lambda^2 \Omega_3, \Omega_3) = 2$
$\dim G_2 = 14$	* $\dim \Omega_1 = 7$	$\dim \text{Hom}_L(\Lambda^2 \Omega_1, \Omega_1) = 1$
	$\dim \Omega_2 = 14$	$\dim \text{Hom}_L(\Lambda^2 \Omega_2, \Omega_2) = 2$

TABLE 13.

10.7 Exceptional

For the exceptional types E , F and G , the results are shown in Table 13; all other multiplicities are zero. The nonadjoint representations are starred.

We can summarize the information in these lists in the following result.

Theorem 10.1. *The multiplicities $\dim \text{Hom}_L(\Lambda^2 \Omega_i, \Omega_i)$ are nonzero only in the orthogonal and exceptional types. The multiplicity of the adjoint representation is always exactly 1.*

We also observe the following fact which distinguishes E_8 :

Theorem 10.2. *The only simple Lie algebra of rank $\ell \leq 8$ which has a fundamental representation which occurs more than twice as a summand in its own exterior square is E_8 .*

It seems to be a natural conjecture that the restriction on the rank in this last result is not necessary.

The only well-understood algebra structure in the above lists (apart from the adjoint representations which recover the original Lie algebras) is the seven-dimensional

representation of G_2 , which gives the simple non-Lie Malcev algebra discussed in Section 5.

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