COMMUNICATIONS IN ALGEBRA, 29(8), 3523–3534 (2001)

IDENTITIES FOR ALGEBRAS OBTAINED FROM THE CAYLEY-DICKSON PROCESS

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ABSTRACT

The Cayley-Dickson process gives a recursive method of constructing a nonassociative algebra of dimension 2^n for all $n \geq 0$, beginning with any ring of scalars. The algebras in this sequence are known to be flexible quadratic algebras; it follows that they are noncommutative Jordan algebras: they satisfy the flexible identity in degree 3 and the Jordan identity in degree 4. For the integral sedenion algebra (the double of the octonions) we determine a complete set of generators for the multilinear identities in degrees ≤ 5. Since these identities are satisfied by all flexible quadratic algebras, it follows that a multilinear identity of degree ≤ 5 is satisfied by all the algebras obtained from the Cayley-Dickson process if and only if it is satisfied by the sedenions.

THE CAYLEY-DICKSON PROCESS

Let A_0 denote a commutative associative ring with unity, the conjugate being the identity: $\bar{x} = x$. We inductively construct an infinite sequence A_n ($n \geq 0$) of nonassociative (that is, not necessarily associative) A_0-algebras by the Cayley-Dickson process; for details see Jacobson [16], pages 417–427. As an A_0-module $A_{n+1} = A_n \oplus A_n$, the direct sum of two copies of A_n; this also defines the addition on A_{n+1}. We define the product and the conjugate on A_{n+1} in terms of A_n as follows:

$$(u, v)(x, y) = (ux - \bar{y}v, yu + v\bar{x}), \qquad \overline{(u, v)} = (\bar{u}, -v).$$

The submodule of A_{n+1} consisting of elements of the form $(u, 0)$ is a subalgebra isomorphic to A_n. It is clear that A_n is a free A_0-module of rank 2^n.

These algebras have been studied by various authors over the last sixty years. The first paper on this topic, by Albert [1], shows that the algebras in this sequence are central simple quadratic algebras which satisfy the flexible identity $(xy)x = x(yx)$. Since they are flexible and quadratic they also satisfy the Jordan identity $(x^2y)x = x^2(yx)$; they are therefore noncommutative Jordan algebras. Other authors have studied the derivation algebras and automorphism groups of these algebras: see the papers of Schafer [21], Brown [6], Erdmann [9], and McCrimmon [19]. To the best of our knowledge, no research has been done on the identities satisfied by the algebras A_n for $n \geq 4$ (apart from the known fact that for all n the algebra A_n satisfies the flexible and Jordan identities).

The purpose of this paper is to determine all the multilinear identities of degree ≤ 5 satisfied by the integral sedenion algebra (the double of the octonions). We determine a complete set of generators for the multilinear identities in degrees ≤ 5. Since these identities are satisfied by all flexible quadratic algebras, it follows that a multilinear identity of degree ≤ 5 is satisfied by all the algebras obtained from the Cayley-Dickson process if and only if it is satisfied by the sedenions.

The name "sedenions" (which comes from *sedecim*, the Latin word for 16) is also used for other 16-dimensional structures which are not isomorphic to the double of the octonions. The papers of Carmody [7–8] discuss a 16-dimensional algebra which contains the octonions as a subalgebra, but in which the non-octonion basis elements are square roots of $+1$ not of -1. The papers of Lohmus and Sorgsepp [17–18] discuss a modification of the Cayley-Dickson process which results in a 16-dimensional ternary algebra. The paper of Smith [22] discusses a 16-dimensional composition semi-algebra which is left distributive but not right distributive.

COMPUTATIONAL METHODS

The computer calculations described in this paper were programmed in C, Pascal, Maple, Mathematica, and Albert (see [15]). The methods were developed by the authors in references [2–5] and [10–12]; see also [13–14].

The technique we use to find identities is to use the representation theory of the symmetric group to convert the problem into questions of linear algebra on large matrices. The row canonical form of such a matrix gives us information about the structure of the space of identities. We can then translate this linear-algebraic information back into multilinear non-associative polynomials which are the identities we seek.

The process of studying identities through group representations is indirect and complicated. It does, however, have two tremendous advantages. Because the process can be run separately on each representation of the symmetric group, the calculations can be broken up into smaller, more manageable portions. Also, the basic unit of the group algebra approach is the identity, rather than all substitutions in an identity. Since there are $n!$ possible substitutions, one can see that it is better to work with one object rather than $n!$ objects.

Let $f(x_1, x_2, \ldots, x_n)$ be a multilinear nonassociative polynomial of degree n in n indeterminates with coefficients from a field F. We first sort the terms of f by association type. Thus we can write $f = f_1 + \cdots + f_t$ where t is the number of association types. Within each association type, the terms can be specified by giving the coefficient $c \in F$ of the term and the permutation $\pi \in S_n$ of the n factors in the term. Thus each f_i for $1 \leq i \leq t$ can be expressed as an element $g_i = \sum_\pi c_{i\pi}\pi$ of the group algebra FS_n of the symmetric group S_n. Hence f may be identified with the element $(g_1, g_2, \ldots, g_t)$ of $M = FS_n \oplus \cdots \oplus FS_n$, the direct sum of t copies of FS_n. If π is any permutation in S_n, then $(\pi g_1, \pi g_2, \ldots, \pi g_t)$ is also an identity since it represents the identity f applied to a permutation of its arguments. Since linear combinations of identities are identities, one also gets that $(gg_1, gg_2, \ldots, gg_t)$ is an identity for all elements g of the group algebra over S_n. From this it is clear that M is a module over the group algebra and the set of all identities we seek is a submodule of M.

Any partition λ of n determines an irreducible representation of S_n of dimension d_λ. The group algebra FS_n is isomorphic to a direct sum of matrix algebras of size $d_\lambda \times d_\lambda$ as λ ranges over all partitions of n. Let p_λ denote the projection of the group algebra onto the matrix subalgebra corresponding to the partition λ. Projecting onto this matrix subalgebra we see that each element of the group algebra corresponds to a matrix of size $d_\lambda \times d_\lambda$. Combining the t association types we put together (horizontally) the t matrices of size $d_\lambda \times d_\lambda$ to obtain a matrix of size

$d_\lambda \times td_\lambda$. Thus $(p_\lambda(g_1), p_\lambda(g_2), \dots, p_\lambda(g_t))$ is a matrix of size $d_\lambda \times td_\lambda$ which represents the identity f in representation type λ. Furthermore,

$$(p_\lambda(gg_1), p_\lambda(gg_2), \dots, p_\lambda(gg_t)) = p_\lambda(g)(p_\lambda(g_1), p_\lambda(g_2), \dots, p_\lambda(g_t)),$$

which is a sequence of row operations applied to the matrix which represents f (including the possibility of zeroing out a row as a row operation). Two identities f and f' of degree n are equivalent (by definition) if they generate the same FS_n-submodule of M; in other words, in each representation type λ, the two matrices representing the two identities have the same row space.

A minimal identity is an identity whose matrix has rank one in one representation and rank zero in all the rest. Any identity is equivalent to a collection of minimal identities. If we are looking for identities, we only need to locate all the minimal identities.

Stacking the matrices (vertically) for a number k of identities $f^{(1)}, \dots, f^{(k)}$ gives a matrix of size $kd_\lambda \times td_\lambda$. Each row of this matrix represents an identity implied by $f^{(1)}, \dots, f^{(k)}$. Row operations on this matrix replace rows with linear combinations of rows, and so the rows of the new matrix also represent identities implied by $f^{(1)}, \dots, f^{(k)}$. The nonzero rows of the row-canonical form of this matrix are a set of independent module generators for the submodule of M in representation type λ generated by the identities $f^{(1)}, \dots, f^{(k)}$. Since the rank of this matrix can be no greater than td_λ, the number of independent generators is at most td_λ.

An identity in degree n implies identities in higher degrees. Given an identity $f(x_1, \dots, x_n)$ of degree n, we obtain $n+2$ identities of degree $n+1$ implied by f either by replacing x_i by $x_i x_{n+1}$ for some i $(1 \leq i \leq n)$ or by multiplying f on the left or the right by x_{n+1}. Any identity in degree $n+1$ which is implied by this set of $n+2$ identities is called a *lifted identity* obtained from f. This process may be repeated to obtain liftings of any degree $> n$ of an identity of degree n.

We now present a brief outline of the group algebra process for determining the identities satisfied in degree n by a particular nonassociative algebra of dimension q. Any minimal identity can be thought of as a single nonzero row in one representation and zeros in all other representations. The single nonzero row can be placed in the first row. By the isomorphism between the group algebra FS_n and the direct sum of matrix algebras of sizes $d_\lambda \times d_\lambda$ we can speak of a correspondence between elements in the group algebra and the matrix representation of these group algebra elements. Let $e_{11}, e_{12}, \dots, e_{1d}$ $(d = d_\lambda)$ represent the elements of the group algebra which the projection p_λ (corresponding to the partition λ) maps to the matrix units $E_{11}, E_{12}, \dots, E_{1d}$. We consider the functions $f_k(g)$ which evaluate the group algebra element g by applying the permutations in g to the elements

$x_1, x_2, \ldots, x_n$ as monomials in association type k. Then a minimal identity can be viewed as

$$\sum_{k=1}^{t} \sum_{j=1}^{d} c_{1j}^{k} f_k(e_{1j}),$$

where c_{1j}^{k} is the coefficient of e_{1j} in the kth association type. Since this function is an identity, it has to evaluate to zero for all choices of elements $x_1, x_2, \ldots, x_n$ in the q-dimensional algebra under consideration. In particular, for any random choice $x_1, x_2, \ldots, x_n$ in the algebra, the dt elements $V_{1j}^{k} = f_k(e_{1j})$ are elements in the algebra which are column vectors of length q. If we list these column vectors in the matrix form

$$\begin{bmatrix} V_{11}^{1} & V_{12}^{1} & \cdots & V_{1d}^{1} & V_{11}^{2} & V_{12}^{2} & \cdots & V_{1d}^{2} & \cdots & V_{11}^{t} & V_{12}^{t} & \cdots & V_{1d}^{t} \end{bmatrix} \qquad (*)$$

then the row vector

$$\begin{bmatrix} c_{11}^{1} & c_{12}^{1} & \cdots & c_{1d}^{1} & c_{11}^{2} & c_{12}^{2} & \cdots & c_{1d}^{2} & \cdots & c_{11}^{t} & c_{12}^{t} & \cdots & c_{1d}^{t} \end{bmatrix}$$

is in the null space of $(*)$. The process of finding the identities satisfied by an algebra amounts to creating $(*)$ for several choices of elements in the algebra. The process involves filling a matrix with $(*)$, then reducing it to row canonical form, then adding a second fill of $(*)$ on the bottom and then again reducing it. The process of adding a new fill of $(*)$ to the bottom and reducing to row canonical form continues until the rank seems to have stabilized. At this point the null space represents the minimal identities.

Any true identity will be found in the null space, but if we have not chosen our random elements properly, or not continued long enough to reach the full rank, we may still have elements in the null space that are not identities. We avoid these false identities by checking the members of the null space to see that they are actually identities.

We did these computations using arithmetic modulo 103; this was necessary in order to be able to store each matrix entry in a single byte. In this way we also avoided integer overflow when computing the row-canonical forms of very large matrices.

All of the algebras A_n have a basis with integral structure constants (in fact the Cayley-Dickson process naturally produces the structure constants 0 and ± 1). In particular the sedenion algebra has a basis with integral structure constants. Therefore, given any commutative associative ring with unity, we have the sedenion algebra over that ring. We wish to obtain information about the integral sedenions (that is, the sedenion algebra over the integers) by comparing them with the $\mathbb{Z}_p$ sedenions (that is, the sedenion algebra over the field with p elements for some prime p). For the purposes of the computations in this paper we take $p = 103$.

Suppose that the integral sedenions satisfy the identities $e_1, e_2, \ldots, e_k$ of degree n: these are multilinear identities with integral coefficients. Let R be the free nonassociative algebra over $\mathbb{Z}$ with n generators. Let S be the $\mathbb{Z}$-submodule of R of multilinear polynomials of degree n. Then S is a free $\mathbb{Z}$-module of rank $n!c_n$ where c_n is the Catalan number. Let I be the submodule of S generated by all $n!$ possible substitutions of the n generators of R into $e_1, e_2, \ldots, e_k$. The quotient group S/I is a finitely generated abelian group with $n!c_n$ generators and with $n!k$ relations which are the generators of I. This means that S/I is a direct sum of cyclic groups, possibly containing both finite (torsion) and infinite (free) summands:

$$S/I = \langle s_1{+}I \rangle \oplus \langle s_2{+}I \rangle \oplus \cdots \oplus \langle s_m{+}I \rangle, \quad s_i \in S \text{ for } 1 \le i \le m.$$

If a generator $s_i + I$ has finite order n_i then we know that $n_i s_i$ is an identity for the integral sedenions, since this element is in I. For any element x in the integral sedenions, if $n \in \mathbb{Z}$ and $n \neq 0$ then $nx = 0$ implies $x = 0$. We conclude that all the s_i from the torsion part of S/I are identities. We will let I^+ be the extension of I by including as generators the s_i from the torsion part of S/I. Thus S/I^+ is a free $\mathbb{Z}$-module.

Suppose we have established that for the $\mathbb{Z}_p$ sedenions there are no multilinear degree n identities other than those in I. We can then claim that there are no multilinear degree n identities for the integral sedenions except those in I^+. Let J be the submodule of all multilinear degree n identities for the integral sedenions. If J is larger than I^+, then there is some combination

$$f = a_{i_1} s_{i_1} + a_{i_2} s_{i_2} + \cdots + a_{i_r} s_{i_r} \qquad (*)$$

in J with each s_{i_j} from the generators of S/I of infinite order and the a_{i_j} nonzero. Since $f \equiv g$ (modulo p) for some $g \in I$, then $f = g + ph_1$ for some $h_1 \in S$. Since $ph_1 = f - g$ is in J, and since J consists of identities for the integral sedenions (which are a free $\mathbb{Z}$-module), we can cancel the p and conclude that h_1 is in J. Now in S/I^+ we have $f \equiv ph_1$ for $h_1 \in J$. Repeating this argument we get $f \equiv ph_1 \equiv p^2 h_2 \equiv p^3 h_3 \equiv \cdots$ where all the h_i are in J. Since S/I^+ is a finitely generated free abelian group, we must have $h_i \equiv 0$ (modulo I^+) for sufficiently large i. Therefore h_i is in I^+ for all i and hence f is in I^+.

We conclude by stating that when we are dealing with identities for an algebra with no torsion (that is, an algebra which is a free $\mathbb{Z}$-module), once we have determined all the identities modulo some prime, we have found all of the integral identities, except possibly those in I^+ but not in I. That is, there may be more identities, but they all have finite order modulo the known identities.

IDENTITIES FOR THE SEDENIONS

Degree Three

For the sedenion algebra in degree 3 we obtain the following table:

P	D	F	S
3	1	1	1
21	2	1	1
111	1	0	0

Column P gives the partitions of 3; these label the irreducible representations of the symmetric group S_3. Column D gives the dimension of the corresponding representation. Column F gives the rank of the S_3-module of identities generated by the flexible identity. Column S gives the rank of the S_3-module of identities satisfied by the sedenion algebra.

From this table we see that in degree 3 the flexible identity implies all the identities satisfied by the sedenions.

Degree Four

For the sedenion algebra in degree 4 we obtain the following table:

P	D	F	FJ	S
4	1	3	4	4
31	3	9	10	10
22	2	5	5	5
211	3	6	6	6
1111	1	1	1	1

Column P gives the partitions of 4 which label the irreducible representations of the symmetric group S_4. Column D gives the dimension of the corresponding representation. Column F gives the rank of the S_4-module of identities in degree 4 implied by the flexible identity in degree 3. Column FJ gives the rank of the S_4-module of identities generated by the liftings of the flexible identity together with the Jordan identity. Column S gives the rank of the S_4-module of identities satisfied by the sedenion algebra.

From this table we see that in degree 4 the flexible and Jordan identities imply all the identities satisfied by the sedenions.

Degree Five

The most obvious degree-5 identities for the sedenions are those identities which are implied by the flexible and the Jordan identities; these are the lifted flexible and lifted Jordan identities.

Since the sedenions contain the octonions as a subalgebra, a natural place to look for further degree-5 identities is a list of degree-5 identities for the octonions. The degree-5 octonion identities were classified by Racine [20]; these results were confirmed by a different method and extended to degree 6 by Hentzel and Peresi [13]. There are two degree-5 octonion identities which do not follow from identities of lower degree: the identity

$$[[a,b]\circ[c,d],e], \quad \text{where} \quad [y,z] = yz - zy, \quad y\circ z = yz + zy;$$

and the complete linearization of the identity

$$\begin{aligned} &P(x^2) - P(x)\circ x, \quad \text{where} \\ &\quad P = V_aV_bV_c - V_aV_cV_b - V_bV_aV_c + V_bV_cV_a + V_cV_aV_b - V_cV_bV_a, \\ &\quad V_y(z) = y\circ z. \end{aligned}$$

We call these the *little Racine* and *big Racine* identities. Both of these identities are satisfied by the sedenions, since they are satisfied by every flexible quadratic algebra, as shown in another paper of Hentzel and Peresi [14].

The identities of degrees ≤ 5 satisfied by quadratic algebras have been classified in [14]. Two of these identities (numbers 23 and 26 in Proposition 3 on pages 11–12 of [14]) satisfy the conditions:

(1) they do not follow from the flexible, Jordan, little and big Racine identities, and
(2) together with those four identities, they generate all the identities in degree 5 for the sedenions.

HP23 is the linearization of the identity

$$\begin{aligned} &2((a,b,a),c,d) + 2(c,(a,b,a),d) - ([a,b]\circ a,c,d) \\ &\quad - ([a,b]\circ c,a,d) + ([a,b],a\circ c,d); \end{aligned}$$

and HP26 is the identity

$$\sum_{b,c,e} \{(a, \langle c, d, e\rangle, b) + (a, b, \langle c, d, e\rangle)\}.$$

In these identities we use the following notation for the associator and the Jordan associator:

$$(a, b, c) = (ab)c - a(bc), \quad \langle a, b, c\rangle = (a \circ b) \circ c - a \circ (b \circ c).$$

In the second identity the sum denotes the alternating sum over all permutations of b, c and e. Here is a table of the cumulative ranks generated by the identities discussed so far:

P	D	F	FJ	FJR	FJR^2	FJR^2 HP	FJR^2 HP^2	S
5	1	11	13	13	13	13	13	13
41	4	43	48	48	48	48	48	48
32	5	51	55	56	56	57	57	57
311	6	57	62	62	63	63	64	64
221	5	45	47	48	48	49	49	49
2111	4	31	32	33	34	35	36	36
11111	1	6	6	7	7	7	7	7

The column headers in this table mean the following:

P = partition
D = dimension of irreducible representation
F = lifted flexible identity
FJ = F and lifted Jordan identity
FJR = FJ and little Racine identity
FJR^2 = FJR and big Racine identity
FJR^{2}HP = FJR^2 and HP23
FJR^{2}HP^2 = FJR^{2}HP and HP26
S = all identities for the sedenions

From this table we see that the S_5-module of identities in degree 5 satisfied by the sedenions is generated by the lifted flexible and lifted Jordan identities, together with the two Racine identities and the two Hentzel-Peresi identities.

We have established the following result:

Theorem. *The (complete linearizations of the) following six identities are a non-redundant set of generators of the set of all multilinear identities of degree* ≤ 5 *satisfied by the sedenion algebra over the field* $\mathbb{Z}_{103}$. *Any integral identity of degree* ≤ 5 *has finite order modulo these identities*:

$$(ab)a - a(ba),$$
$$((a^2)b)a - a^2(ba),$$
$$[[a,b]\circ[c,d],e],$$
$$P(x^2) - P(x)\circ x, \quad \text{where}$$
$$P = V_aV_bV_c - V_aV_cV_b - V_bV_aV_c + V_bV_cV_a + V_cV_aV_b - V_cV_bV_a,$$
$$V_y(z) = y\circ z,$$
$$2((a,b,a),c,d) + 2(c,(a,b,a),d)$$
$$-([a,b]\circ a, c, d) - ([a,b]\circ c, a, d) + ([a,b], a\circ c, d),$$
$$\sum_{b,c,e}\{(a,\langle c,d,e\rangle,b) + (a,b,\langle c,d,e\rangle)\}.$$

(*In the last identity the sum denotes the alternating sum over all permutations of b, c and e*). *All of these identities are satisfied by every flexible quadratic algebra, and hence all are satisfied by every algebra obtained by the Cayley-Dickson process.* □

Degree Six

We computed the ranks for the S_6-module of all degree-6 identities implied by the known identities in degree 6, and the ranks for the S_6-module of all degree-6 identities for the sedenions:

P	6	51	42	411	33	321	3111	222	2211	21111	111111
D	1	5	9	10	5	16	10	5	9	5	1
L	41	199	352	382	194	598	362	177	319	169	31
S	41	199	352	382	194	599	362	178	319	169	32

(The format of this table is the transpose of the format of the previous tables). As before, P is the partition, D is the dimension of the irreducible representation, L is the rank of the lifted known identities from lower degrees, and S is the rank of the sedenion identities. From this we see that a single new degree-6 identity for the sedenions exists in each of the representations 321 and 222 and 111111. We have not explicitly determined these identities.

CONCLUDING REMARKS

It is possible that the three new degree-6 identities are also identities for every flexible quadratic algebra, and hence for all the algebras obtained from the Cayley-Dickson process. To check this, we computed the ranks of the identities in degree 6 for the double of the sedenions. (This algebra is called the *trigintaduonion* algebra, from the Latin word *trigintaduo*, meaning 32). However these ranks are the same as those in row S of the previous table. Since any identity for the trigintaduonions is also an identity for the sedenions, it follows that the degree-6 identities for these two algebras are the same.

This raises an important question: What is the largest n for which all identities of degree $\leq n$ satisfied by the sedenions are also satisfied by every algebra in the sequence obtained by the Cayley-Dickson process? We have shown in this paper that $n \geq 5$.

ACKNOWLEDGMENTS

Murray Bremner thanks the Department of Mathematics of Iowa State University for its hospitality, the University of Saskatchewan for a sabbatical leave from July to December 1999, and the Natural Sciences and Engineering Research Council of Canada (NSERC) for financial support.

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Received March 2000
Revised August 2000