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Identities for algebras of matrices over the octonions

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To J. Marshall Osborn for his inspiring work on identities of algebras

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Abstract

This paper describes all the identities of degree $\leqslant 7$ satisfied by algebras of 2×2 matrices over the octonions. There are three cases: (1) the full matrix algebra under the usual matrix product, (2) the algebra of Hermitian matrices under the symmetric product, and (3) the algebra of skew-Hermitian matrices under the antisymmetric product. In case (1) we present seven new identities in degree 7 which were discovered by a computer search but which are proved to hold for matrices with entries in any alternative ring. In case (2) we recover the identities of Vasilovsky in degrees 5 and 6 for the special Jordan algebra of a nondegenerate symmetric bilinear form. In case (3) we describe a computational proof that there are no identities in degree $\leqslant 7$ which are not implied by anticommutativity.

Keywords: Nonassociative rings; Matrix algebras; Alternative algebras; Jordan algebras; Malcev algebras; Polynomial identities; Computational linear algebra; Representations of the symmetric group

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1. Introduction

Let R be an *algebra* (not necessarily associative) over the field $\mathbb{Q}$ of rational numbers: that is, R is a vector space over $\mathbb{Q}$ with a bilinear multiplication $R \times R \to R$ denoted $(a,b) \mapsto ab$. By an *involution* on R we mean an anti-automorphism of order 2; that is, a linear map $R \to R$ denoted $a \mapsto \bar{a}$ with the properties

$$\overline{ab} = \bar{b}\bar{a}, \qquad \bar{\bar{a}} = a. \tag{1}$$

If X and Y are matrices with entries in R, of sizes $m \times n$ and $n \times p$ respectively, then we can define the product XY by the usual formula

$$(XY)_{ik} = \sum_{j=1}^{n} X_{ij} Y_{jk}, \tag{2}$$

since this does not depend on associativity. If R has an involution then $M_n(R)$ has the *conjugate transpose* involution $X \mapsto X^*$ defined by

$$\bigl(X^*\bigr)_{ij} = \overline{X_{ji}}. \tag{3}$$

For any integer $n \geqslant 1$ we consider three matrix algebras with entries in R. The first is

> $M_n(R)$: the algebra of $n \times n$ matrices under the usual matrix product.

If R has an involution then the second and third matrix algebras are

> $H_n(R)$: the subspace of all $n \times n$ *Hermitian* matrices X over R (meaning $X^* = X$) under the *symmetric* product $X \circ Y = \frac{1}{2}(XY + YX)$;
>
> $SH_n(R)$: the subspace of all $n \times n$ *skew-Hermitian* matrices X over R (meaning $X^* = -X$) under the *antisymmetric* product $[X,Y] = XY - YX$.

We avoid using the term Jordan or Lie product, since this may suggest that the resulting algebra is a Jordan or Lie algebra, which in general is not the case. We may also regard $M_n(R)$ itself as an algebra under the symmetric or antisymmetric product.

If R is associative then $M_n(R)$ is associative for all n. The subspace $H_n(R)$ is a *Jordan algebra*; that is, it satisfies the commutative and Jordan identities:

$$X \circ Y = Y \circ X, \tag{4}$$

$$(X \circ X) \circ (X \circ Y) = X \circ \bigl((X \circ X) \circ Y\bigr). \tag{5}$$

The subspace $SH_n(R)$ is a *Lie algebra*; that is, it satisfies the anticommutative and Jacobi identities:

$$[X, X] = 0, \tag{6}$$

$$\big[[X, Y], Z\big] + \big[[Y, Z], X\big] + \big[[Z, X], Y\big] = 0. \tag{7}$$

For R not associative the algebras $M_n(R)$, $H_n(R)$ and $SH_n(R)$ have received very little attention. In this paper we will consider the case in which R is an *alternative algebra*; that is, R satisfies the left and right alternative identities:

$$X^2Y = X(XY), \qquad YX^2 = (YX)X. \tag{8}$$

Equivalently (by Artin's Theorem) R satisfies the condition that any subalgebra generated by two elements is associative. The *associator* in R is defined by

$$(X, Y, Z) = (XY)Z - X(YZ), \tag{9}$$

and the ring R is alternative if and only if the associator is an alternating function of its three arguments. If R is alternative then a subspace of R which is closed under the symmetric product is still a Jordan algebra. A subspace of R which is closed under the antisymmetric product is not necessarily a Lie algebra but is a *Malcev algebra*; that is, it satisfies anticommutativity and the Malcev identity:

$$\begin{aligned}\big[[W, Y], [X, Z]\big] = {} & \big[\big[[W, X], Y\big], Z\big] + \big[\big[[X, Y], Z\big], W\big] + \big[\big[[Y, Z], W\big], X\big] \\ & + \big[\big[[Z, W], X\big], Y\big]. \end{aligned} \tag{10}$$

For the purposes of this paper, the most important varieties of nonassociative algebras are the alternative, Jordan and Malcev algebras, and generalizations of these. Basic references on nonassociative algebras are [4] and [10].

Let $\mathbb{O}$ denote the *octonion division algebra*; that is, the vector space over $\mathbb{Q}$ with basis

$$1,\ i,\ j,\ k,\ l,\ m,\ n,\ p \tag{11}$$

and multiplication given in Table 1. The involution on $\mathbb{O}$ is defined by

Table 1
Octonion multiplication

$\times$	1	i	j	k	l	m	n	p
1	1	i	j	k	l	m	n	p
i	i	-1	k	$-j$	m	$-l$	$-p$	n
j	j	$-k$	-1	i	n	p	$-l$	$-m$
k	k	j	$-i$	-1	p	$-n$	m	$-l$
l	l	$-m$	$-n$	$-p$	-1	i	j	k
m	m	l	$-p$	n	$-i$	-1	$-k$	j
n	n	p	l	$-m$	$-j$	k	-1	$-i$
p	p	$-n$	m	l	$-k$	$-j$	i	-1

Table 2
Octonion matrix algebras

n	$M_n(\mathbb{O})$	$\dim M_n(\mathbb{O})$	$H_n(\mathbb{O})$	$\dim H_n(\mathbb{O})$	$SH_n(\mathbb{O})$	$\dim SH_n(\mathbb{O})$
1	$\mathbb{O}$	8	$\mathbb{Q}$	1	$\mathbb{M}$	7
2	?	32	$\mathbb{SJ}$	10	?	22
3	??	72	$\mathbb{EJ}$	27	??	45
4	???	128	???	52	???	76

$$\overline{a+bi+cj+dk+el+fm+gn+hp} = a-bi-cj-dk-el-fm-gn-hp. \tag{12}$$

Over an algebraically closed field, the octonion algebra is the only simple nonassociative alternative algebra. An excellent survey article on the octonions and their applications in algebra and geometry is [2].

Table 2 lists the most important nonassociative algebras constructed from $n \times n$ octonion matrix algebras for $1 \leqslant n \leqslant 4$. We use the following notation:

$\mathbb{SJ}$: the simple special Jordan algebra of 2×2 Hermitian matrices over $\mathbb{O}$, which is isomorphic to the Jordan algebra of a bilinear form on a 9-dimensional vector space;

$\mathbb{EJ}$: the simple exceptional Jordan algebra of 3×3 Hermitian matrices over $\mathbb{O}$;

$\mathbb{M}$: the simple non-Lie Malcev algebra of “pure imaginary” octonions.

Over an algebraically closed field, $\mathbb{EJ}$ is the only simple exceptional Jordan algebra, and $\mathbb{M}$ is the only simple non-Lie Malcev algebra. The question marks indicate that for these particular algebras, the structure of the identities has not been studied (with the exception of the new identities for $M_2(\mathbb{O})$ presented in Section 3). The identities satisfied by the algebras indicated by question marks will provide varieties to study which generalize the family of algebras in that particular column (respectively alternative, Jordan and Malcev). We have the following obvious result.

Proposition 1. *The dimension formulas for the octonion matrix algebras are*

$$\dim M_n(\mathbb{O}) = 8n^2, \tag{13}$$

$$\dim H_n(\mathbb{O}) = 8\binom{n}{2} + n = n(4n-3), \tag{14}$$

$$\dim SH_n(\mathbb{O}) = 8\binom{n}{2} + 7n = n(4n+3). \tag{15}$$

Remark. The case $H_4(\mathbb{O})$ is especially interesting because of some numerical coincidences with simple Lie algebras of type F_4. Both have dimension 52, both have a 4-dimensional subalgebra (diagonal matrices for $H_4(\mathbb{O})$, Cartan subalgebra for F_4) which gives a decomposition of the algebra into eigenspaces (Peirce spaces for $H_4(\mathbb{O})$, root spaces for F_4) which combine into two isomorphic 24-dimensional subalgebras (upper and lower triangular matrices for $H_4(\mathbb{O})$, positive and negative root spaces for F_4) such that the algebra is the direct sum (as a vector space) of the three subalgebras: $24 + 4 + 24 = 52$. Also, $H_4(\mathbb{O})$ is commutative but F_4 is anticommutative, so $H_4(\mathbb{O})$ looks very much like a "commutative version" of F_4. There are two interesting open problems here: (1) to find the simplest identity for $H_4(\mathbb{O})$ which does not follow from commutativity, and (2) to define a natural anticommutative algebra structure on the vector space $H_4(\mathbb{O})$ which makes it isomorphic to a Lie algebra of type F_4.

Main results. The purpose of this paper is threefold:

(1) to determine the identities of degree $\leqslant 7$ satisfied by the algebra $M_2(\mathbb{O})$: there are no identities in degree $\leqslant 6$ but seven new identities in degree 7, all of which hold for $M_2(R)$ where R is any alternative ring;
(2) to determine the identities of degree $\leqslant 7$ satisfied by the algebra $H_2(\mathbb{O})$: here we recover Vasilovsky's identities of degrees 5 and 6 for the special Jordan algebra of a nondegenerate symmetric bilinear form;
(3) to determine the identities of degree $\leqslant 7$ satisfied by the algebra $SH_2(\mathbb{O})$: here we give a computational demonstration that there are no identities of degree $\leqslant 7$ that do not follow from anticommutativity.

These three cases will be discussed separately following the next section.

2. Computational methods

Let A be any algebra (not necessarily associative) over a field F. That is, A is a vector space over F, together with a bilinear map $A \times A \to A$ (equivalently, a linear map $A \otimes A \to A$). We are interested in the polynomial identities satisfied by the algebra A. To simplify the discussion, we will assume initially that the base field F has characteristic 0. This assumption implies that any polynomial identity over F is equivalent to a family of homogeneous multilinear identities.

2.1. *Associative polynomials*

We fix a positive integer n and a set of n indeterminates

$$X = \{a_1, a_2, \ldots, a_n\}.$$

We let S_n be the symmetric group on $\{1, 2, \ldots, n\}$. We will write elements of S_n as monomials of degree n in the indeterminates X. That is, the permutation $\sigma \in S_n$ corresponds to the monomial

$$p_\sigma = a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}.$$

Here we apply σ to the subscripts not the positions, so we have to multiply permutations from right to left. This convention assumes that we are dealing with multilinear identities;

we regard an identity with a variable repeated k times as shorthand for the symmetric sum over k new variables in the positions of the original repeated variable.

An example will make this clear. Suppose we want to look at

$$f(x_1, x_2, x_3, x_4, x_5) = x_5x_1x_3x_2x_4 + x_1x_5x_3x_2x_4.$$

In terms of the action on subscripts we would write this identity as

$$(1542)(3) + (1)(254)(3) \quad \text{or} \quad (1542) + (254).$$

The left ideal generated by this element of the group ring will include

$$(123)\big[(1542) + (254)\big] = (1543) + (12543) = x_5x_2x_1x_3x_4 + x_2x_5x_1x_3x_4.$$

This is just what we want: those portions of the monomials which do not move are substituted for the same element in all terms. The parts which switch are switched into new elements but the pattern of the movement will be the same. But we have to multiply from right to left because the right hand permutation hits the subscript first.

In terms of the action of permutations on positions we would write the same identity as $I + (12)$. If $I + (12)$ acts on $x_5x_1x_3x_2x_4$ we get

$$x_5x_1x_3x_2x_4 + x_1x_5x_3x_2x_4.$$

Here we must multiply from left to right because we put the permutation in first, and then apply the identity:

$$(123)\big[I + (12)\big] = (123) + (1)(23),$$

and $(123) + (23)$ acting on $x_5x_1x_3x_2x_4$ is

$$x_3x_5x_1x_2x_4 + x_5x_3x_1x_2x_4.$$

This is also correct if we make the assumption that permutations are acting on positions.

The action on subscripts gives a left ideal when we multiply from right to left. The action on positions gives a left ideal when we multiply from left to right. The first method (action on subscripts) is easier to program and corresponds directly to the representation theory given in standard references such as [3].

Let P_n be the linear span of all $n!$ monomials p_σ as σ ranges over S_n. We regard P_n as a module over S_n under the action induced by $\pi p_\sigma = p_{\pi\sigma}$. If A is an associative algebra then the elements of P_n correspond to the homogeneous multilinear identities of degree n that could be satisfied by A. More precisely, let $f \in P_n$ be a homogeneous multilinear associative polynomial of degree n. We say that f is an identity for A if it vanishes identically on A; that is, if

$$f(x_1, x_2, \ldots, x_n) = 0 \quad \text{for every } x_1, x_2, \ldots, x_n \in A.$$

An identity can be identified with the submodule that it generates: an algebra A satisfies the identity f if and only if it satisfies all the identities in the submodule generated by f.

The group ring FS_n decomposes as the direct sum of full matrix rings of size $d_\lambda \times d_\lambda$, where d_λ is the dimension of the irreducible $\mathbb{Q}S_n$-module corresponding to λ, and λ ranges over all partitions of n. The module P_n decomposes into the same direct sum. Any submodule is a direct sum of irreducible modules. Since we can perform computations one representation at a time, it is possible to study an identity by breaking the problem down into smaller pieces, each of which corresponds to a partition of the degree n.

2.2. Nonassociative polynomials

If A is a nonassociative algebra then we also have to keep track of the possible association types that may occur in a monomial of degree n. The number of distinct association types (that is, the number of distinct ways to parenthesize n factors) is the Catalan number

$$t_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Here is a short table:

n	1	2	3	4	5	6	7	8
t_n	1	1	2	5	14	42	132	429

The total number of homogeneous multilinear monomials of degree n for a nonassociative algebra A is the Catalan number t_n (the number of association types) times the factorial $n!$ (the number of associative words, or unparenthesized monomials):

$$\frac{1}{n}\binom{2n-2}{n-1}n! = \frac{(2n-2)!}{(n-1)!}.$$

This is the dimension of the vector space Q_n of all possible multilinear homogeneous nonassociative identities of degree n. We can think of Q_n as the direct sum of t_n copies of P_n, one copy for each association type.

2.3. Random vectors

Suppose we want to find the simplest identity satisfied by a nonassociative algebra A of dimension s. We represent elements of A as s-tuples with respect to some convenient basis, and we assume that we have an explicit procedure for computing the product in A with respect to this basis. (Such a procedure can be obtained from the structure constants for the algebra A.)

For each degree n and each partition $n = n_1 + \cdots + n_k$ we list all the corresponding monomials involving k variables $x_1, \ldots, x_k$ with x_i occurring n_i times in each monomial.

(We are no longer assuming multilinearity.) Let c be the number of these monomials; we have

$$c = \frac{1}{n}\binom{2n-2}{n-1}\binom{n}{n_1, \ldots, n_k} = \frac{(2n-2)!}{(n-1)!} \cdot \frac{1}{n_1! n_2! \cdots n_k!}.$$

We set up a matrix M with $c + s$ rows and c columns, initialized to zero.

We now begin the following iterative procedure: We generate k random elements of A. Each column of M is labelled by one of the c monomials, and each monomial involves k variables. We set the k variables equal to the k random elements of A and evaluate each of the c monomials using the product in A. For each column of M we obtain another element of A which we view as an $s \times 1$ column vector. In column j of M, in rows $c + 1$ to $c + s$ (at the bottom of the matrix), we put the $s \times 1$ column vector obtained by evaluating the jth monomial on the k random elements of A. After the $s \times c$ bottom segment of M is filled in this way, each of the last s rows contains a linear relation which must be satisfied by the coefficients of any identity satisfied by A. To see this, let T_i $(1 \leqslant i \leqslant c)$ be the monomials labelling the columns of the matrix M. Let

$$\sum_{i=1}^{c} a_i T_i$$

be the general linear combination of the monomials T_i where the coefficients a_i are indeterminates. When we evaluate the T_i on the k random elements of A, each T_i becomes an $s \times 1$ column vector:

$$T_i = (t_{i1}, t_{i2}, \ldots, t_{is})^T.$$

Writing out the components of the general linear combination of the T_i we get s linear relations that must be satisfied by the a_i:

$$\sum_{i=1}^{c} t_{ij} a_i, \quad 1 \leqslant j \leqslant s.$$

These are the relations that occupy the last s rows of the matrix M. (Another way of viewing this process is to say that we are generating random counterexamples to the possible identities satisfied by A; we thank Don Pigozzi for pointing this out.) We now compute the row canonical form of M. Since M has size $(c + s) \times c$, its rank must be $\leqslant c$, and so the bottom $s \times c$ submatrix will now be zero.

We repeat this fill-and-reduce process; in our experience each iteration tends to increase the rank of M by $s = \dim A$. The process is continued until the rank of M stops increasing. We perform a few more iterations to be sure that M has reached full rank. If the nullspace of M at this point is nonzero, it contains candidates for non-trivial identities satisfied by A. We now test the candidates by seeing if they evaluate to zero on further choices of random arguments. Finally, we attempt to prove them directly.

2.4. Using the symmetric group ring

Another technique we use to find identities is the representation theory of the symmetric group. The process of studying identities through group representations is indirect and complicated. It does, however, have two tremendous advantages. Because the process can be run separately on each representation of the symmetric group, the calculations can be broken up into smaller, more manageable portions. Also, the basic unit of the group algebra approach is the identity, rather than all substitutions in an identity. Since there are $n!$ possible substitutions, one can see that it is better to work with one object rather than $n!$ objects.

Let $f(x_1, x_2, \ldots, x_n)$ be a multilinear nonassociative polynomial of degree n in n indeterminates with coefficients from a field F. We first sort the terms of f by association type. Thus we can write $f = f_1 + \cdots + f_t$ where t is the number of association types. Within each association type, the terms can be specified by giving the coefficient $c \in F$ of the term and the permutation $\pi \in S_n$ of the n factors in the term. Thus each f_i for $1 \leqslant i \leqslant t$ can be expressed as an element $g_i = \sum_\pi c_{i\pi}\pi$ of the group algebra FS_n of the symmetric group S_n. Hence f may be identified with the element $(g_1, g_2, \ldots, g_t)$ of

$$M = FS_n \oplus \cdots \oplus FS_n \quad (t \text{ summands}),$$

the direct sum of t copies of FS_n. If π is any permutation, then $(\pi g_1, \pi g_2, \ldots, \pi g_t)$ is also an identity since it represents the identity f applied to a permutation of its arguments. Since linear combinations of identities are identities, one also gets that $(gg_1, gg_2, \ldots, gg_t)$ is an identity for all elements g of the group algebra over S_n. From this it is clear that M is a module over the group algebra and the set of all identities we seek is a submodule of M.

Any partition λ of n determines an irreducible representation of S_n of dimension d_λ. The group algebra FS_n is isomorphic to a direct sum of matrix algebras of size $d_\lambda \times d_\lambda$ as λ ranges over all partitions of n. Let p_λ denote the projection of the group algebra onto the matrix subalgebra corresponding to the partition λ. Projecting onto this matrix subalgebra we see that each element of the group algebra corresponds to a matrix of size $d_\lambda \times d_\lambda$. Combining the t association types we put together (horizontally) the t matrices of size $d_\lambda \times d_\lambda$ to obtain a matrix of size $d_\lambda \times td_\lambda$. Thus $(p_\lambda(g_1), p_\lambda(g_2), \ldots, p_\lambda(g_t))$ is a matrix of size $d_\lambda \times td_\lambda$ which represents the identity f in representation type λ. Furthermore,

$$\bigl(p_\lambda(gg_1), p_\lambda(gg_2), \ldots, p_\lambda(gg_t)\bigr) = p_\lambda(g)\bigl(p_\lambda(g_1), p_\lambda(g_2), \ldots, p_\lambda(g_t)\bigr),$$

which is a sequence of row operations applied to the matrix which represents f (including the possibility of zeroing out a row). Two identities f and f' of degree n are equivalent (by definition) if they generate the same FS_n-submodule of M; in other words, in each representation type λ, the two matrices representing the two identities have the same row space.

A minimal identity is an identity whose matrix has rank one in one representation and rank zero in all the rest. Any identity is equivalent to a collection of minimal identities. If we are looking for identities, we only need to locate all the minimal identities.

Stacking the matrices (vertically) for a number k of identities $f^{(1)}, \dots, f^{(k)}$ gives a matrix of size $kd_\lambda \times td_\lambda$. Each row of this matrix represents an identity implied by $f^{(1)}$, $\dots, f^{(k)}$. Row operations on this matrix replace rows with linear combinations of rows, and so the rows of the new matrix also represent identities implied by $f^{(1)}, \dots, f^{(k)}$. The nonzero rows of the row-canonical form of this matrix are a set of independent module generators for the submodule of M in representation type λ generated by the identities $f^{(1)}, \dots, f^{(k)}$. Since the rank of this matrix can be no greater than td_λ, the number of independent generators is at most td_λ.

An identity in degree n implies identities in higher degrees. Given an identity $f(x_1, \dots, x_n)$ of degree n, we obtain $n+2$ identities of degree $n+1$ implied by f either by replacing x_i by $x_i x_{n+1}$ for some i $(1 \leqslant i \leqslant n)$ or by multiplying f on the left or the right by x_{n+1}. Any identity in degree $n+1$ which is implied by this set of $n+2$ identities is called a "lifted identity" obtained from f. This process may be repeated to obtain liftings of any degree $> n$ of an identity of degree n.

2.5. *Further details of the group ring approach*

We now present a brief outline of the group algebra process for determining the identities satisfied in degree n by a particular nonassociative algebra of dimension s. Any minimal identity can be thought of as a single nonzero row in one representation and zeros in all other representations. The single nonzero row can be placed in the first row. By the isomorphism between the group algebra FS_n and the direct sum of matrix algebras of sizes $d_\lambda \times d_\lambda$ we can speak of a correspondence between elements in the group algebra and the matrix representation of these group algebra elements. Let $e_{11}, e_{12}, \dots, e_{1d}$ $(d = d_\lambda)$ represent the elements of the group algebra which are mapped by the projection p_λ (corresponding to the partition λ) to the matrix units $E_{11}, E_{12}, \dots, E_{1d}$. We consider the functions $f_k(g)$ which evaluate the group algebra element g by applying the permutations in g to the elements $x_1, x_2, \dots, x_n$ as monomials in association type k. Then a minimal identity can be viewed as

$$\sum_{k=1}^{t} \sum_{j=1}^{d} c_{1j}^{k} f_k(e_{1j}),$$

where c_{1j}^k is the coefficient of e_{1j} in the kth association type. Since this function is an identity, it has to evaluate to zero for all choices of elements $x_1, x_2, \dots, x_n$ in the s-dimensional algebra under consideration. In particular, for any random choice $x_1, x_2, \dots, x_n$ in the algebra, the dt elements $V_{1j}^k = f_k(e_{1j})$ are elements in the algebra which are column vectors of length s. If we list these column vectors in the matrix form

$$\begin{bmatrix} V_{11}^1 & V_{12}^1 & \cdots & V_{1d}^1 & V_{11}^2 & V_{12}^2 & \cdots & V_{1d}^2 & \cdots & V_{11}^t & V_{12}^t & \cdots & V_{1d}^t \end{bmatrix} \tag{16}$$

then the row vector

$$\begin{bmatrix} c_{11}^1 & c_{12}^1 & \cdots & c_{1d}^1 & c_{11}^2 & c_{12}^2 & \cdots & c_{1d}^2 & \cdots & c_{11}^t & c_{12}^t & \cdots & c_{1d}^t \end{bmatrix}$$

is in the null space of (16). The process of finding the identities satisfied by an algebra amounts to creating (16) for several choices of elements in the algebra. The process involves filling a matrix with (16), then reducing it to row canonical form, then adding a second fill of (16) on the bottom and then again reducing it. The process of adding a new fill of (16) to the bottom and reducing to row canonical form continues until the rank seems to have stabilized. At this point the null space represents the minimal identities. Any true identity will be found in the null space, but if we have not chosen our random elements properly, or not continued long enough to reach the full rank, we may still have elements in the null space that are not identities. We avoid these false identities by checking the members of the null space to see that they are actually identities.

We are confident that our stochastic process reached full rank in every case. If we had stopped the iterations too early before reaching the matrix $\overline{M}$ of maximal rank, we would have

$$\operatorname{rowspace}(M) \subset \operatorname{rowspace}\bigl(\overline{M}\bigr), \qquad \operatorname{rank}(M) < \operatorname{rank}\bigl(\overline{M}\bigr). \tag{17}$$

Since our algebras are elements of a known variety, for each degree the identities of the variety are contained in the identities of the algebra. Thus the rank of the identities of the variety is less than or equal to the rank of the identities of the algebra. The rank of $\overline{M}$, plus the rank of the identities of the algebra, equals the total dimension of the space of polynomials. In each case we found that the rank of M, plus the rank of the identities of the variety, equals the total dimension of the space of polynomials. Thus we know that the rank of M equals the rank of $\overline{M}$ and that the rank of the identities of the variety equals the rank of the identities of the algebra. In the cases where we found identities, we were able either to prove them directly or to use results from the literature.

To save space and time these computational methods were implemented over the field with p elements where p is a prime larger than the degree of the identities under consideration. This guarantees that the group ring will be semisimple, and effectively ensures that the results will be equivalent to the characteristic 0 case; that is, the dimension of a submodule of identities will be equal to the dimension of the corresponding submodule over $\mathbb{Q}$, and the basis which is computed will be formally the same in the two cases if the coefficients of the monomials are expressed as small integers.

The computer code implementing these algorithms is available from the authors.

3. The full matrix algebra

For $M_2(\mathbb{O})$ the dimension is $s = 32$. Using the computational methods described in Section 2, no identities were found in degree $d \leqslant 6$. In degree 7, the algebra $M_2(\mathbb{O})$ satisfies no identities in partition 7, but it does satisfy identities in partition 61. See Theorem 3 for the proof of these statements.

While working out a proof of the identities in partition 61, it was found that they can be generalized to hold in partition 22111, and that these more general identities hold with the matrix entries coming from any alternative algebra. These are the identities stated in

Theorem 1. Three other identities, also in partition 22111, were discovered; these are stated in Theorem 2.

To describe these identities we first use the associator to define five homogeneous nonassociative polynomials of degree 3:

$$\mathcal{LA}(X,Y,Z) = (X,Y,Z) + (Y,X,Z), \tag{18}$$

$$\mathcal{RA}(X,Y,Z) = (X,Y,Z) + (X,Z,Y), \tag{19}$$

$$\mathcal{LS}(X,Y,Z) = (X,Y,Z) - (Y,X,Z), \tag{20}$$

$$\mathcal{RS}(X,Y,Z) = (X,Y,Z) - (X,Z,Y), \tag{21}$$

$$\mathcal{F}(X,Y,Z) = (X,Y,Z) + (Z,Y,X). \tag{22}$$

The abbreviations indicate that these are the *left-alternative*, *right-alternative*, *left-symmetric*, *right-symmetric*, and *flexible* identities. We can now state the two main theorems of this section.

Theorem 1. *Let R be an alternative ring, and let $M_2(R)$ be the algebra of* 2×2 *matrices with entries from R. Then $M_2(R)$ satisfies the identities*

$$\mathcal{LA}\big(\mathcal{F}\big(\mathcal{LS}(A,B,B),C,C\big),D,E\big) = 0, \tag{23}$$

$$\mathcal{LA}\big(\mathcal{F}\big(\mathcal{RS}(B,B,A),C,C\big),D,E\big) = 0, \tag{24}$$

$$\mathcal{RA}\big(D,E,\mathcal{F}\big(\mathcal{LS}(A,B,B),C,C\big)\big) = 0, \tag{25}$$

$$\mathcal{RA}\big(D,E,\mathcal{F}\big(\mathcal{RS}(B,B,A),C,C\big)\big) = 0, \tag{26}$$

where A, B, C, D, E are any elements of $M_2(R)$.

Theorem 2. *Let R be an alternative ring, and let $M_2(R)$ be the algebra of* 2×2 *matrices with entries from R. Then $M_2(R)$ satisfies the identities*

$$\mathcal{F}\big(\mathcal{LS}(A,B,B),\mathcal{RS}(D,D,C),E\big) - \mathcal{F}\big(\mathcal{RS}(D,D,C),\mathcal{LS}(A,B,B),E\big) = 0, \tag{27}$$

$$\mathcal{F}\big(\mathcal{RS}(B,B,A),\mathcal{RS}(D,D,C),E\big) - \mathcal{F}\big(\mathcal{RS}(D,D,C),\mathcal{RS}(B,B,A),E\big) = 0, \tag{28}$$

$$\mathcal{F}\big(\mathcal{LS}(A,B,B),\mathcal{LS}(C,D,D),E\big) - \mathcal{F}\big(\mathcal{LS}(C,D,D),\mathcal{LS}(A,B,B),E\big) = 0, \tag{29}$$

where A, B, C, D, E are any elements of $M_2(R)$.

These two theorems may be distinguished by the ternary association types of their terms when the associators are expanded. The terms of the identities in Theorem 1 have the association types

$$\begin{array}{lll} \big(\big((xxx)xx\big)xx\big), & \big(\big(x(xxx)x\big)xx\big), & \big(\big(xx(xxx)\big)xx\big), \\ \big(x\big((xxx)xx\big)x\big), & \big(x\big(x(xxx)x\big)x\big), & \big(x\big(xx(xxx)\big)x\big), \\ \big(xx\big((xxx)xx\big)\big), & \big(xx\big(x(xxx)x\big)\big), & \big(xx\big(xx(xxx)\big)\big) \end{array} \tag{30}$$

whereas the terms of the identities in Theorem 2 have the association types

$$\big((xxx)(xxx)x\big), \qquad \big((xxx)x(xxx)\big), \qquad \big(x(xxx)(xxx)\big). \tag{31}$$

Theorem 1 follows immediately from Propositions 2, 3 and 4 below. The proof of Theorem 2 is very similar.

We first prove a formula for the associator in $M_2(R)$ where R is an arbitrary nonassociative ring.

Lemma 1. *Let R be a ring, not necessarily associative, and let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \qquad C = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \tag{32}$$

be 2×2 matrices over R. The associator of these matrices is

$$(A, B, C) = \begin{pmatrix} aep + bgp + afr + bhr & aeq + bgq + afs + bhs \\ cep + dgp + cfr + dhr & ceq + dgq + cfs + dhs \end{pmatrix} \tag{33}$$

where xyz denotes the associator (x, y, z) for any $x, y, z \in R$.

Proof. We have

$$(AB)C = \begin{pmatrix} (ae)p + (bg)p + (af)r + (bh)r & (ae)q + (bg)q + (af)s + (bh)s \\ (ce)p + (dg)p + (cf)r + (dh)r & (ce)q + (dg)q + (cf)s + (dh)s \end{pmatrix}, \tag{34}$$

$$A(BC) = \begin{pmatrix} a(ep) + a(fr) + b(gp) + b(hr) & a(eq) + a(fs) + b(gq) + b(hs) \\ c(ep) + c(fr) + d(gp) + d(hr) & c(eq) + c(fs) + d(gq) + d(hs) \end{pmatrix}. \tag{35}$$

From this we see that the entries of $(AB)C - A(BC)$ are the given sums of associators of the matrix entries. □

Corollary 1. *If R is an alternative ring then in $M_2(R)$ we have*

$$(A, A, A) = \begin{pmatrix} 2abc - bcd & abd \\ -acd & abc - 2bcd \end{pmatrix}. \tag{36}$$

From this we see that if R is an alternative ring which is not associative, then $M_2(R)$ is not even third-power associative.

Proof. If we take $A = B = C$ in Lemma 1 we obtain

$$(A, A, A) = \begin{pmatrix} aaa + bca + abc + bdc & aab + bcb + abd + bdd \\ caa + dca + cbc + ddc & cab + dcb + cbd + ddd \end{pmatrix}. \tag{37}$$

If R is alternative then an associator with a repeated factor is zero, and the others can be rearranged (with appropriate sign changes) so that the factors are in alphabetical order. This gives the result. □

Proposition 2. *If R is an alternative ring and A, B are elements of the matrix ring $M_2(R)$ then $\mathcal{LS}(A, B, B)$ and $\mathcal{RS}(B, B, A)$ are matrices of trace* 0.

Proof. We write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}. \tag{38}$$

Using Lemma 1 we obtain

$$(A, B, B) = \begin{pmatrix} aee + bge + afg + bhg & aef + bgf + afh + bhh \\ cee + dge + cfg + dhg & cef + dgf + cfh + dhh \end{pmatrix}, \tag{39}$$

$$(B, A, B) = \begin{pmatrix} eae + fce + ebg + fdg & eaf + fcf + ebh + fdh \\ gae + hce + gbg + hdg & gaf + hcf + gbh + hdh \end{pmatrix}. \tag{40}$$

Since R is alternative we may simplify the associators in R as in the proof of Corollary 1:

$$(A, B, B) = \begin{pmatrix} -beg + afg - bgh & aef - bfg + afh \\ -deg + cfg - dgh & cef - dfg + cfh \end{pmatrix}, \tag{41}$$

$$(B, A, B) = \begin{pmatrix} cef - beg - dfg & -aef - beh - dfh \\ aeg + ceh + dgh & afg + cfh - bgh \end{pmatrix}. \tag{42}$$

Subtracting the last two matrix associators gives

$$\mathcal{LS}(A, B, B) = \begin{pmatrix} afg - bgh - cef + dfg & 2aef + afh + beh - bfg + dfh \\ -aeg - ceh + cfg - deg - 2dgh & -afg + bgh + cef - dfg \end{pmatrix} \tag{43}$$

in which the sum of the diagonal entries is zero. The proof of the result for $\mathcal{RS}(B, B, A)$ is similar; if R has an involution, we may simply apply the conjugate transpose on $M_2(R)$. □

Proposition 3. *If R is an alternative ring and B, C are elements of the matrix ring $M_2(R)$ with B of trace* 0 *then $\mathcal{F}(B, C, C) = tI$ where t is in R and I is the identity matrix.*

Proof. We write

$$B = \begin{pmatrix} e & f \\ g & -e \end{pmatrix}, \qquad C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}. \tag{44}$$

Using Lemma 1 we obtain

$$(B, C, C) = \begin{pmatrix} epp + frp + eqr + fsr & epq + frq + eqs + fss \\ gpp - erp + gqr - esr & gpq - erq + gqs - ess \end{pmatrix}, \tag{45}$$

$$(C, C, B) = \begin{pmatrix} ppe + qre + pqg + qsg & ppf + qrf - pqe - qse \\ rpe + sre + rqg + ssg & rpf + srf - rqe - sse \end{pmatrix}. \tag{46}$$

Simplifying the associators in R gives

$$(B, C, C) = \begin{pmatrix} -fpr + eqr - frs & epq - fqr + eqs \\ epr + gqr + ers & gpq + eqr + gqs \end{pmatrix}, \tag{47}$$

$$(C, C, B) = \begin{pmatrix} eqr + gpq + gqs & fqr - epq - eqs \\ -epr - ers - gqr & -fpr - frs + eqr \end{pmatrix}. \tag{48}$$

Adding the last two matrix associators gives

$$\mathcal{F}(B, C, C) = \begin{pmatrix} 2eqr - fpr - frs + gpq + gqs & 0 \\ 0 & 2eqr - fpr - frs + gpq + gqs \end{pmatrix} \tag{49}$$

which is an R-multiple of the identity matrix. □

Proposition 4. *If R is an alternative ring and C, D are elements of the matrix ring $M_2(R)$ with $C = tI$ for some t in R then $\mathcal{LA}(C, D, D)$ and $\mathcal{RA}(D, D, C)$ are the zero matrix.*

Proof. We write

$$C = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \qquad D = \begin{pmatrix} w & x \\ y & z \end{pmatrix}. \tag{50}$$

Using Lemma 1 we obtain

$$(C, D, D) = \begin{pmatrix} tww + txy & twx + txz \\ tyw + tzy & tyx + tzz \end{pmatrix}, \tag{51}$$

$$(D, C, D) = \begin{pmatrix} wtw + xty & wtx + xtz \\ ytw + zty & ytx + ztz \end{pmatrix}. \tag{52}$$

Simplifying the associators in R gives

$$(C, D, D) = \begin{pmatrix} txy & twx + txz \\ -twy - tyz & -txy \end{pmatrix}, \tag{53}$$

$$(D, C, D) = \begin{pmatrix} -txy & -twx - txz \\ twy + tyz & txy \end{pmatrix}. \tag{54}$$

Adding the last two matrix associators gives

$$\mathcal{LA}(C, D, D) = \mathcal{O}. \tag{55}$$

The proof of the result for $\mathcal{RA}(D, D, C)$ is similar; if R has an involution, we may simply apply the conjugate transpose on $M_2(R)$. □

Theorem 3. *The algebra $M_2(\mathbb{O})$ satisfies no identities of degree $\leqslant 6$. Every identity of degree* 7 *satisfied by $M_2(\mathbb{O})$ follows from the seven identities in Theorems* 1 *and* 2.

Proof. The proof employs the computational methods described in Section 2.

Table 3 displays the ranks of the S_5-modules for the identities of degree 5. The first column gives a partition λ and the second column gives the dimension d_λ of the corresponding irreducible S_5-module. The group algebra $\mathbb{Q}S_5$ contains a full matrix subalgebra of size $d_\lambda \times d_\lambda$. Column 3 gives the product $t_5 d_\lambda$ where $t_5 = 14$ is the number of association types; this product is the rank (multiplicity) of the irreducible $\mathbb{Q}S_5$-module for λ in the direct sum of t_5 copies of $\mathbb{Q}S_5$. Column 4, labelled $\bar{I}$ to indicate linear relations satisfied by the identities, gives the rank (multiplicity) of each irreducible module in the *complement* of the submodule of identities satisfied by the algebra $M_2(\mathbb{O})$. Since column 3 equals column 4, we see that there are no identities in degree 5 satisfied by $M_2(\mathbb{O})$.

Table 4 displays the ranks of the S_6-modules for the identities of degree 6. Since column 3 equals column 4, we see that there are no identities in degree 6 satisfied by $M_2(\mathbb{O})$.

For $d = 7$ the number of association types is 132, and for partition 61 there are 7 monomials in each association type. Each monomial involves 6 x's and 1 y. This gives a total of $132 \cdot 7 = 924$ monomials. The matrix M has size $(924 + 32) \times 924 = 956 \times 924$. After $\lceil 924/32 \rceil = 29$ iterations the rank stabilizes at 920, giving a 4-dimensional nullspace of identities.

Table 3
Ranks for $M_2(\mathbb{O})$ in degree 5

λ	d_λ	$t_5 d_\lambda$	$\bar{I}$
5	1	14	14
41	4	56	56
32	5	70	70
311	6	84	84
221	5	70	70
2111	4	56	56
11111	1	14	14

Table 4
Ranks for $M_2(\mathbb{O})$ in degree 6

λ	d_λ	$t_6 d_\lambda$	$\bar{I}$
6	1	42	42
51	5	210	210
42	9	378	378
411	10	420	420
33	5	210	210
321	16	672	672
3111	10	420	420
222	5	210	210
2211	9	378	378
21111	5	210	210
111111	1	42	42

Table 5 displays the ranks of the S_7-modules for the identities of degree 7. The first three columns are analogous to the corresponding columns in the previous tables. Column 4, labelled "new" to indicate the new identities in Theorems 1 and 2, gives the rank (multiplicity) of each irreducible module in the submodule generated by these seven identities in degree 7; that is, the submodule of identities which are consequences of the new identities. Column 5, labelled $\bar{I}$ to indicate linear relations satisfied by the identities, gives the rank of each irreducible module in the *complement* of the submodule of identities satisfied by the algebra $M_2(\mathbb{O})$. Since column 3 equals the sum of columns 4 and 5, we see that the seven new identities generate all the identities in degree 7 satisfied by $M_2(\mathbb{O})$. □

Similar calculations in degree 8 give the following result.

Table 5
Ranks for $M_2(\mathbb{O})$ in degree 7

λ	d_λ	$t_7 d_\lambda$	new	$\bar{I}$
7	1	132	0	132
61	6	792	4	788
52	14	1848	13	1835
511	15	1980	15	1965
43	14	1848	16	1832
421	35	4620	36	4584
4111	20	2640	16	2624
331	21	2772	23	2749
322	21	2772	21	2751
3211	35	4620	28	4592
31111	15	1980	5	1975
2221	14	1848	12	1836
22111	14	1848	7	1841
211111	6	792	0	792
1111111	1	132	0	132

Theorem 4. *In the* 429*-dimensional space of homogeneous nonassociative polynomials in one variable in degree* 8 *there is a* 12*-dimensional subspace of identities for the algebra* $M_2(\mathbb{O})$. *In this* 12*-dimensional space, a* 4*-dimensional subspace consists of identities implied by the identities in degree* 7. *Therefore there exists an* 8*-dimensional space of new identities for* $M_2(\mathbb{O})$ *in degree* 8.

An attempt to find a convenient basis for a complementary 8-dimensional subspace of new identities in degree 8 has so far proved unsuccessful.

4. The Hermitian matrix algebra

In this and the next section we study commutative and anticommutative algebras, and so we need only consider one representative of each equivalence class of association types with respect to the symmetry $yx = \pm xy$. If s_d denotes the number of such representatives (the *Wedderburn number* [9]), then we have the recursion formula

$$s_1 = 1, \quad s_d = \sum_{i=1}^{\lfloor \frac{d-1}{2} \rfloor} s_{d-i} s_i + \binom{s_{d/2}}{2}, \tag{56}$$

where the last term only occurs when d is even. We obtain

$$s_1 = 1, \quad s_2 = 1, \quad s_3 = 1, \quad s_4 = 2, \quad s_5 = 3, \quad s_6 = 6, \quad s_7 = 11, \quad s_8 = 23. \tag{57}$$

In this section we study the algebra $H_2(\mathbb{O})$ of 2 × 2 Hermitian matrices over the octonions. This algebra is a 10-dimensional special Jordan algebra, and is isomorphic to the Jordan algebra of a bilinear form on a 9-dimensional vector space. The identities for Jordan algebras of nondegenerate symmetric bilinear forms have been determined by Vasilovsky; see [6,7] and [8].

Theorem 5 (Vasilovsky). *Let* A *denote the Jordan algebra of a nondegenerate symmetric bilinear form on a vector space of dimension* n *over a field of characteristic* 0. *Then* A *satisfies these two identities in degrees* 5 *and* 6:

$$\sum_{\sigma \in S_3} \varepsilon_\sigma \big[x_{\sigma(1)}, [x_{\sigma(2)}, y, x_{\sigma(3)}], y\big] = 0, \tag{58}$$

$$\big[T(x, x, y, y), z, t\big] = 0, \tag{59}$$

where we define the Jordan associator and the T*-polynomial by*

$$[x, y, z] = (x \circ y) \circ z - x \circ (y \circ z), \tag{60}$$

$$T(x, y, z, t) = [x \circ y, z, t] - x \circ [y, z, t] - y \circ [x, z, t]. \tag{61}$$

Furthermore, every identity of degree $\leqslant 2n$ *satisfied by* A *follows from these two identities.*

Table 6
Ranks for $H_2(\mathbb{O})$ in degree 5

λ	d_λ	$s_5 d_\lambda$	$CJ5$	$\bar{I}$
5	1	3	2	1
41	4	12	10	2
32	5	15	12	3
311	6	18	17	1
221	5	15	12	3
2111	4	12	12	0
11111	1	3	3	0

Proof. We give a computational demonstration of this result in the special case $A = H_2(\mathbb{O})$ with an upper bound of degree 7. This simultaneously reconfirms Vasilovsky's result and provides evidence of the reliability of our computational methods.

Table 6 displays the ranks of the S_5-modules (as described in Section 2) for the identities of degree 5. The first column gives a partition λ and the second column gives the dimension d_λ of the corresponding irreducible S_5-module. The group algebra $\mathbb{Q}S_5$ contains a full matrix subalgebra of size $d_\lambda \times d_\lambda$. Column 3 gives the product $s_5 d_\lambda$ where $s_5 = 3$ is the number of commutative association types; this product is the rank (multiplicity) of the irreducible S_5-module for λ in the direct sum of s_5 copies of the group algebra $\mathbb{Q}S_5$. Column 4, labelled $CJ5$ to indicate the commutative and Jordan identities together with the degree-5 Vasilovsky identity, gives the rank (multiplicity) of each irreducible module in the submodule generated by these three identities in degree 5; that is, the submodule of degree-5 identities which are consequences of the commutative, Jordan and degree-5 Vasilovsky identities. Column 5, labelled $\bar{I}$ to indicate linear relations satisfied by the identities, gives the rank of each irreducible module in the *complement* of the submodule of identities satisfied by the commutative algebra $H_2(\mathbb{O})$. Since column 3 equals the sum of columns 4 and 5, we see that the three given identities generate all the identities in degree 5 satisfied by $H_2(\mathbb{O})$.

Table 7 gives analogous results for degree 6: column 4 (labelled $CJ56$) gives the ranks of the submodules generated by the commutative and Jordan identities together with the

Table 7
Ranks for $H_2(\mathbb{O})$ in degree 6

λ	d_λ	$s_6 d_\lambda$	$CJ56$	$\bar{I}$
6	1	6	5	1
51	5	30	28	2
42	9	54	49	5
411	10	60	58	2
33	5	30	29	1
321	16	96	92	4
3111	10	60	60	0
222	5	30	27	3
2211	9	54	53	1
21111	5	30	30	0
111111	1	6	6	0

Table 8
Ranks for $H_2(\mathbb{O})$ in degree 7

λ	d_λ	$s_7 d_\lambda$	$CJ56$	$\bar{I}$
7	1	11	10	1
61	6	66	63	3
52	14	154	148	6
511	15	165	163	2
43	14	154	150	4
421	35	385	378	7
4111	20	220	220	0
331	21	231	230	1
322	21	231	226	5
3211	35	385	384	1
31111	15	165	165	0
2221	14	154	151	3
22111	14	154	154	0
211111	6	66	66	0
1111111	1	11	11	0

Vasilovsky identities of degree 5 and 6. As before, column 3 equals the sum of columns 4 and 5, and so these four identities generate all the identities in degree 6 satisfied by $H_2(\mathbb{O})$.

Table 8 summarizes the results for degree 7: column 4 (labelled $CJ56$) gives the ranks of the submodules generated by the commutative and Jordan identities together with the Vasilovsky identities of degree 5 and 6. As before, column 3 equals the sum of columns 4 and 5, and so these four identities generate all the identities in degree 7 satisfied by $H_2(\mathbb{O})$: there are no new identities appearing in degree 7. □

Similar computations give the following three results.

Theorem 6. *For the full matrix algebra $M_2(\mathbb{O})$, regarded as a commutative algebra under the symmetric product $\frac{1}{2}(XY + YX)$, there are no identities in degree $\leqslant 7$ that do not follow from commutativity.*

Theorem 7. *For the exceptional Jordan algebra $H_3(\mathbb{O})$ there are no identities in degree $\leqslant 7$ that do not follow from commutativity and the Jordan identity.*

Theorem 8. *For the commutative algebra $H_4(\mathbb{O})$ there are no identities in degree $\leqslant 6$ that do not follow from commutativity.*

In the last result, space considerations prevented us from continuing the computations through degree 7.

Table 9
Ranks for
$mathitSH_2(\mathbb{O})$ in degree 5

λ	d_λ	$s_5 d_\lambda$	x^2	$\overline{I}$
5	1	3	3	0
41	4	12	11	1
32	5	15	12	3
311	6	18	13	5
221	5	15	9	6
2111	4	12	6	6
11111	1	3	1	2

5. The skew-Hermitian matrix algebra

In this section we study the algebra $mathitSH_2(\mathbb{O})$ of 2×2 skew-Hermitian matrices over the octonions. This is an anticommutative algebra of dimension 22.

Theorem 9. *For the algebra $mathitSH_2(\mathbb{O})$ there are no identities in degree $\leqslant 7$ that do not follow from anticommutativity.*

Proof. Table 9 displays the ranks of the modules for the identities of degree 5. The first three columns are analogous to those in the previous tables. Column 4, labelled x^2 to indicate the anticommutative identity, gives the rank (multiplicity) of each irreducible module in the submodule generated by the anticommutative identity in degree 5; that is, the submodule of degree-5 identities which are consequences of the anticommutative identity. Column 5, labelled $\overline{I}$ to indicate linear relations satisfied by the identities, gives the rank (multiplicity) of each irreducible module in the *complement* of the submodule of identities satisfied by the anticommutative algebra $mathitSH_2(\mathbb{O})$. Since, for every partition λ, column 3 equals the sum of columns 4 and 5, it follows that every identity satisfied by the algebra $mathitSH_2(\mathbb{O})$ in degree 5 is a consequence of the anticommutative identity.

Similar remarks apply to Tables 10 and 11. As in degree 5, every identity satisfied by the algebra $mathitSH_2(\mathbb{O})$ in degrees 6 and 7 is a consequence of the anticommutative identity. □

Theorem 10. *For the full matrix algebra $M_2(\mathbb{O})$, regarded as an anticommutative algebra under the antisymmetric product $XY - YX$, there are no identities of degree $\leqslant 7$ that do not follow from anticommutativity.*

Proof. This follows immediately from the previous theorem since any identity for $M_2(\mathbb{O})$ under $XY - YX$ would also be an identity for $mathitSH_2(\mathbb{O})$. □

6. Concluding remarks

We have the isomorphism

$$M_2(\mathbb{O}) \approx M_2(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{O}. \tag{62}$$

Both $M_2(\mathbb{Q})$ and $\mathbb{O}$ are composition algebras, and the tensor product of composition algebras is a *structurable* algebra in the sense of Allison [1] (see also [5]): that is, a unital algebra with involution which satisfies the identity

$$[T_z, V_{x,y}] = V_{T_z x,y} - V_{x,T_{\bar{z}} y} \tag{63}$$

Table 10
Ranks for
$mathitSH_2(\mathbb{O})$ in degree 6

λ	d_λ	$s_6 d_\lambda$	x^2	$\bar{I}$
6	1	6	6	0
51	5	30	29	1
42	9	54	49	5
411	10	60	52	8
33	5	30	25	5
321	16	96	75	21
3111	10	60	44	16
222	5	30	21	9
2211	9	54	34	20
21111	5	30	17	13
111111	1	6	2	4

Table 11
Ranks for
$mathitSH_2(\mathbb{O})$ in degree 7

λ	d_λ	$s_7 d_\lambda$	x^2	$\bar{I}$
7	1	11	11	0
61	6	66	65	1
52	14	154	146	8
511	15	165	154	11
43	14	154	139	15
421	35	385	335	50
4111	20	220	185	35
331	21	231	190	41
322	21	231	182	49
3211	35	385	294	91
31111	15	165	117	48
2221	14	154	105	49
22111	14	154	98	56
211111	6	66	36	30
1111111	1	11	4	7

where

$$V_{a,b}(c) = \bigl(a\bar{b}\bigr)c + \bigl(c\bar{b}\bigr)a - \bigl(c\bar{a}\bigr)\bar{b}, \qquad T_a = V_{a,1}. \tag{64}$$

This identity involves an involution and hence it is not an identity in the sense considered in this paper. But the above isomorphism shows that $M_2(\mathbb{O})$ satisfies the structurable identity, and hence it is clear that the class of identities with involution is a natural place to search for further identities satisfied by the octonion matrix algebras $M_n(\mathbb{O})$, $H_n(\mathbb{O})$ and $mathitSH_n(\mathbb{O})$.

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References

[1] B.N. Allison, A class of nonassociative algebras with involution containing the class of Jordan algebras, Math. Ann. 237 (1978) 133–156 (MR 81h:17003).

[2] J.C. Baez, The octonions, Bull. Amer. Math. Soc. 39 (2002) 145–205 (MR 2003f:17003).

[3] G. James, A. Kerber, The Representation Theory of the Symmetric Group, Addison–Wesley, Reading, MA, 1981 (MR 83k:20003).

[4] R.D. Schafer, An Introduction to Nonassociative Algebras, Dover Publications, New York, 1995 (MR 96j:17001).

[5] O.N. Smirnov, Simple and semisimple structurable algebras, Algebra and Logic 29 (1990) 377–394; Russian original: 571–596 (MR 93b:17006).

[6] S.Yu. Vasilovsky, Basis of identities of a Jordan algebra of a bilinear form over an infinite field, Tr. Inst. Mat. (Novosibirsk), Issled. Teor. Kolets i Algebr 16 (1989) 5–37 (MR 91i:17050); see also Siberian Adv. Math. 1 (1991) 142–185 (MR 1 138 008).

[7] S.Yu. Vasilovsky, Identities with a trace of the Jordan algebra of a bilinear form, Siberian Math. J. 32 (1991) 916–919; Russian original: 19–23 (MR 93c:17054).

[8] S.Yu. Vasilovsky, Graded polynomial identities of the Jordan superalgebra of a bilinear form, J. Algebra 184 (1996) 255–296 (MR 97j:17034).

[9] J.H.M. Wedderburn, The functional equation $g(x^2) = 2\alpha x + [g(x)]^2$, Ann. of Math. 24 (1922–3) 121–140.

[10] K.A. Zhevlakov, A.M. Slin'ko, I.P. Shestakov, A.I. Shirshov, Rings That Are Nearly Associative, Academic Press, New York, 1982 (MR 83i:17001, see also 80h:17002).