UNIVERSITY OF SASKATCHEWAN

MATHEMATICS 124.3 (02 & 04)- Solutions to Midterm #2

Time: 80 minutes March 25, 1999 *Instructors: D.W. MacLean & H.S. Yin*

NO BOOKS OR CALCULATORS ALLOWED.
One 8.5×11 inch formula sheet (both sides) is permitted.

Show all of your work. No credit will be given for unsubstantiated correct answers.

Each question is worth 10%.

Part I: Evaluate the definite integrals or show they are divergent:

(1) $\displaystyle\int_0^3 \frac{dx}{x^2-x-2}$

Solution: $\displaystyle\frac{1}{x^2-x-2} = \frac{1}{(x-2)(x+1)} = A\frac{1}{x-2} + B\frac{1}{x+1},$

so we know we will end up with the problem of determining the convergence of

$$\int_0^3 \frac{dx}{x-2} = \int_0^2 \frac{dx}{x-2} + \int_2^3 \frac{dx}{x-2} = \lim_{T\to 2^-}\int_0^T \frac{dx}{x-2} + \lim_{T\to 2^+}\int_T^3 \frac{dx}{x-2}$$

Considering just the first summand, we find

$$\lim_{T\to 2^-}\int_0^T \frac{dx}{x-2} = \lim_{T\to 2^-}\int_0^T \frac{dx}{x-2} = \lim_{T\to 2^-}(\ln(2-T)-\ln(2-0)) = \lim_{T\to 2^-}(\ln(2-T)-\ln 2) = -\infty,$$

so the definite integral does not converge .

Considerable partial marks weregiven for partial fraction work:
For all x we must have:

$1 = A(x+1)+B(x-2)$. Setting $x=2$, we get $A=\frac{1}{3}$, and setting $x=-1$, we get $B=-\frac{1}{3}$.

Therefore $\displaystyle\frac{1}{x^2-x-2} = \frac{1}{3}\frac{1}{x-2} - \frac{1}{3}\frac{1}{x+1}$, so

$$\int \frac{dx}{x^2-x-2} = \frac{1}{3}\int\frac{dx}{x-2} - \frac{1}{3}\int\frac{dx}{x+1} = \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C$$

Therefore we have $\displaystyle\int_0^3 \frac{dx}{x^2-x-2} = \frac{1}{3}\int_0^3\frac{dx}{x-2} - \frac{1}{3}\int_0^3\frac{dx}{x+1}$, **if these integrals converge!**

Since $y = \dfrac{1}{x-2}$ has a vertical asymptote at $x=2$, we have an improper integral of type II.
We must evaluate

$$\int_0^3 \frac{dx}{x-2} = \int_0^2 \frac{dx}{x-2} + \int_2^3 \frac{dx}{x-2} = \lim_{T\to 2^-}\int_0^T \frac{dx}{x-2} + \lim_{T\to 2^+}\int_T^3 \frac{dx}{x-2} =$$

$\lim_{T\to 2^-} \ln|x-2|\Big|_0^T + \lim_{T\to 2^+} \ln|x-2|\Big|_T^3 = \lim_{T\to 2^-} (\ln(2-T) - \ln(2-0)) + \lim_{T\to 2^+} (\ln(T-2) - \ln(3-2)) = \lim_{T\to 2^-} (\ln(2-T) - \ln 2) + \lim_{T\to 2^+} (\ln(T-2)) = -\infty + (-\infty)$

so the definite integral does not converge .

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(2) $\int_{-1}^{1} \frac{x+1}{\sqrt[3]{x^4}} dx$

Solution: Since the y-axis is a vertical asymptote, this is a type II improper integral.

$\int_{-1}^{1} \frac{x+1}{\sqrt[3]{x^4}} dx = \int_{-1}^{1} \left(\frac{x}{\sqrt[3]{x^4}} + \frac{1}{\sqrt[3]{x^4}}\right) dx = \int_{-1}^{1} \left(x^{-\frac{1}{3}} + x^{-\frac{4}{3}}\right) dx = \int_{-1}^{1} x^{-\frac{1}{3}} dx + \int_{-1}^{1} x^{-\frac{4}{3}} dx$

We have $\int_{-1}^{1} x^{-\frac{4}{3}} dx = \lim_{T\to 0-} \int_{-1}^{T} x^{-\frac{4}{3}} dx + \lim_{T\to 0+} \int_{T}^{1} x^{-\frac{4}{3}} dx = \lim_{T\to 0-} \frac{x^{-\frac{1}{3}}}{-\frac{1}{3}}\Bigg|_{-1}^{T} + \lim_{T\to 0+} \frac{x^{-\frac{1}{3}}}{-\frac{1}{3}}\Bigg|_{T}^{1} =$

$\lim_{T\to 0-} -3x^{-\frac{1}{3}}\Big|_{-1}^{T} + \lim_{T\to 0+} -3x^{-\frac{1}{3}}\Big|_{T}^{1} = \lim_{T\to 0-} -3T^{-\frac{1}{3}} - (-3) + \lim_{T\to 0+} -3 - \left(-3T^{-\frac{1}{3}}\right)$

so the definite integral diverges .

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(3) $\int_0^{\frac{\pi}{4}} \frac{\sin^3 x}{\cos x} dx$

Solution: $\int_0^{\frac{\pi}{4}} \frac{\sin^3 x}{\cos x} dx = \int_{x=0}^{x=\frac{\pi}{4}} \frac{\sin^2 x}{\cos x} \sin x dx = \int_{x=0}^{x=\frac{\pi}{4}} \frac{1-\cos^2 x}{\cos x} \sin x dx =$(letting $u = \cos x$)

$-\int_{u=1}^{u=\frac{\sqrt{2}}{2}} \frac{1-u^2}{u} du = \int_{u=1}^{u=\frac{\sqrt{2}}{2}} u - \frac{1}{u} du = \left(\frac{u^2}{2} - \ln u\right)\Big|_{u=1}^{u=\frac{\sqrt{2}}{2}} = \left(\frac{\left(\frac{\sqrt{2}}{2}\right)^2}{2} - \ln\frac{\sqrt{2}}{2}\right) - \left(\frac{1^2}{2} - \ln 1\right) =$

$\left(\frac{1}{4} + \frac{1}{2}\ln 2\right) - \frac{1}{2} = \frac{1}{2}\ln 2 - \frac{1}{4} \doteq 0.1$

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(4) $\int_1^2 x \ln x \, dx$

Solution: $\int_{x=1}^{x=2} x \ln x \, dx$=(using Integration by Parts, with $u = \ln x$ and $dv = xdx$)

$\int_{x=1}^{x=2} x \ln x \, dx = \int_{x=1}^{x=2} udv = uv|_{x=1}^{x=2} - \int_{x=1}^{x=2} vdu = \ln x \left(\frac{x^2}{2}\right)\Big|_{x=1}^{x=2} - \int_{x=1}^{x=2} \left(\frac{x^2}{2}\right)\frac{1}{x} dx =$

$\left(\frac{x^2}{2}\right)\ln x\Big|_{x=1}^{x=2} - \int_{x=1}^{x=2} \frac{x}{2} dx = \left(\frac{x^2}{2}\right)\ln x - \frac{x^2}{4}\Big|_{x=1}^{x=2} =$

$\left(\frac{2^2}{2}\ln 2 - \frac{2^2}{4}\right) - \left(\frac{1^2}{2}\ln 1 - \frac{1^2}{4}\right) = 2\ln 2 - 1 + \frac{1}{4} = 2\ln 2 - \frac{3}{4} \doteq 0.6$

Evaluate:

(5) $\int \frac{dx}{x^2\sqrt{4+x^2}}$

Solution: Let $x = 2\tan\theta$, so that $4 + x^2 = 4 + 4\tan^2\theta = 4\left(1+\tan^2\theta\right) = 4\sec^2\theta = (2\sec\theta)^2$ and $dx = 2\sec^2\theta d\theta$, and

$$\int \frac{dx}{x^2\sqrt{4+x^2}} = \int \frac{2\sec^2\theta d\theta}{(2\tan\theta)^2\sqrt{(2\sec\theta)^2}} = \int \frac{2\sec^2\theta d\theta}{4\tan^2\theta\,(2\sec\theta)} = \frac{1}{4}\int \frac{\sec\theta d\theta}{\tan^2\theta} =$$

$$\frac{1}{4}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}} d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta} d\theta = \text{(letting } u = \sin\theta\text{)} = \frac{1}{4}\int \frac{1}{u^2} du = \frac{1}{4}\int u^{-2} du = \frac{1}{4}\frac{u^{-1}}{-1} + C = -\frac{1}{4u} + C =$$

$$-\frac{1}{4\sin\theta} + C$$

We must express $\sin\theta$ in terms of x:

$\tan\theta = \frac{x}{2}$, $\cot\theta = \frac{2}{x}$, $\csc^2\theta = 1 + \cot^2\theta = 1 + \frac{4}{x^2} = \frac{x^2+4}{x^2}$, so $\csc\theta = \frac{\sqrt{x^2+4}}{x}$
and therefore $\sin\theta = \frac{1}{\csc\theta} = \frac{x}{\sqrt{x^2+4}}$

Thus we have

$$\int \frac{dx}{x^2\sqrt{4+x^2}} = -\frac{1}{4\sin\theta} + C = -\frac{1}{4\frac{x}{\sqrt{x^2+4}}} + C = -\frac{\sqrt{x^2+4}}{4x} + C$$

Part II:

(6) Find the length of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$, $1 \le x \le 2$

Solution: We have $y' = \frac{x^2}{2} - \frac{1}{2x^2}$, so

$$L = \int_1^2 \sqrt{1 + (y')^2}dx = \int_1^2 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}dx = \int_1^2 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}dx = \int_1^2 \frac{x^2}{2} + \frac{1}{2x^2}dx =$$

$$\frac{1}{2}\int_1^2 x^2 + x^{-2}dx = \frac{1}{2}\left(\frac{x^3}{3} + \frac{x^{-1}}{-1}\right)\Bigg|_1^2 = \frac{1}{2}\left(\frac{x^3}{3} - \frac{1}{x}\right)\Bigg|_1^2 = \frac{1}{2}\left[\left(\frac{2^3}{3} - \frac{1}{2}\right) - \left(\frac{1^3}{3} - \frac{1}{1}\right)\right] =$$

$$\frac{1}{2}\left[\left(\frac{8}{3} - \frac{1}{2}\right) - \left(\frac{1}{3} - \frac{1}{1}\right)\right] = \frac{1}{2}\left[\left(\frac{7}{3} + \frac{1}{2}\right)\right] = \frac{1}{2}\left[\left(\frac{14}{6} + \frac{3}{6}\right)\right] = \frac{17}{12}$$

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(7) Find the area obtained by rotating the curve $y = \frac{x^3}{6} + \frac{1}{2x}$, $1 \le x \le 2$ about the x-axis.

Solution: We have $y' = \frac{x^2}{2} - \frac{1}{2x^2}$, so

$$S = \int_1^2 2\pi y\sqrt{1 + (y')^2}dx = 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}dx =$$

$$2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}dx = 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)dx =$$

$$\frac{\pi}{2}\int_1^2 \left(\frac{x^3}{3} + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)dx = \frac{\pi}{2}\int_1^2 \frac{x^5}{3} + 4\frac{x}{3} + x^{-3}dx =$$

$$\frac{\pi}{2}\left(\frac{x^6}{18} + 4\frac{x^2}{6} + \frac{x^{-2}}{-2}\right)\Bigg|_1^2 = \frac{\pi}{2}\left(\frac{x^6}{18} + 2\frac{x^2}{3} - \frac{1}{2x^2}\right)\Bigg|_1^2 =$$

$$\frac{\pi}{2}\left[\left(\frac{2^6}{18} + 2\frac{2^2}{3} - \frac{1}{2(2)^2}\right) - \left(\frac{1^6}{18} + 2\frac{1^2}{3} - \frac{1}{2(1)^2}\right)\right] =$$

$$\frac{\pi}{2}\left[\left(\frac{64}{18} + \frac{8}{3} - \frac{1}{8}\right) - \left(\frac{1}{18} + \frac{2}{3} - \frac{1}{2}\right)\right] =$$

$$\frac{\pi}{2}\left(\frac{63}{18} + \frac{6}{3} + \frac{3}{8}\right) = \frac{\pi}{2}\left(\frac{7}{2} + 2 + \frac{3}{8}\right) = \frac{\pi}{2}\left(\frac{28}{8} + \frac{16}{8} + \frac{3}{8}\right) = \frac{\pi}{2}\left(\frac{28 + 16 + 3}{8}\right) = \frac{47\pi}{16}$$

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(8) Find the solution of the differential equation $e^y y' = \dfrac{3x^2}{1+y}$ which satisfies the initial condition $y(2) = 0$

Solution: Separating variables, $e^y \dfrac{dy}{dx} = \dfrac{3x^2}{1+y}$ becomes

$(1+y)e^y dy = 3x^2 dx$ which can then be integrated:

$$\int (1+y)e^y dy = \int 3x^2 dx$$

$ye^y = x^3 + C$

Substituting $x = 2$ and $y = 0$ we get

$0e^0 = 2^3 + C$, so $C = -8$. The required solution is $ye^y = x^3 - 8$ or $x = \sqrt[3]{ye^y - 8}$

Part III:

Let $\mathcal{R}$ be the region in the x-y plane bounded by $y = \cos x$, and the x-axis, $-\frac{\pi}{2} \leq x =\leq \frac{\pi}{2}$.

(9) Find the volume of the solid obtained by rotating $\mathcal{R}$ about the x-axis.

Solution: $V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi (\cos x)^2 \, dx = 2\pi \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2x}{2} dx = \pi \left(x + \frac{1}{2} \sin 2x \right) \Big|_0^{\frac{\pi}{2}} = \frac{1}{2}\pi^2$

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(10) Find the volume of the solid obtained by rotating $\mathcal{R}$ about the y-axis.

Solution: $V = \int_0^{\frac{\pi}{2}} 2\pi x \cos x dx = 2\pi \int_0^{\frac{\pi}{2}} x \cos x dx =$(letting $u = x$, $dv = \cos x dx$)=

$2\pi \left(x \sin x|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x dx \right) = 2\pi \left(x \sin x + \cos x|_0^{\frac{\pi}{2}} \right)$

$2\pi \left(\frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2} \right) - 2\pi \left(0 \sin 0 + \cos 0 \right) = \pi^2 - 2\pi$

The End