Alternating triple systems with simple Lie algebras of derivations*

Murray R. Bremner†
Research Unit in Algebra and Logic
University of Saskatchewan
McLean Hall (Room 142), 106 Wiggins Road
Saskatoon, SK, S7N 5E6, Canada

Irvin R. Hentzel‡
Department of Mathematics
Iowa State University
Carver Hall (Room 400)
Ames, IA, 50011-2064, U.S.A.

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Abstract

We prove a formula for the multiplicity of the irreducible representation $V(n)$ of $sl(2,\mathbb{C})$ as a direct summand of its own exterior cube $\Lambda^3 V(n)$. From this we determine that $V(n)$ occurs exactly once as a summand of $\Lambda^3 V(n)$ if and only if $n = 3, 5, 6, 7, 8, 10$. These representations admit a unique $sl(2)$-invariant alternating ternary structure obtained from the projection $\Lambda^3 V(n) \to V(n)$. We calculate the structure constants for each of these alternating triple systems and use computer algebra to determine their polynomial identities of degree ≤ 7. We discover a remarkable 14-term identity in degree 7. The variety defined by this identity contains $V(3)$, $V(5)$ and $V(7)$.

Introduction

An irreducible representation of a simple Lie algebra can be a direct summand of its own exterior cube. In this case, the representation admits the structure of

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†Email address: bremner@math.usask.ca
‡Email address: hentzel@iastate.edu

an alternating triple system which is invariant in the sense that the Lie algebra acts as ternary derivations. This paper studies this situation in detail for the Lie algebra $sl(2, \mathbb{C})$.

In section 1 we review the basic representation theory of $sl(2)$. In section 2 we prove a general formula for the multiplicity of an irreducible representation in its own exterior cube. From this we determine all representations for which the multiplicity equals 1; such representations admit an $sl(2)$-invariant alternating ternary structure which is unique up to a scalar multiple. In section 3 we review basic material about ternary operations.

In the following six sections we describe computer searches for polynomial identities satisfied by the six representations which admit a unique alternating ternary structure; we determine all their identities of degree ≤ 7. A detailed discussion of our computational methods for discovering identities satisfied by nonassociative algebras may be found in three previous articles by the authors [3], [4], [5]. These methods involve expressing the identities as the nullspace of a large linear system, and then solving the system by using a computer algebra system to compute the row canonical form of the coefficient matrix.

In section 10 we go beyond $sl(2)$ and use the computer algebra package `LiE` [6] to determine all fundamental representations of simple Lie algebras of rank ≤ 8 which occur as summands of their own exterior cubes. This demonstrates the existence of a large number of new alternating triple systems, with simple Lie algebras in their derivation algebras, which deserve further study.

1 Representations of the Lie algebra $sl(2)$

We first recall some standard facts about $sl(2)$ and its representations. All vector spaces and tensor products are over $\mathbb{F}$, an algebraically closed field of characteristic zero. Our basic reference is Humphreys [8], especially §II.7.

1.1 The Lie algebra $sl(2)$

As an abstract Lie algebra, $sl(2)$ has basis $\{E, H, F\}$ and commutation relations

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

All other relations between basis elements follow from anticommutativity. Since the Lie bracket is bilinear these relations determine the product $[X, Y]$ for all $X, Y \in sl(2)$.

1.2 The irreducible representation $V(n)$

For any nonnegative integer n, there is an irreducible representation of $sl(2)$ containing a nonzero vector v_n (called the highest weight vector) satisfying the conditions

$$H.v_n = nv_n, \qquad E.v_n = 0.$$

This representation is unique up to isomorphism of $sl(2)$-modules. It is denoted $V(n)$ and is called the representation with highest weight n. Its dimension is $n+1$; a basis of $V(n)$ consists of the $n+1$ vectors v_{n-2i} where

$$v_{n-2i} = \frac{1}{i!} F^i.v_n, \qquad 0 \leq i \leq n.$$

The action of $sl(2)$ on $V(n)$ is then as follows:

$$\begin{aligned} E.v_{n-2i} &= (n-i+1)v_{n-2i+2}, \\ H.v_{n-2i} &= (n-2i)v_{n-2i}, \\ F.v_{n-2i} &= (i+1)v_{n-2i-2}. \end{aligned}$$

The basis vectors v_{n-2i} are called weight vectors since they are eigenvectors for H.

1.3 Exterior cubes

In this paper we are primarily concerned with the multiplicity of $V(n)$ as a direct summand of its exterior cube $\Lambda^3 V(n)$. A basis of $\Lambda^3 V(n)$ consists of the $\binom{n+1}{3}$ alternating sums

$$\sum_{\text{alt}} v_p \otimes v_q \otimes v_r =$$
$$v_p \otimes v_q \otimes v_r + v_q \otimes v_r \otimes v_p + v_r \otimes v_p \otimes v_q - v_p \otimes v_r \otimes v_q - v_q \otimes v_p \otimes v_r - v_r \otimes v_q \otimes v_p,$$

where p, q, r are decreasing distinct weights of $V(n)$; that is,

$$n \geq p > q > r \geq -n, \quad p, q, r \equiv n \ (\mathrm{mod}\ 2).$$

If $D \in sl(2)$ then D acts on these alternating sums by the derivation rule

$$D.\sum_{\text{alt}} v_p \otimes v_q \otimes v_r =$$
$$\sum_{\text{alt}} (D.v_p) \otimes v_q \otimes v_r + \sum_{\text{alt}} v_p \otimes (D.v_q) \otimes v_r + \sum_{\text{alt}} v_p \otimes v_q \otimes (D.v_r),$$

and this action extends linearly to all of $\Lambda^3 V(n)$.

If $V(n)$ occurs as a summand of $\Lambda^3 V(n)$ then the multiplicity

$$m = \dim \mathrm{Hom}_{sl(2)}(\Lambda^3 V(n), V(n))$$

will be positive. Any $sl(2)$-module homomorphism $p \colon \Lambda^3 V(n) \to V(n)$ gives $V(n)$ the structure of an alternating triple system in the sense that

$$p(u^\sigma, v^\sigma, w^\sigma) = \epsilon(\sigma) p(u, v, w),$$

where $\sigma \in S_3$ is any permutation of u, v, w and $\epsilon(\sigma)$ is its sign. This alternating triple product is $sl(2)$-invariant in the sense that the elements of $sl(2)$ act as ternary derivations:

$$D.p(u, v, w) = p(D.u, v, w) + p(u, D.v, w) + p(u, v, D.w).$$

Two projections p_1 and p_2 which differ by a nonzero scalar multiple define isomorphic triple systems on $V(n)$ and so the family of non-isomorphic alternating ternary structures on $V(n)$ has $m - 1$ projective parameters.

1.4 Relation between binary and ternary structures

By the Clebsch-Gordan Theorem (see Bremner and Hentzel [5] for an explicit version of this classical result) we know that $V(n)$ occurs as a summand of its exterior square if and only if $n \equiv 2 \pmod 4$. In this case the multiplicity is 1, and so $V(n)$ has an $sl(2)$-invariant anticommutative binary product. For $n = 2$ we recover $sl(2)$ and for $n = 6$ we recover the simple non-Lie Malcev algebra. For $n \geq 10$ we obtain less familiar anticommutative algebras; the $n = 10$ case is studied in detail in Bremner and Hentzel [5]. On any anticommutative algebra, the Jacobian

$$J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y]$$

defines an alternating ternary operation, which is identically zero if and only if the algebra is a Lie algebra.

2 A general multiplicity formula

In this section we prove a formula for the multiplicity of $V(n)$ as a summand of its exterior cube $\Lambda^3 V(n)$: see Theorem 1. We first prove four Lemmas.

Lemma 1. *Let M be a finite dimensional $sl(2)$-module. For every integer n define $M_n = \{v \in M \mid H.v = nv\}$; that is, the subspace of M consisting of all vectors of weight n (together with 0). Then for any non-negative integer n the multiplicity of $V(n)$ as a direct summand of M equals* $\dim M_n - \dim M_{n+2}$.

Proof. We know that M is completely reducible; this follows from Weyl's Theorem (any finite dimensional module over any semisimple Lie algebra is completely reducible), but we only need the special case when the Lie algebra is $sl(2)$. So we can write

$$M = m_1 V(n_1) \oplus m_2 V(n_2) \oplus \cdots \oplus m_k V(n_k),$$

where $m_1, \ldots, m_k$ are positive integers and $n_1 > n_2 > \cdots > n_k \geq 0$; furthermore, the integers k, m_i, n_i are uniquely determined. (The notation $mV(n)$ abbreviates "the direct sum of m copies of $V(n)$".) We know that $V(n)$ has a one-dimensional weight space for any weight n' with $n \geq n' \geq -n$ and $n \equiv n'$

(mod 2). Therefore, we need to separate the "even" and "odd" parts of M, so we define

$$M^0 = \bigoplus_{n_i \equiv n_1 \pmod 2} m_i V(n_i), \qquad M^1 = \bigoplus_{n_i \not\equiv n_1 \pmod 2} m_i V(n_i).$$

Now assume that $n' \equiv n_1 \pmod 2$; equivalently, we will assume that $M^0 = M$ and $M^1 = \{0\}$. The direct sum decomposition of M now gives

$$\begin{aligned}
&\dim M_{n'} = m_1 \text{ for } n_1 \geq n' > n_2,\\
&\dim M_{n'} = m_1 + m_2 \text{ for } n_2 \geq n' > n_3,\\
&\qquad\qquad \dots\\
&\dim M_{n'} = m_1 + m_2 + \cdots + m_i \text{ for } n_i \geq n' > n_{i+1},\\
&\qquad\qquad \dots\\
&\dim M_{n'} = m_1 + m_2 + \cdots + m_k \text{ for } n_k \geq n' \geq 0.
\end{aligned}$$

From this it is obvious that

$$m_i = (m_1 + \cdots + m_i) - (m_1 + \cdots + m_{i-1}) = \dim M_{n_i} - \dim M_{n_i+2}.$$

The proof in the case $n' \not\equiv n_1 \pmod 2$ is similar. □

In order to apply Lemma 1 to $M = \Lambda^3 V(n)$ we need to know the dimensions of the weight spaces $M_{n'}$. We state the following result for any exterior power.

Lemma 2. *Let $M = \Lambda^k V(n)$ and assume that $nk \geq n' \geq -nk$ with $n' \equiv nk$ (mod 2). Then the dimension of the weight space $M_{n'}$ equals the number of sequences $(n'_1, n'_2, \dots, n'_k)$ satisfying the conditions*

$$n \geq n'_1 > n'_2 > \cdots > n'_k \geq -n, \quad n'_1 + n'_2 + \cdots + n'_k = n'.$$

Proof. A basis of $\Lambda^k V(n)$ consists of the alternating sums

$$\sum_{\sigma \in S_k} \epsilon(\sigma)\, v_{n'_{\sigma(1)}} \otimes v_{n'_{\sigma(2)}} \otimes \cdots \otimes v_{n'_{\sigma(k)}}, \quad n \geq n'_1 > n'_2 > \cdots > n'_k \geq -n,$$

where $\epsilon(\sigma)$ is the sign of the permutation σ and $v_{n'_i}$ is a fixed vector of weight n'_i in $V(n)$ (such a vector is unique up to a nonzero scalar multiple). Since

$$H.v = (n'_1 + n'_2 + \cdots + n'_k)\, v, \quad \text{for} \quad v = v_{n'_{\sigma(1)}} \otimes v_{n'_{\sigma(2)}} \otimes \cdots \otimes v_{n'_{\sigma(k)}},$$

the result follows. □

To compute the dimensions of the weight spaces in $M = \Lambda^3 V(n)$, by Lemma 2 we need to determine the number of triples (n'_1, n'_2, n'_3) satisfying the conditions

$$n \geq n'_1 > n'_2 > n'_3 \geq -n, \quad n'_1 + n'_2 + n'_3 = n',$$

for the cases $n' = n$ and $n' = n + 2$; then we can apply Lemma 1.

We first simplify the notation. Let n be a non-negative integer, let n' be an integer satisfying

$$3n \geq n' \geq -3n, \quad n' \equiv 3n \,(\mathrm{mod}\, 2),$$

and let p, q, r be integers satisfying

$$n \geq p > q > r \geq -n, \quad p+q+r = n', \quad p, q, r \equiv n \,(\mathrm{mod}\, 2).$$

Now define

$$P = \frac{1}{2}(p+n), \quad Q = \frac{1}{2}(q+n), \quad R = \frac{1}{2}(r+n).$$

Then (P, Q, R) is a triple of integers satisfying

$$n \geq P > Q > R \geq 0, \quad P+Q+R = N, \quad N = \frac{1}{2}(n'+3n).$$

So we need to count the number of partitions of N into two or three distinct parts less than or equal to n. We only need to consider the cases $N = 2n$ ($n' = n$) and $N = 2n+1$ ($n' = n+2$).

Lemma 3. *The number of triples (P, Q, R) of integers satisfying the conditions*

$$n \geq P > Q > R \geq 0, \quad P+Q+R = 2n,$$

is given by the formula

$$\sum_{i=0}^{\lfloor n/3 \rfloor - 1} \left(\left\lfloor \frac{n+i-1}{2} \right\rfloor - 2i \right).$$

Proof. We enumerate the triples as follows. First, those with $P = n$:

$$(n, n-1, 1), \quad (n, n-2, 2), \quad \ldots.$$

The last triple with $P = n$ is

$$(n, \tfrac{n}{2}+1, \tfrac{n}{2}-1) \ (n \text{ even}); \quad (n, \tfrac{n+1}{2}, \tfrac{n-1}{2}) \ (n \text{ odd}).$$

For $P = n$ there are altogether $\lfloor \frac{n-1}{2} \rfloor$ triples.

Second, the triples with $P = n-1$:

$$(n-1, n-2, 3), \quad (n-1, n-3, 4), \quad \ldots.$$

The last triple with $P = n-1$ is

$$(n-1, \tfrac{n}{2}+1, \tfrac{n}{2}) \ (n \text{ even}); \quad (n-1, \tfrac{n+3}{2}, \tfrac{n-1}{2}) \ (n \text{ odd}).$$

For $P = n-1$ there are altogether $\lfloor \frac{n}{2} - 2 \rfloor$ triples.

In the general case, when $P = n-i$, we have

$$(n-i, n-(i+1), 2i+1), \quad (n-i, n-(i+2), 2i+2), \quad \ldots.$$

The last triple with $P = n - i$ is

$$(n-i, \tfrac{n}{2}+\tfrac{i}{2}+1, \tfrac{n}{2}+\tfrac{i}{2}-1) \ (n \text{ even}, i \text{ even});$$
$$(n-i, \tfrac{n}{2}+\tfrac{i+1}{2}, \tfrac{n}{2}+\tfrac{i+1}{2}-1) \ (n \text{ even}, i \text{ odd});$$
$$(n-i, \tfrac{n+1}{2}+\tfrac{i}{2}, \tfrac{n+1}{2}+\tfrac{i}{2}-1) \ (n \text{ odd}, i \text{ even});$$
$$(n-i, \tfrac{n+1}{2}+\tfrac{i+1}{2}, \tfrac{n+1}{2}+\tfrac{i+1}{2}-2) \ (n \text{ odd}, i \text{ odd}).$$

These four triples can be reduced to two cases:

$$\begin{array}{ll} (n-i, \frac{n+i+1}{2}, \frac{n+i-1}{2}) & \text{when } n \text{ and } i \text{ have opposite parity;} \\ (n-i, \frac{n+i}{2}+1, \frac{n+i}{2}-1) & \text{when } n \text{ and } i \text{ have the same parity.} \end{array}$$

For $P = n - i$ the total number of triples is

$$\left\lfloor \frac{n+i-1}{2} \right\rfloor - 2i.$$

Since $P > Q$ we obtain the condition $n \geq 3(i+1)$, or equivalently

$$0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor - 1.$$

Summing the number of triples for each i over this range gives the result. □

Lemma 4. *The number of triples (P, Q, R) of integers satisfying the conditions*

$$n \geq P > Q > R \geq 0, \quad P + Q + R = 2n + 1,$$

is given by the formula

$$\sum_{i=0}^{\lfloor (n-1)/3 \rfloor - 1} \left(\left\lfloor \frac{n+i}{2} \right\rfloor - (2i+1) \right).$$

Proof. Very similar to the proof of Lemma 3. □

Theorem 1. *Let $V(n)$ denote the simple $sl(2)$-module with highest weight n. Write $n = 6k + \ell$ where $0 \leq \ell \leq 5$. Then*

$$\dim \mathrm{Hom}_{sl(2)}(\Lambda^3 V(n), V(n)) = \begin{cases} k, & \text{if } \ell = 0, 1, 2, 4; \\ k+1, & \text{if } \ell = 3, 5. \end{cases}$$

Proof. By Lemma 1, the multiplicity is the difference between the formulas of Lemmas 3 and 4. Since the expressions $\lfloor \frac{n}{3} \rfloor$ and $\lfloor \frac{n}{2} \rfloor$ occur, we need to distinguish 6 cases, depending on the congruence class of n modulo 6. So we write $n = 6k + \ell$ where $0 \leq \ell \leq 5$.

First assume that $\ell = 0$. Then $\lfloor \frac{n}{3} \rfloor = 2k$ and $\lfloor \frac{n-1}{3} \rfloor = 2k - 1$. Therefore the multiplicity is

$$\sum_{i=0}^{2k-1} \left(\left\lfloor \frac{6k+i-1}{2} \right\rfloor - 2i \right) - \sum_{i=0}^{2k-2} \left(\left\lfloor \frac{6k+i}{2} \right\rfloor - (2i+1) \right).$$

Separating the last term of the first sum, and combining the other terms in one sum, gives

$$\sum_{i=0}^{2k-2} \left(\left\lfloor \frac{6k+i-1}{2} \right\rfloor - 2i - \left\lfloor \frac{6k+i}{2} \right\rfloor + (2i+1) \right) \\ + \left(\left\lfloor \frac{6k+(2k-1)-1}{2} \right\rfloor - 2(2k-1) \right).$$

Simplifying this, we obtain

$$\sum_{i=0}^{2k-2} \left(3k + \left\lfloor \frac{i-1}{2} \right\rfloor - 3k - \left\lfloor \frac{i}{2} \right\rfloor \right) + (2k-1) + (4k-1) - (4k-2) \\ = \sum_{i=0}^{2k-2} \left(\left\lfloor \frac{i-1}{2} \right\rfloor - \left\lfloor \frac{i}{2} \right\rfloor \right) + 2k.$$

Splitting the remaining sum into two parts for i even and i odd gives

$$\sum_{i=0,\,\text{even}}^{2k-2} \left(\left\lfloor \frac{i-1}{2} \right\rfloor - \left\lfloor \frac{i}{2} \right\rfloor \right) + \sum_{i=1,\,\text{odd}}^{2k-3} \left(\left\lfloor \frac{i-1}{2} \right\rfloor - \left\lfloor \frac{i}{2} \right\rfloor \right) + 2k \\ = \sum_{i=0,\,\text{even}}^{2k-2} (-1) + \sum_{i=1,\,\text{odd}}^{2k-3} (0) + 2k = -k + 0 + 2k = k.$$

This completes the proof for $\ell = 0$. The other cases ($1 \leq \ell \leq 5$) are similar. □

Corollary 1. *The simple $sl(2)$-module $V(n)$ admits a unique $sl(2)$-invariant alternating ternary structure if and only if $n = 3, 5, 6, 7, 8, 10$.*

Proof. By Theorem 1 the module $V(n)$ occurs exactly once as a summand of $\Lambda^3 V(n)$ if and only if $n = 3, 5, 6, 7, 8, 10$. Hence (up to a nonzero scalar multiple) there is a unique $sl(2)$-module homomorphism from $\Lambda^3 V(n)$ to $V(n)$ exactly in these cases. □

3 Ternary operations

Let A be a vector space over a field $\mathbb{F}$, together with a trilinear map $p\colon A \times A \times A \to A$. We call the pair (A, p) a triple system (or ternary algebra) over $\mathbb{F}$. We often suppress p and write $[a, b, c]$ for $p(a, b, c)$. We are interested in the (ternary) polynomial identities satisfied by A. To simplify the discussion, we will assume initially that the base field $\mathbb{F}$ has characteristic 0. This implies that any polynomial identity over $\mathbb{F}$ is equivalent to a family of homogeneous multilinear identities. See Bremner [1, 2] for the following results.

Proposition 1. *Up to alternating equivalence, there is one association type in degree 5 for an alternating ternary product* $[a, b, c]$*:*

$$[[a, b, c], d, e].$$

There are 10 distinct multilinear monomials:

$$\begin{gathered}[[a,b,c],d,e],\ [[a,b,d],c,e],\ [[a,b,e],c,d],\ [[a,c,d],b,e],\ [[a,c,e],b,d],\\ [[a,d,e],b,c],\ [[b,c,d],a,e],\ [[b,c,e],a,d],\ [[b,d,e],a,c],\ [[c,d,e],a,b].\end{gathered}$$

The decomposition of the 10-dimensional representation S_5 *with these monomials as basis is*

$$[221] \oplus [2111] \oplus [11111].$$

Here $[\lambda]$ *denotes the irreducible representation of* S_5 *corresponding to the partition* λ*.*

Proposition 2. *Up to alternating equivalence, there are two association types in degree 7:*

$$[[[a, b, c], d, e], f, g], \qquad [[a, b, c], [d, e, f], g].$$

For the first association type the number of distinct multilinear monomials is

$$\frac{7!}{3!2!2!} = 210,$$

and for the second association type the number is

$$\frac{1}{2} \cdot \frac{7!}{3!3!} = 70;$$

altogether there are 280 multilinear monomials in degree 7. The decomposition of the 280-dimensional representation of S_7 *with these monomials as basis is*

$$[322] \oplus [3211] \oplus 3[31111] \oplus [2221] \oplus 3[22111] \oplus 4[211111] \oplus 2[1111111].$$

Here $m[\lambda]$ *denotes the direct sum of* m *copies of the irreducible representation of* S_7 *corresponding to the partition* λ*.*

3.1 Notation

Many of the identities we present in this paper contain alternating sums over all permutations of certain sets of variables. We will use the notation

$$\sum_{\mathrm{alt}(S)} [a_{i_1} a_{i_2} \cdots a_{i_d}],$$

to denote the alternating sum over all permutations of the variables a_s for $s \in S$. (The other variables remain in the same position in each term of the sum.) The term $[a_{i_1} a_{i_2} \cdots a_{i_d}]$ denotes a multilinear alternating ternary monomial of degree

d in some association. If $S = \{1, \dots, k\}$ where $2 \leq k \leq d$ then we will use the notation

$$\sum_{\mathrm{alt}(k)} [a_{i_1} a_{i_2} \cdots a_{i_d}].$$

If $|S| = k$ then such an alternating sum contains $k!$ terms each with a coefficient $+1$ or -1. The alternating ternary laws will cause many of these terms to be equal. So the actual identity will have many fewer terms when expressed as a linear combination of the basic monomials.

4 Representation $V(3)$: dimension 4

The decomposition of the exterior cube is

$$\Lambda^3 V(3) \cong V(3).$$

The exterior cube, which is this case is isomorphic to the original representation, has a basis consisting of the 4 alternating sums corresponding to the triples of weights

$$[p, q, r] = [3, 1, -1],\ [3, 1, -3],\ [3, -1, -3],\ [1, -1, -3].$$

Any highest weight vector in $\Lambda^3 V(3)$ is a nonzero scalar multiple of

$$z_3 = \sum_{\mathrm{alt}} v_3 \otimes v_1 \otimes v_{-1}.$$

Applying $F \in sl(2)$ repeatedly we obtain the other weight vectors forming a basis of $V(3)$:

$$z_1 = \sum_{\mathrm{alt}} 3 v_3 \otimes v_1 \otimes v_{-3}, \quad z_{-1} = \sum_{\mathrm{alt}} 3 v_3 \otimes v_{-1} \otimes v_{-3}, \quad z_{-3} = \sum_{\mathrm{alt}} v_1 \otimes v_{-1} \otimes v_{-3}.$$

Let A be the change of basis matrix in which the ij-entry is the coefficient of the i-th alternating sum in the j-th weight vector. Then

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore the structure constants for the alternating ternary product on $V(3)$ are

$$\begin{aligned} [v_3, v_1, v_{-1}] &= v_3, & [v_3, v_1, v_{-3}] &= \tfrac{1}{3} v_1, \\ [v_3, v_{-1}, v_{-3}] &= \tfrac{1}{3} v_{-1}, & [v_1, v_{-1}, v_{-3}] &= v_{-3}, \end{aligned}$$

and all others follow from the alternating property.

4.1 Identities of degree 5

Theorem 2. *Every identity in degree 5 satisfied by the alternating ternary structure on $V(3)$ is a consequence of the identity*

$$
\begin{aligned}
&[[abc]de] - [[abd]ce] + [[abe]cd] + [[acd]be] - [[ace]bd] \\
&\quad + [[ade]bc] - [[bcd]ae] + [[bce]ad] - [[bde]ac] + [[cde]ab].
\end{aligned}
$$

This identity is ($\frac{1}{12}$-th of) the alternating sum over all permutations of the arguments in the basic monomial $[[abc]de]$.

Proof. We create a matrix of size 14×10 in which the columns are labelled by the ordered basis of multilinear monomials in degree 5 for an alternating ternary operation. We generate five random elements of $V(3)$ (random vectors with 4 components) and evaluate the 10 monomials on these five elements. We put the 10 resulting 4×1 vectors into the bottom of the matrix (in rows 11 through 14). Each of the last four rows of the matrix now contains a linear relation that must be satisfied by the coefficients of any identity for the triple system. We compute the row canonical form of the matrix. Now the last four rows are zero, so we can repeat the "fill and reduce" process. We keep repeating this process until the rank of the matrix stabilizes. In this case the rank reached 9 and did not increase further. The nullspace of the matrix at this point is one-dimensional. A basis for the nullspace is the identity in the Theorem. □

The identity of Theorem 2 can also be written using our alternating sum notation in the more compact form

$$
\frac{1}{12} \sum_{\mathrm{alt}(5)} [[a_1 a_2 a_3] a_4 a_5].
$$

We will use this notation in the rest of this paper.

4.2 Identities of degree 7

Theorem 3. *Every identity in degree 7 satisfied by the alternating triple system $V(3)$ follows from the identity in degree 5 from Theorem 2 and the 14-term identity displayed in Theorem 4.*

Proof. Using the random element method described in our previous papers (Bremner and Hentzel [3, 4, 5]) we found that the vector space of all identities for $V(3)$ in degree 7 is a subspace of dimension 210 in the 280-dimensional space of all possible degree 7 identities. We need to determine which of these identities are consequences of the known identity in degree 5 given in Theorem 2.

We write $I = I(a, b, c, d, e)$ for the identity of Theorem 2. This is an alternating function of its five arguments, so there are only two inequivalent ways of lifting this identity to degree 7:

$$
I([a, f, g], b, c, d, e), \qquad [I(a, b, c, d, e), f, g].
$$

There are $\binom{7}{3} = 35$ different forms of the first lifting, and $\binom{7}{2} = 21$ different forms of the second. We put these 56 different liftings into a matrix of size 56×280. (The ij-entry is the coefficient of the j-th monomial in degree 7 in the i-th lifting.) We computed the rank of this matrix: the rank is 56, so the different liftings are linearly independent. They form a basis for the subspace of the degree 7 identities which are consequences of the degree 5 identity of Theorem 2.

We took the 14-term identity in Theorem 4 and added it as the last row of a matrix of size 211×280 in which the first 210 rows are the basis vectors of the identities for $V(3)$ in degree 7. We reduced this matrix and found that its rank is 210. It follows that the 14-term identity is satisfied by the alternating ternary structure on $V(3)$.

We then took the 14-term identity and determined a basis for the submodule of identities in degree 7 that it generates. (For the method we used see the proof of Theorem 5.) This gives 189 linearly independent vectors of length 210. We stacked two matrices: the first matrix of size 56×280 contains a basis for the lifted identities from degree 5; the second matrix of size 189×280 contains a basis for the submodule generated by the 14-term identity. This gives a matrix of size 245×280. We reduced this matrix and found that its rank is 210. This implies that the lifted identities together with the 14-term identity generate the entire 210-dimensional space of identities satisfied by $V(3)$ in degree 7. □

4.3 Another alternating triple system of dimension 4

Let V be a vector space over the field $\mathbb{F}$ with basis I, J, K, L. Let

$$X = (x_1, x_2, x_3, x_4), \ Y = (y_1, y_2, y_3, y_4), \ Z = (z_1, z_2, z_3, z_4),$$

be any three elements of V expressed as quadruples with respect to the given basis. Define the ternary cross product on V by the formal determinant

$$[X, Y, Z] = \begin{vmatrix} I & J & K & L \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

This gives the following triple products of basis vectors:

$$[I, J, K] = -L, \quad [I, J, L] = K, \quad [I, K, L] = -J, \quad [J, K, L] = I.$$

It is shown in Bremner and Hentzel [3] that every identity of degree ≤ 7 satisfied by this triple system follows from the alternating property and the ternary Jacobi identity

$$[V, W, [X, Y, Z]] = [[V, W, X], Y, Z] + [X, [V, W, Y], Z] + [X, Y, [V, W, Z]].$$

In terms of the standard association type this identity can be written as

$$[[X, Y, Z], V, W] - [[X, V, W], Y, Z] + [[Y, V, W], X, Z] - [[Z, V, W], X, Y].$$

It is shown in Bremner [1] (Theorem 2, page 83) that this identity implies the identity of Theorem 2 but is not implied by that identity. (This proves that these two four-dimensional alternating triple systems are not isomorphic.) The ternary Jacobi identity and its n-ary generalization were introduced by Filippov [7] in his study of n-Lie algebras.

5 Representation $V(5)$: dimension 6

The decomposition of the exterior cube is

$$\Lambda^3 V(5) \cong V(9) \oplus V(5) \oplus V(3).$$

In this case, to illustrate our computational methods, we will present the calculations in greater detail than in the following sections.

The multiplicities of the submodules in this decomposition can be read off by applying Lemma 1 to Table 1 which contains the ordered triples $[p, q, r]$ of distinct weights of $V(5)$: that is, p, q, r are odd integers satisfying $5 \geq p > q > r \geq -5$. The triple $[p, q, r]$ represents the alternating sum

$$\sum_{\text{alt}} v_p \otimes v_q \otimes v_r.$$

The $\binom{6}{3} = 20$ alternating sums in Table 1 form a basis of $\Lambda^3 V(5)$.

WEIGHT	TRIPLE 1	TRIPLE 2	TRIPLE 3
9	$[5, 3, 1]$		
7	$[5, 3, -1]$		
5	$[5, 3, -3]$	$[5, 1, -1]$	
3	$[5, 3, -5]$	$[5, 1, -3]$	$[3, 1, -1]$
1	$[5, 1, -5]$	$[3, 1, -3]$	$[3, 1, -3]$
-1	$[5, -1, -5]$	$[3, 1, -5]$	$[3, -1, -3]$
-3	$[5, -3, -5]$	$[3, -1, -5]$	$[1, -1, -3]$
-5	$[3, -3, -5]$	$[1, -1, -5]$	
-7	$[1, -3, -5]$		
-9	$[-1, -3, -5]$		

Table 1: Alternating sums forming a basis of $\Lambda^3 V(5)$

5.1 The $V(9)$ summand

A highest weight vector for the $V(9)$ summand of $\Lambda^3 V(5)$ is

$$x_9 = \sum_{\text{alt}} v_5 \otimes v_3 \otimes v_1.$$

Applying F repeatedly we obtain the other basis vectors for the $V(9)$ summand:

$$x_7 = \sum_{\text{alt}} 3v_5 \otimes v_3 \otimes v_{-1}, \qquad x_5 = \sum_{\text{alt}} (6v_5 \otimes v_3 \otimes v_{-3} + 3v_5 \otimes v_1 \otimes v_{-1}),$$
$$x_3 = \sum_{\text{alt}} (10v_5 \otimes v_3 \otimes v_{-5} + 8v_5 \otimes v_1 \otimes v_{-3} + v_3 \otimes v_1 \otimes v_{-1}),$$
$$x_1 = \sum_{\text{alt}} (15v_5 \otimes v_1 \otimes v_{-5} + 6v_5 \otimes v_{-1} \otimes v_{-3} + 3v_3 \otimes v_1 \otimes v_{-3}),$$
$$x_{-1} = \sum_{\text{alt}} (15v_5 \otimes v_{-1} \otimes v_{-5} + 6v_3 \otimes v_1 \otimes v_{-5} + 3v_3 \otimes v_{-1} \otimes v_{-3}),$$
$$x_{-3} = \sum_{\text{alt}} (10v_5 \otimes v_{-3} \otimes v_{-5} + 8v_3 \otimes v_{-1} \otimes v_{-5} + v_1 \otimes v_{-1} \otimes v_{-3}),$$
$$x_{-5} = \sum_{\text{alt}} (6v_3 \otimes v_{-3} \otimes v_{-5} + 3v_1 \otimes v_{-1} \otimes v_{-5}),$$
$$x_{-7} = \sum_{\text{alt}} 3v_1 \otimes v_{-3} \otimes v_{-5}, \qquad x_{-9} = \sum_{\text{alt}} v_{-1} \otimes v_{-3} \otimes v_{-5}.$$

5.2 The $V(5)$ summand

A highest weight vector for the $V(5)$ summand of $\Lambda^3 V(5)$ has the form

$$y_5 = a \sum_{\text{alt}} v_5 \otimes v_3 \otimes v_{-3} + b \sum_{\text{alt}} v_5 \otimes v_1 \otimes v_{-1}.$$

Applying E to this vector we obtain

$$E.y_5 = (2a + 4b) \sum_{\text{alt}} v_5 \otimes v_3 \otimes v_{-1}.$$

Therefore $2a + 4b = 0$ so we take $a = 2$, $b = -1$ to get the highest weight vector

$$y_5 = \sum_{\text{alt}} (2v_5 \otimes v_3 \otimes v_{-3} - v_5 \otimes v_1 \otimes v_{-1}).$$

Applying F repeatedly we obtain the other basis vectors for the $V(5)$ summand:

$$y_3 = \sum_{\text{alt}} (10v_5 \otimes v_3 \otimes v_{-5} - v_3 \otimes v_1 \otimes v_{-1}),$$
$$y_1 = \sum_{\text{alt}} (10v_5 \otimes v_1 \otimes v_{-5} - 2v_3 \otimes v_1 \otimes v_{-3}),$$
$$y_{-1} = \sum_{\text{alt}} (10v_5 \otimes v_{-1} \otimes v_{-5} - 2v_3 \otimes v_{-1} \otimes v_{-3}),$$
$$y_{-3} = \sum_{\text{alt}} (10v_5 \otimes v_{-3} \otimes v_{-5} - v_1 \otimes v_{-1} \otimes v_{-3}),$$
$$y_{-5} = \sum_{\text{alt}} (2v_3 \otimes v_{-3} \otimes v_{-5} - v_1 \otimes v_{-1} \otimes v_{-5}).$$

5.3 The $V(3)$ summand

A highest weight vector for the $V(3)$ summand of $\Lambda^3 V(5)$ has the form

$$z_3 = a \sum_{\text{alt}} v_5 \otimes v_3 \otimes v_{-5} + b \sum_{\text{alt}} v_5 \otimes v_1 \otimes v_{-3} + c \sum_{\text{alt}} v_3 \otimes v_1 \otimes v_{-1}.$$

Applying E to this vector we obtain

$$E.z_3 = (a + 4b) \sum_{\text{alt}} v_5 \otimes v_3 \otimes v_{-3} + (2b + 5c) \sum_{\text{alt}} v_5 \otimes v_1 \otimes v_{-1}.$$

Therefore $a + 4b = 2b + 5c = 0$ so we take $a = 20$, $b = -5$, $c = 2$ to get the highest weight vector

$$z_3 = \sum_{\text{alt}} \left(20 v_5 \otimes v_3 \otimes v_{-5} - 5 v_5 \otimes v_1 \otimes v_{-3} + 2 v_3 \otimes v_1 \otimes v_{-1} \right).$$

Applying F repeatedly we obtain the other basis vectors for the $V(3)$ summand:

$$z_1 = \sum_{\text{alt}} \left(15 v_5 \otimes v_1 \otimes v_{-5} - 15 v_5 \otimes v_{-1} \otimes v_{-3} + 3 v_3 \otimes v_1 \otimes v_{-3} \right),$$
$$z_{-1} = \sum_{\text{alt}} \left(-15 v_5 \otimes v_{-1} \otimes v_{-5} + 15 v_3 \otimes v_1 \otimes v_{-5} - 3 v_3 \otimes v_{-1} \otimes v_{-3} \right),$$
$$z_{-3} = \sum_{\text{alt}} \left(-20 v_5 \otimes v_{-3} \otimes v_{-5} + 5 v_3 \otimes v_{-1} \otimes v_{-5} - 2 v_1 \otimes v_{-1} \otimes v_{-3} \right).$$

5.4 The change of basis matrix

We now consider two ordered bases of the exterior cube $\Lambda^3 V(5)$. The tensor basis consists of the 20 alternating sums listed in Table 1 in lexicographical order:

$$\sum_{\text{alt}} v_5 \otimes v_3 \otimes v_1, \quad \sum_{\text{alt}} v_5 \otimes v_3 \otimes v_{-1}, \quad \sum_{\text{alt}} v_5 \otimes v_3 \otimes v_{-3}, \quad \sum_{\text{alt}} v_5 \otimes v_3 \otimes v_{-5},$$
$$\sum_{\text{alt}} v_5 \otimes v_1 \otimes v_{-1}, \quad \sum_{\text{alt}} v_5 \otimes v_1 \otimes v_{-3}, \quad \sum_{\text{alt}} v_5 \otimes v_1 \otimes v_{-5}, \quad \sum_{\text{alt}} v_5 \otimes v_{-1} \otimes v_{-3},$$
$$\sum_{\text{alt}} v_5 \otimes v_{-1} \otimes v_{-5}, \quad \sum_{\text{alt}} v_5 \otimes v_{-3} \otimes v_{-5}, \quad \sum_{\text{alt}} v_3 \otimes v_1 \otimes v_{-1}, \quad \sum_{\text{alt}} v_3 \otimes v_1 \otimes v_{-3},$$
$$\sum_{\text{alt}} v_3 \otimes v_1 \otimes v_{-5}, \quad \sum_{\text{alt}} v_3 \otimes v_{-1} \otimes v_{-3}, \quad \sum_{\text{alt}} v_3 \otimes v_{-1} \otimes v_{-5}, \quad \sum_{\text{alt}} v_3 \otimes v_{-3} \otimes v_{-5},$$
$$\sum_{\text{alt}} v_1 \otimes v_{-1} \otimes v_{-3}, \quad \sum_{\text{alt}} v_1 \otimes v_{-1} \otimes v_{-5}, \quad \sum_{\text{alt}} v_1 \otimes v_{-3} \otimes v_{-5}, \quad \sum_{\text{alt}} v_{-1} \otimes v_{-3} \otimes v_{-5}.$$

The module basis consists of the weight vectors in the summands computed above, in the order

$$x_9, x_7, x_5, x_3, x_1, x_{-1}, x_{-3}, x_{-5}, x_{-7}, x_{-9}, y_5, y_3, y_1, y_{-1}, y_{-3}, y_{-5}, z_3, z_1, z_{-1}, z_{-3}.$$

We now create a 20×20 matrix A in which the rows are labelled by the tensor basis sums and the columns are labelled by the module basis vectors; the ij-entry is the coefficient of the i-th tensor basis sum in the j-th module basis vector. The matrix A is displayed in Table 2.

$$
\left(\begin{array}{cccccccccc|cccccc|cccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&6&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0\\
0&0&0&10&0&0&0&0&0&0&0&10&0&0&0&0&20&0&0&0\\
0&0&3&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&8&0&0&0&0&0&0&0&0&0&0&0&0&-5&0&0&0\\
0&0&0&0&15&0&0&0&0&0&0&0&10&0&0&0&0&15&0&0\\
0&0&0&0&6&0&0&0&0&0&0&0&0&0&0&0&0&-15&0&0\\
0&0&0&0&0&15&0&0&0&0&0&0&0&10&0&0&0&0&-15&0\\
0&0&0&0&0&0&10&0&0&0&0&0&0&0&10&0&0&0&0&-20\\
0&0&0&1&0&0&0&0&0&0&0&-1&0&0&0&0&2&0&0&0\\
0&0&0&0&3&0&0&0&0&0&0&0&-2&0&0&0&0&3&0&0\\
0&0&0&0&0&6&0&0&0&0&0&0&0&0&0&0&0&0&15&0\\
0&0&0&0&0&3&0&0&0&0&0&0&0&-2&0&0&0&0&-3&0\\
0&0&0&0&0&0&8&0&0&0&0&0&0&0&0&0&0&0&0&5\\
0&0&0&0&0&0&0&6&0&0&0&0&0&0&0&2&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&-1&0&0&0&0&-2\\
0&0&0&0&0&0&0&3&0&0&0&0&0&0&0&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&3&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0
\end{array}\right)
$$

Table 2: Change of basis matrix for $\Lambda^3 V(5)$

5.5 The structure constants for the alternating triple system

The columns of the inverse matrix A^{-1} give the coefficients of the tensor basis sums in terms of the module basis vectors. From this we can read off the matrices representing the projection maps from $\Lambda^3 V(5)$ onto its simple submodules. Rows 11–16 of A^{-1} give the structure constants for the alternating triple system obtained from the projection $p\colon \Lambda^3 V(5) \to V(5)$. These rows are displayed in Table 3.

Let h be the isomorphism from the original copy of $V(5)$ (with basis vectors v_p) to the $V(5)$ summand (with basis vectors y_p). Then $h^{-1} \circ p$ is the alternating ternary product on $V(5)$. If we write

$$[v_p, v_q, v_r] = (h^{-1} \circ p) \left(\sum_{\text{alt}} v_p \otimes v_q \otimes v_r \right),$$

$$
\frac{1}{20}\begin{pmatrix}
0 & 0 & 5 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & -10 & 0 & 0
\end{pmatrix}
$$

Table 3: Structure constants for $V(5)$ triple system

then the structure constants for the $sl(2)$-invariant alternating triple system obtained from $V(5)$ are as follows; here we have scaled the basis vectors to clear the denominators:

$$
\begin{array}{lll}
[v_5, v_3, v_1] = 0, & [v_5, v_3, v_{-1}] = 0, & [v_5, v_3, v_{-3}] = 5v_5, \\
[v_5, v_3, v_{-5}] = v_3, & [v_5, v_1, v_{-1}] = -10v_5, & [v_5, v_1, v_{-3}] = 0, \\
[v_5, v_1, v_{-5}] = v_1, & [v_5, v_{-1}, v_{-3}] = 0, & [v_5, v_{-1}, v_{-5}] = v_{-1}, \\
[v_5, v_{-3}, v_{-5}] = v_{-3}, & [v_3, v_1, v_{-1}] = -10v_3, & [v_3, v_1, v_{-3}] = -5v_1, \\
[v_3, v_1, v_{-5}] = 0, & [v_3, v_{-1}, v_{-3}] = -5v_{-1}, & [v_3, v_{-1}, v_{-5}] = 0, \\
[v_3, v_{-3}, v_{-5}] = 5v_{-5}, & [v_1, v_{-1}, v_{-3}] = -10v_{-3}, & [v_1, v_{-1}, v_{-5}] = -10v_{-5}, \\
[v_1, v_{-3}, v_{-5}] = 0, & [v_{-1}, v_{-3}, v_{-5}] = 0. &
\end{array}
$$

All other triple products follow from the alternating property.

5.6 Identities of degree 5

Computations with $p = 101$ show that this triple system satisfies no identity in degree 5. By the discussion in Bremner and Hentzel [5] this implies that it also satisfies no identities of degree 5 in characteristic 0.

5.7 Identities of degree 7

Theorem 4. *The space of identities in degree 7 for the alternating ternary structure on $V(5)$ has this decomposition into irreducible representations of S_7:*

$$
[322] \oplus 2[3211] \oplus [31111] \oplus 2[2221] \oplus 3[22111] \oplus 2[211111] \oplus 2[1111111].
$$

This space of identities is generated by the 14-term identity

$$
\begin{aligned}
&([[[aef]bc]dg] - [[[abc]dg]ef]) - ([[[aef]bd]cg] - [[[abd]cg]ef]) \\
&- ([[[aef]bg]cd] - [[[abg]cd]ef]) + ([[[aef]cd]bg] - [[[acd]bg]ef]) \\
&+ ([[[aef]cg]bd] - [[[acg]bd]ef]) - ([[[aef]dg]bc] - [[[adg]bc]ef]) \\
&+ 2[[[bcd]ef]ag] + 2[[bcd][efg]a],
\end{aligned}
$$

together with the identity

$$\sum_{\text{alt}(7)} [[[a_1a_2a_3]a_4a_5]a_6a_7].$$

The first identity has exactly one term in the second association type. It shows that any term in the second association type can be written as a linear combination of terms in the first association type.

Computations with $p = 101$ show that this triple system satisfies 190 linearly independent identities in degree 7. These computations were repeated in characteristic 0 and confirmed the existence of 190 linearly independent identities. One of these identities generates a 189-dimensional S_7-submodule of identities.

The first identity in the last Theorem implies the identity

$$\sum_{\text{alt}(7)} [[[a_1a_2a_3]a_4a_5]a_6a_7] - \sum_{\text{alt}(7)} [[a_1a_2a_3][a_4a_5a_6]a_7].$$

This is the difference of the alternating sums over the two association types in degree 7. The second identity is the alternating sum over the first association type. Thus the triple system $V(5)$ satisfies both alternating sums over the two association types in degree 7.

The first identity is the same as the generating identity for the degree 7 identities of the alternating triple system $V(7)$. (For details of the method by which this identity was discovered, see Section 7.3.) It is remarkable that the two alternating triple systems $V(5)$ and $V(7)$ satisfy almost the same identities in degree 7. They both satisfy all the identities in the 189-dimensional space generated by the first identity in Theorem 4. In addition, the system $V(5)$ satisfies the second identity in Theorem 4.

6 Representation $V(6)$: dimension 7

The decomposition of the exterior cube is

$$\Lambda^3 V(6) \cong V(12) \oplus V(8) \oplus V(6) \oplus V(4) \oplus V(0).$$

Before stating the highest weight vectors we introduce the notation

$$[p, q, r] = \sum_{\text{alt}} v_p \otimes v_q \otimes v_r.$$

With this convention, the highest weight vectors for the summands on the right side of the isomorphism are

$$\begin{aligned}
u_{12} &= [6, 4, 2], \qquad w_8 = 5\,[6, 4, -2] - 3\,[6, 2, 0], \\
x_6 &= 5\,[6, 4, -4] - 2\,[6, 2, -2] + [4, 2, 0], \\
y_4 &= 10\,[6, 4, -6] - 2\,[6, 2, -4] + [6, 0, -2], \\
z_0 &= 30\,[6, 0, -6] - 20\,[6, -2, -4] - 20\,[4, 2, -6] + 5\,[4, 0, -4] - 2\,[2, 0, -2].
\end{aligned}$$

6.1 Structure constants

The structure constants for the alternating triple system on $V(6)$ are presented in the following list. Here we use the compact notation $[p,q,r]=c$ to abbreviate the equation $[v_p,v_q,v_r]=cv_{p+q+r}$.

$[6,4,2]=0,$	$[6,4,0]=0,$	$[6,4,-2]=0,$	$[6,4,-4]=6,$
$[6,4,-6]=1,$	$[6,2,0]=0,$	$[6,2,-2]=-15,$	$[6,2,-4]=0,$
$[6,2,-6]=1,$	$[6,0,-2]=-10,$	$[6,0,-4]=-4,$	$[6,0,-6]=0,$
$[6,-2,-4]=-3,$	$[6,-2,-6]=-1,$	$[6,-4,-6]=-1,$	$[4,2,0]=60,$
$[4,2,-2]=15,$	$[4,2,-4]=6,$	$[4,2,-6]=3,$	$[4,0,-2]=0,$
$[4,0,-4]=0,$	$[4,0,-6]=4,$	$[4,-2,-4]=-6,$	$[4,-2,-6]=0,$
$[4,-4,-6]=-6,$	$[2,0,-2]=0,$	$[2,0,-4]=0,$	$[2,0,-6]=10,$
$[2,-2,-4]=-15,$	$[2,-2,-6]=15,$	$[2,-4,-6]=0,$	$[0,-2,-4]=-60,$
$[0,-2,-6]=0,$	$[0,-4,-6]=0,$	$[-2,-4,-6]=0.$	

6.2 Identities of degrees 5 and 7

The anticommutative binary structure on $V(6)$ obtained from the projection $\Lambda^2 V(6) \to V(6)$ produces an algebra isomorphic to the 7-dimensional simple non-Lie Malcev algebra; see Bremner and Hentzel [5]. Since this is not a Lie algebra, the Jacobian of the binary Malcev product gives a non-trivial alternating ternary product on $V(6)$. Since $V(6)$ occurs only once as a summand of $\Lambda^3 V(6)$, the alternating ternary product on $V(6)$ obtained from the projection $\Lambda^3 V(6) \to V(6)$ must be equal (up to a nonzero scalar multiple) to the Jacobian of the Malcev product.

In any alternative algebra (over a field of characteristic not 2 or 3) the associator is a multiple of the Jacobian. Therefore the Jacobian on the 7-dimensional non-Lie Malcev algebra equals (up to a scalar multiple which does not affect the identities) the associator on the 7-dimensional subspace of pure imaginary Cayley numbers.

The alternating ternary structure on $V(6)$ is therefore isomorphic to that obtained from the associator on the Cayley numbers (after factoring out the one-dimensional ideal of scalars).

The identities of degree 7 for the associator on the Cayley numbers have been described in Bremner and Hentzel [3]. See especially Theorem 1 (page 262) and Theorem 2 (page 267). There are no identities in degree 5, and seven identities in degree 7.

In that paper the results are stated for a field of characteristic $p = 103$ since we used arithmetic modulo 103 for the computations. However, all the coefficients of the identities can be regarded as integers which are small in absolute value. In this way the identities are meaningful over any field. As long as the characteristic is greater than 7 (the degree of the identities in question) the results will remain valid.

7 Representation $V(7)$: dimension 8

The decomposition of the exterior cube is

$$\Lambda^3 V(7) \cong V(15) \oplus V(11) \oplus V(9) \oplus V(7) \oplus V(5) \oplus V(3).$$

Highest weight vectors for the summands on the right side are

$$\begin{aligned}
t_{15} &= [7,5,3], \qquad u_{11} = 3\,[7,5,-1] - 2\,[7,3,1], \\
w_9 &= 14\,[7,5,-3] - 7\,[7,3,-1] + 4\,[5,3,1], \\
x_7 &= 15\,[7,5,-5] - 5\,[7,3,-3] + 3\,[7,1,-1], \\
y_5 &= 210\,[7,5,-7] - 35\,[7,3,-5] + 7\,[7,1,-3] + 5\,[5,3,-3] - 3\,[5,1,-1], \\
z_3 &= 7\,[7,1,-5] - 7\,[7,-1,-3] - 5\,[5,3,-5] + 2\,[5,1,-3] - [3,1,-1].
\end{aligned}$$

7.1 Structure constants

The structure constants for the alternating triple system on $V(7)$ are displayed in the following list (again using the compact notation introduced in the previous section).

$[7,5,3]=0,$	$[7,5,1]=0,$	$[7,5,-1]=0,$	$[7,5,-3]=0,$
$[7,5,-5]=7,$	$[7,5,-7]=1,$	$[7,3,1]=0,$	$[7,3,-1]=0,$
$[7,3,-3]=-21,$	$[7,3,-5]=0,$	$[7,3,-7]=1,$	$[7,1,-1]=35,$
$[7,1,-3]=0,$	$[7,1,-5]=0,$	$[7,1,-7]=1,$	$[7,-1,-3]=0,$
$[7,-1,-5]=0,$	$[7,-1,-7]=1,$	$[7,-3,-5]=0,$	$[7,-3,-7]=1,$
$[7,-5,-7]=1,$	$[5,3,1]=0,$	$[5,3,-1]=0,$	$[5,3,-3]=-21,$
$[5,3,-5]=-7,$	$[5,3,-7]=0,$	$[5,1,-1]=35,$	$[5,1,-3]=0,$
$[5,1,-5]=-7,$	$[5,1,-7]=0,$	$[5,-1,-3]=0,$	$[5,-1,-5]=-7,$
$[5,-1,-7]=0,$	$[5,-3,-5]=-7,$	$[5,-3,-7]=0,$	$[5,-5,-7]=7,$
$[3,1,-1]=35,$	$[3,1,-3]=21,$	$[3,1,-5]=0,$	$[3,1,-7]=0,$
$[3,-1,-3]=21,$	$[3,-1,-5]=0,$	$[3,-1,-7]=0,$	$[3,-3,-5]=-21,$
$[3,-3,-7]=-21,$	$[3,-5,-7]=0,$	$[1,-1,-3]=35,$	$[1,-1,-5]=35,$
$[1,-1,-7]=35,$	$[1,-3,-5]=0,$	$[1,-3,-7]=0,$	$[1,-5,-7]=0,$
$[-1,-3,-5]=0,$	$[-1,-3,-7]=0,$	$[-1,-5,-7]=0,$	$[-3,-5,-7]=0.$

7.2 Identities of degree 5

Computations with $p = 101$ show that this triple system satisfies no identity in degree 5; this implies that it also satisfies no identities of degree 5 in characteristic 0.

7.3 Identities of degree 7

Computations with $p = 101$ show that this triple system satisfies 189 linearly independent identities in degree 7. The simplest of these identities has 14 terms. All of its nonzero coefficients are in the set $\{1, 50, 51, 100\}$. These residue classes modulo 101 correspond to the rational numbers 1, $-1/2$, $1/2$, -1. We can therefore regard this as an identity with rational coefficients. We repeated our computations in characteristic 0 and confirmed that this triple system satisfies 189 linearly independent identities. The 14-term identity of Theorem 4 generates all 189 of the identities in characteristic 0.

Theorem 5. *The space of identities in degree 7 for the alternating ternary structure on $V(7)$ has this decomposition into irreducible representations of S_7:*

$$[322] \oplus 2[3211] \oplus [31111] \oplus 2[2221] \oplus 3[22111] \oplus 2[211111] \oplus [1111111].$$

This space of identities is generated by the single 14-term identity which was displayed in Theorem 4.

Proof. To show that this single identity generates the entire 189-dimensional space of identities we proceed as follows. We create a matrix of size 400×280. We generate all 5040 permutations of seven letters and divide them into 42 groups of 120 permutations. For each of the 42 groups we apply each of the 120 permutations to the given identity. This gives us 120 new identities which we place in the bottom of the matrix (in rows 281 through 400). We then compute the row canonical form of the matrix. At this point the last 120 rows are zero. We then repeat this "fill and reduce" process with the next group of 120 permutations. After we have processed all 42 groups the nonzero rows of the row canonical form give us a basis of the S_7-submodule generated by the original identity. When the last group had been processed the rank of the matrix was 189. This shows that the original identity generates the entire space of identities in degree 7 for the triple system $V(7)$. □

The multiplicities of the irreducible representations in this case are almost the same as for $V(5)$. The only difference is that here the last representation occurs only once.

8 Representation $V(8)$: dimension 9

The decomposition of the exterior cube is

$$\Lambda^3 V(8) \cong V(18) \oplus V(14) \oplus V(12) \oplus V(10) \oplus V(8) \oplus 2V(6) \oplus V(2).$$

Highest weight vectors for the summands on the right side are

$s_{18} = [8, 6, 4], \qquad t_{14} = 7\,[8, 6, 0] - 5\,[8, 4, 2],$
$u_{12} = 14\,[8, 6, -2] - 8\,[8, 4, 0] + 5\,[6, 4, 2],$

$$
\begin{aligned}
w_{10} &= 7\,[8,6,-4] - 3\,[8,4,-2] + 2\,[8,2,0],\\
x_8 &= 56\,[8,6,-6] - 16\,[8,4,-4] + 4\,[8,2,-2] + 3\,[6,4,-2] - 2\,[6,2,0],\\
y_6 &= 105\,[8,6,-8] - 15\,[8,4,-6] + 5\,[8,2,-4] - 3\,[8,0,-2],\\
y_6' &= 2352\,[8,6,-8] - 336\,[8,4,-6] + 56\,[8,2,-4] + 42\,[6,4,-4] - 21\,[6,2,-2]\\
&\qquad + 12\,[4,2,0],\\
z_2 &= 280\,[8,2,-8] - 210\,[6,4,-8] - 112\,[8,0,-6] + 35\,[6,2,-6] + 70\,[8,-2,-4]\\
&\qquad - 7\,[6,0,-4] - 5\,[4,2,-4] + 3\,[4,0,-2].
\end{aligned}
$$

In this case the module $V(6)$ occurs with multiplicity 2, so we have two linearly independent highest weight vectors of weight 6.

8.1 Structure constants for the triple system

The structure constants for the alternating triple system on $V(8)$ are displayed in the following list:

$[8,6,4]=0$,	$[8,6,2]=0$,	$[8,6,0]=0$,	$[8,6,-2]=0$,
$[8,6,-4]=0$,	$[8,6,-6]=16$,	$[8,6,-8]=2$,	$[8,4,2]=0$,
$[8,4,0]=0$,	$[8,4,-2]=0$,	$[8,4,-4]=-56$,	$[8,4,-6]=0$,
$[8,4,-8]=2$,	$[8,2,0]=0$,	$[8,2,-2]=56$,	$[8,2,-4]=-21$,
$[8,2,-6]=-6$,	$[8,2,-8]=1$,	$[8,0,-2]=35$,	$[8,0,-4]=0$,
$[8,0,-6]=-5$,	$[8,0,-8]=0$,	$[8,-2,-4]=0$,	$[8,-2,-6]=-4$,
$[8,-2,-8]=-1$,	$[8,-4,-6]=-3$,	$[8,-4,-8]=-2$,	$[8,-6,-8]=-2$,
$[6,4,2]=0$,	$[6,4,0]=0$,	$[6,4,-2]=168$,	$[6,4,-4]=7$,
$[6,4,-6]=2$,	$[6,4,-8]=3$,	$[6,2,0]=-280$,	$[6,2,-2]=42$,
$[6,2,-4]=0$,	$[6,2,-6]=-6$,	$[6,2,-8]=4$,	$[6,0,-2]=70$,
$[6,0,-4]=35$,	$[6,0,-6]=0$,	$[6,0,-8]=5$,	$[6,-2,-4]=28$,
$[6,-2,-6]=6$,	$[6,-2,-8]=6$,	$[6,-4,-6]=-2$,	$[6,-4,-8]=0$,
$[6,-6,-8]=-16$,	$[4,2,0]=-245$,	$[4,2,-2]=-98$,	$[4,2,-4]=-49$,
$[4,2,-6]=-28$,	$[4,2,-8]=0$,	$[4,0,-2]=0$,	$[4,0,-4]=0$,
$[4,0,-6]=-35$,	$[4,0,-8]=0$,	$[4,-2,-4]=49$,	$[4,-2,-6]=0$,
$[4,-2,-8]=21$,	$[4,-4,-6]=-7$,	$[4,-4,-8]=56$,	$[4,-6,-8]=0$,
$[2,0,-2]=0$,	$[2,0,-4]=0$,	$[2,0,-6]=-70$,	$[2,0,-8]=-35$,
$[2,-2,-4]=98$,	$[2,-2,-6]=-42$,	$[2,-2,-8]=-56$,	$[2,-4,-6]=-168$,
$[2,-4,-8]=0$,	$[2,-6,-8]=0$,	$[0,-2,-4]=245$,	$[0,-2,-6]=280$,
$[0,-2,-8]=0$,	$[0,-4,-6]=0$,	$[0,-4,-8]=0$,	$[0,-6,-8]=0$,
$[-2,-4,-6]=0$,	$[-2,-4,-8]=0$,	$[-2,-6,-8]=0$,	$[-4,-6,-8]=0$.

8.2 Identities of degree 5

Computations with $p = 101$ show that this triple system satisfies no identity in degree 5; this implies that it also satisfies no identities of degree 5 in characteristic 0.

8.3 Identities of degree 7

Theorem 6. *In characteristic* 0 *the alternating triple system on* $V(8)$ *satisfies eight linearly independent multilinear identities. These eight identities span a representation of the symmetric group* S_7 *which decomposes into the direct sum of irreducible components*

$$[211111] \oplus 2[1111111].$$

This representation is generated by these two identities:

$$\begin{aligned} I_1 &= 6 \sum_{\mathrm{alt}(6)} [[[a_1 a_2 a_3] a_4 b] a_5 a_6] - \sum_{\mathrm{alt}(6)} [[a_1 a_2 a_3][a_4 a_5 a_6] b], \\ I_2 &= \sum_{\mathrm{alt}(7)} [[[a_1 a_2 a_3] a_4 a_5] a_6 a_7]. \end{aligned}$$

The first identity along with the second identity implies that the alternating sum over the second association type is also an identity.

The computational methods used in the proof of this result are very similar to those discussed elsewhere in this paper, so we omit the details.

9 Representation $V(10)$: dimension 11

The decomposition of the exterior cube is

$$\begin{aligned} \Lambda^3 V(10) \cong\; & V(24) \oplus V(20) \oplus V(18) \oplus V(16) \oplus V(14) \oplus 2V(12) \\ & \oplus V(10) \oplus 2V(8) \oplus V(6) \oplus V(4) \oplus V(0). \end{aligned}$$

Highest weight vectors for the summands on the right side are

$$\begin{aligned} o_{24} &= [10,8,6], \qquad p_{20} = 9\,[10,8,2] - 7\,[10,6,4], \\ q_{18} &= 15\,[10,8,0] - 10\,[10,6,2] + 7\,[8,6,4], \\ r_{16} &= 36\,[10,8,-2] - 20\,[10,6,0] + 15\,[10,4,2], \\ s_{14} &= 36\,[10,8,-4] - 16\,[10,6,-2] + 5\,[10,4,0] + 4\,[8,6,0] - 3\,[8,4,2], \\ t_{12} &= 42\,[10,8,-6] - 14\,[10,6,-4] + 7\,[10,4,-2] - 5\,[10,2,0], \\ t'_{12} &= 360\,[10,8,-6] - 120\,[10,6,-4] + 30\,[10,4,-2] + 24\,[8,6,-2] - 15\,[8,4,0] \\ &\quad + 10\,[6,4,2], \\ u_{10} &= 630\,[10,8,-8] - 140\,[10,6,-6] + 35\,[10,4,-4] + 14\,[8,6,-4] \\ &\quad - 10\,[10,2,-2] - 7\,[8,4,-2] + 5\,[8,2,0], \end{aligned}$$

$$
\begin{aligned}
w_8 = {} & 252\,[10,8,-10] - 28\,[10,6,-8] + 7\,[10,4,-6] - 3\,[10,2,-4] \\
& + 2\,[10,0,-2], \\
w_8' = {} & 18900\,[10,8,-10] - 2100\,[10,6,-8] + 315\,[10,4,-6] + 168\,[8,6,-6] \\
& - 45\,[10,2,-4] - 63\,[8,4,-4] + 18\,[8,2,-2] + 14\,[6,4,-2] - 10\,[6,2,0], \\
x_6 = {} & 315\,[10,4,-8] - 252\,[8,6,-8] - 180\,[10,2,-6] + 63\,[8,4,-6] \\
& + 75\,[10,0,-4] + 9\,[8,2,-4] - 28\,[6,4,-4] - 30\,[8,0,-2] + 16\,[6,2,-2] \\
& - 10\,[4,2,0], \\
y_4 = {} & 1890\,[10,4,-10] - 1512\,[8,6,-10] - 540\,[10,2,-8] + 189\,[8,4,-8] \\
& + 240\,[10,0,-6] - 36\,[8,2,-6] - 14\,[6,4,-6] - 162\,[10,-2,-4] \\
& + 9\,[8,0,-4] + 6\,[6,2,-4] - 4\,[6,0,-2], \\
z_0 = {} & 1890\,[10,0,-10] - 1134\,[8,2,-10] + 882\,[6,4,-10] - 1134\,[10,-2,-8] \\
& + 378\,[8,0,-8] - 126\,[6,2,-8] + 882\,[10,-4,-6] - 126\,[8,-2,-6] \\
& - 14\,[6,0,-6] + 42\,[4,2,-6] + 42\,[6,-2,-4] - 21\,[4,0,-4] \\
& + 12\,[2,0,-2].
\end{aligned}
$$

In this case the modules $V(12)$ and $V(8)$ occur with multiplicity 2, so for weights 12 and 8 we have two linearly independent highest weight vectors.

The anticommutative binary structure on $V(10)$ obtained from the projection $\Lambda^2 V(10) \to V(10)$ has been studied in detail in Bremner and Hentzel [5]. Since this is not a Lie algebra, the Jacobian of this binary product gives a non-trivial alternating ternary product on $V(10)$. Since $V(10)$ occurs only once as a summand of $\Lambda^3 V(10)$, the alternating ternary product on $V(10)$ obtained from the projection $\Lambda^3 V(10) \to V(10)$ must be equal (up to a nonzero scalar multiple) to the Jacobian of the binary product.

9.1 Structure constants for the triple system

The structure constants for the alternating triple system on $V(10)$ are displayed in the following list:

$[10,8,6]=0$,	$[10,8,4]=0$,	$[10,8,2]=0$,
$[10,8,0]=0$,	$[10,8,-2]=0$,	$[10,8,-4]=0$,
$[10,8,-6]=0$,	$[10,8,-8]=30$,	$[10,8,-10]=3$,
$[10,6,4]=0$,	$[10,6,2]=0$,	$[10,6,0]=0$,
$[10,6,-2]=0$,	$[10,6,-4]=0$,	$[10,6,-6]=-135$,
$[10,6,-8]=0$,	$[10,6,-10]=3$,	$[10,4,2]=0$,
$[10,4,0]=0$,	$[10,4,-2]=0$,	$[10,4,-4]=240$,
$[10,4,-6]=-36$,	$[10,4,-8]=-8$,	$[10,4,-10]=2$,
$[10,2,0]=0$,	$[10,2,-2]=-210$,	$[10,2,-4]=84$,
$[10,2,-6]=0$,	$[10,2,-8]=-7$,	$[10,2,-10]=1$,
$[10,0,-2]=-126$,	$[10,0,-4]=0$,	$[10,0,-6]=0$,

$[10,0,-8]=-6,$ $[10,0,-10]=0,$ $[10,-2,-4]=0,$
$[10,-2,-6]=0,$ $[10,-2,-8]=-5,$ $[10,-2,-10]=-1,$
$[10,-4,-6]=0,$ $[10,-4,-8]=-4,$ $[10,-4,-10]=-2,$
$[10,-6,-8]=-3,$ $[10,-6,-10]=-3,$ $[10,-8,-10]=-3,$
$[8,6,4]=0,$ $[8,6,2]=0,$ $[8,6,0]=0,$
$[8,6,-2]=0,$ $[8,6,-4]=360,$ $[8,6,-6]=-27,$
$[8,6,-8]=-6,$ $[8,6,-10]=3,$ $[8,4,2]=0,$
$[8,4,0]=0,$ $[8,4,-2]=-840,$ $[8,4,-4]=192,$
$[8,4,-6]=0,$ $[8,4,-8]=-16,$ $[8,4,-10]=4,$
$[8,2,0]=1260,$ $[8,2,-2]=-168,$ $[8,2,-4]=168,$
$[8,2,-6]=63,$ $[8,2,-8]=-8,$ $[8,2,-10]=5,$
$[8,0,-2]=-252,$ $[8,0,-4]=0,$ $[8,0,-6]=54,$
$[8,0,-8]=0,$ $[8,0,-10]=6,$ $[8,-2,-4]=0,$
$[8,-2,-6]=45,$ $[8,-2,-8]=8,$ $[8,-2,-10]=7,$
$[8,-4,-6]=36,$ $[8,-4,-8]=16,$ $[8,-4,-10]=8,$
$[8,-6,-8]=6,$ $[8,-6,-10]=0,$ $[8,-8,-10]=-30,$
$[6,4,2]=0,$ $[6,4,0]=0,$ $[6,4,-2]=-756,$
$[6,4,-4]=-144,$ $[6,4,-6]=-54,$ $[6,4,-8]=-36,$
$[6,4,-10]=0,$ $[6,2,0]=1134,$ $[6,2,-2]=-126,$
$[6,2,-4]=0,$ $[6,2,-6]=27,$ $[6,2,-8]=-45,$
$[6,2,-10]=0,$ $[6,0,-2]=-378,$ $[6,0,-4]=-216,$
$[6,0,-6]=0,$ $[6,0,-8]=-54,$ $[6,0,-10]=0,$
$[6,-2,-4]=-180,$ $[6,-2,-6]=-27,$ $[6,-2,-8]=-63,$
$[6,-2,-10]=0,$ $[6,-4,-6]=54,$ $[6,-4,-8]=0,$
$[6,-4,-10]=36,$ $[6,-6,-8]=27,$ $[6,-6,-10]=135,$
$[6,-8,-10]=0,$ $[4,2,0]=1008,$ $[4,2,-2]=504,$
$[4,2,-4]=288,$ $[4,2,-6]=180,$ $[4,2,-8]=0,$
$[4,2,-10]=0,$ $[4,0,-2]=0,$ $[4,0,-4]=0,$
$[4,0,-6]=216,$ $[4,0,-8]=0,$ $[4,0,-10]=0,$
$[4,-2,-4]=-288,$ $[4,-2,-6]=0,$ $[4,-2,-8]=-168,$
$[4,-2,-10]=-84,$ $[4,-4,-6]=144,$ $[4,-4,-8]=-192,$
$[4,-4,-10]=-240,$ $[4,-6,-8]=-360,$ $[4,-6,-10]=0,$
$[4,-8,-10]=0,$ $[2,0,-2]=0,$ $[2,0,-4]=0,$
$[2,0,-6]=378,$ $[2,0,-8]=252,$ $[2,0,-10]=126,$
$[2,-2,-4]=-504,$ $[2,-2,-6]=126,$ $[2,-2,-8]=168,$
$[2,-2,-10]=210,$ $[2,-4,-6]=756,$ $[2,-4,-8]=840,$
$[2,-4,-10]=0,$ $[2,-6,-8]=0,$ $[2,-6,-10]=0,$

$[2,-8,-10]=0,$	$[0,-2,-4]=-1008,$	$[0,-2,-6]=-1134,$
$[0,-2,-8]=-1260,$	$[0,-2,-10]=0,$	$[0,-4,-6]=0,$
$[0,-4,-8]=0,$	$[0,-4,-10]=0,$	$[0,-6,-8]=0,$
$[0,-6,-10]=0,$	$[0,-8,-10]=0,$	$[-2,-4,-6]=0,$
$[-2,-4,-8]=0,$	$[-2,-4,-10]=0,$	$[-2,-6,-8]=0,$
$[-2,-6,-10]=0,$	$[-2,-8,-10]=0,$	$[-4,-6,-8]=0,$
$[-4,-6,-10]=0,$	$[-4,-8,-10]=0,$	$[-6,-8,-10]=0.$

9.2 Identities of degree 5

Computations with $p = 101$ show that this triple system satisfies no identity in degree 5; this implies that it also satisfies no identities of degree 5 in characteristic 0.

9.3 Identities of degree 7

Computations with $p = 101$ show that this triple system satisfies no identity in degree 7; this implies that it also satisfies no identities of degree 7 in characteristic 0.

9.4 Open problem

Determine the lowest degree for which the alternating ternary structure on $V(10)$ has non-trivial identities. In particular, are there any identities of degree 9?

10 Other simple Lie algebras

10.1 Tensor products

Let L be a semisimple (finite dimensional) Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. Let U, V, W be three finite dimensional representations of L. By Weyl's Theorem we know that any finite dimensional representation of L is completely reducible; that is, it decomposes as the direct sum of irreducible representations. In particular, this holds for the tensor product $U \otimes V \otimes W$. In the special case $U = V = W$ we are interested in the multiplicity of V as a direct summand of its own tensor cube. If this multiplicity is nonzero then there exists a nonzero L-module homomorphism $p\colon V^{\otimes 3} \to V$ which gives V the structure of a triple system, and this structure is L-invariant in the sense that L is contained in the derivation algebra. This structure is alternating when V occurs as a summand of the exterior cube $\Lambda^3(V)$. The simple Lie algebras are characterized by their Dynkin diagrams. We follow the conventions of Humphreys [8] for the labelling of these diagrams.

B_2	10	Ω_1	5	0	Ω_2	4	1				
B_3	21	Ω_1	7	0	Ω_2	21	0	Ω_3	8	1
B_4	36	Ω_1	9	0	Ω_2	36	0	Ω_3	84	1
		Ω_4	16	0						
B_5	55	Ω_1	11	0	Ω_2	55	0	Ω_3	165	1
		Ω_4	330	1	Ω_5	32	1			
B_6	78	Ω_1	13	0	Ω_2	78	0	Ω_3	286	1
		Ω_4	715	1	Ω_5	1287	2	Ω_6	64	2
B_7	105	Ω_1	15	0	Ω_2	105	0	Ω_3	455	1
		Ω_4	1365	1	Ω_5	3003	2	Ω_6	5005	3
		Ω_7	128	1						
B_8	136	Ω_1	17	0	Ω_2	136	0	Ω_3	680	1
		Ω_4	2380	1	Ω_5	6188	2	Ω_6	12376	3
		Ω_7	19448	4	Ω_8	256	1			

Table 4: Alternating ternary structures in Type B

C_3	21	Ω_1	6	1	Ω_2	14	0	Ω_3	14	1
C_4	36	Ω_1	8	1	Ω_2	27	0	Ω_3	48	2
		Ω_4	42	0						
C_5	55	Ω_1	10	1	Ω_2	44	0	Ω_3	110	1
		Ω_4	165	1	Ω_5	132	1			
C_6	78	Ω_1	12	1	Ω_2	65	0	Ω_3	208	3
		Ω_4	429	1	Ω_5	572	2	Ω_6	429	1
C_7	105	Ω_1	14	1	Ω_2	90	0	Ω_3	350	3
		Ω_4	910	1	Ω_5	1638	3	Ω_6	2002	2
		Ω_7	1430	1						
C_8	136	Ω_1	16	1	Ω_2	119	0	Ω_3	544	3
		Ω_4	1700	1	Ω_5	3808	4	Ω_6	6188	2
		Ω_7	7072	3	Ω_8	4862	1			

Table 5: Alternating ternary structures in Type C

D_4	28	Ω_1	8	0	Ω_2	28	0			
D_5	45	Ω_1	10	0	Ω_2	45	0	Ω_3	120	1
		Ω_4	16	0						
D_6	66	Ω_1	12	0	Ω_2	66	0	Ω_3	220	2
		Ω_4	495	1	Ω_5	32	1			
D_7	91	Ω_1	14	0	Ω_2	91	0	Ω_3	364	1
		Ω_4	1001	2	Ω_5	2002	3	Ω_6	64	0
D_8	120	Ω_1	16	0	Ω_2	120	0	Ω_3	560	1
		Ω_4	1820	1	Ω_5	4368	4	Ω_6	8008	4
		Ω_7	128	0						

Table 6: Alternating ternary structures in Type D

E_6	78	Ω_1	27	0	Ω_2	78	0
		Ω_3	351	0	Ω_4	2925	10
		Ω_5	351	0	Ω_6	27	0
E_7	133	Ω_1	133	0	Ω_2	912	1
		Ω_3	8645	7	Ω_4	365750	209
		Ω_5	27664	23	Ω_6	1539	1
		Ω_7	56	1			
E_8	248	Ω_1	3875	1	Ω_2	147250	9
		Ω_3	6696000	214	Ω_4	6899079264	173159
		Ω_5	146325270	4087	Ω_6	2450240	128
		Ω_7	30380	6	Ω_8	248	0
F_4	52	Ω_1	52	0	Ω_2	1274	7
		Ω_3	273	4	Ω_4	26	0
G_2	14	Ω_1	7	1	Ω_2	14	0

Table 7: Alternating ternary structures in Types E, F, G

10.2 Exterior cubes

A simple Lie algebra of rank ℓ has ℓ fundamental representations, which we will denote by Ω_i for $1 \leq i \leq \ell$. In this section we list, for each simple Lie algebra of rank $2 \leq \ell \leq 8$ and each fundamental representation, the multiplicity

$$\dim \mathrm{Hom}_L(\Lambda^3\Omega_i, \Omega_i)$$

of Ω_i as a direct summand of its exterior cube. This multiplicity is (one more than) the number of (projective) parameters that occur in the classification of the L-invariant alternating ternary structures on Ω_i. To perform these calculations we used the computer algebra package `LiE` [6].

10.3 Special linear: Type A

The only fundamental representation of a simple Lie algebra of type A_ℓ (with $1 \leq \ell \leq 8$) which occurs as a direct summand of its own exterior cube is the 20-dimensional representation Ω_3 of the Lie algebra A_5. The multiplicity is 1, and so Ω_3 admits an A_5-invariant alternating ternary operation which is unique up to a scalar multiple.

10.4 Orthogonal: Type B

The results are displayed in Table 4. In this and the following tables the results are presented in this format: the name of the Lie algebra is followed by its dimension; and the name of each fundamental representation is followed by its dimension and its multiplicity in its own exterior cube.

10.5 Symplectic: Type C

The results are displayed in Table 5.

10.6 Orthogonal: Type D

By the symmetry of the Dynkin diagram the dimensions for $\Omega_{\ell-1}$ and Ω_ℓ are the same, so we omit the latter. By the triality symmetry of the Dynkin diagram of D_4 the dimensions of Ω_1, Ω_3 and Ω_4 are the same so we omit the latter two. The results are displayed in Table 6.

10.7 Exceptional: Types E, F, G

The results are displayed in Table 7. The computation of the largest exterior cube in this list, $\Lambda^3\Omega_4$ for E_8, took 860 seconds; altogether 1436 distinct representations occur in the decomposition.

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