UNIVERSITY OF SASKATCHEWAN

MATHEMATICS 124.3 (02 & 04)-Solutions to Midterm #1

(1) Evaluate the definite integral: $\int_{-2}^{0} x + \sqrt{4 - x^2}dx$(Hint: use geometry)

Solution: $\int_{-2}^{0} x + \sqrt{4-x^2}dx = \int_{-2}^{0} xdx + \int_{-2}^{0} \sqrt{4-x^2}dx =$
-(area of triangle with base 2 and height 2) + (area of quarter circle of radius 2)$=-2 + \pi = \pi - 2$

(2) Evaluate the definite integral: $\int_{e}^{e^2} \frac{\ln(x^2)}{x}dx$

Solution: $\int_{e}^{e^2} \frac{\ln(x^2)}{x}dx = 2\int_{x=e}^{x=e^2} \frac{\ln x}{x}dx = 2\int_{u=1}^{u=2} udu = u^2\Big|_{u=1}^{u=2} = 2^2 - 1^2 = 3$

where we have made the substitution $u = \ln x$.

In questions 3 to 6 evaluate the derivative of each function at the indicated x-value:

(3) $f(x) = xe^{-(x+1)}$ at $x = -1$.

Solution: $f'(x) = \left(xe^{-(x+1)}\right)' = (x)'e^{-(x+1)} + x\left(e^{-(x+1)}\right)' =$

$(1)e^{-(x+1)} + xe^{-(x+1)}(-(x+1))' = e^{-(x+1)} + xe^{-(x+1)}(-1) = (1-x)e^{-(x+1)}$,

so $f'(-1) = (1-(-1))e^{-(-1+1)} = 2$

(4) $f(x) = \dfrac{ex}{\ln(x-1)}$ at $x = e+1$.

Solution: $f'(x) = \dfrac{\ln(x-1)(ex)' - ex(\ln(x-1))'}{(\ln(x-1))^2} = \dfrac{\ln(x-1)e - ex\frac{1}{x-1}}{(\ln(x-1))^2}$, so

$f'(e+1) = \dfrac{\ln(e+1-1)e - e(e+1)\frac{1}{e+1-1}}{(\ln(e+1-1))^2} = \dfrac{\ln(e)e - e(e+1)\frac{1}{e}}{(\ln(e))^2} = \dfrac{(1)e - (e+1)}{(1)^2} = -1$

(5) $f(x) = \arctan(1-x^2)$ at $x = 1$.

Solution: $f'(x) = \dfrac{1}{1+(1-x^2)^2}(1-x^2)' = \dfrac{1}{1+(1-x^2)^2}(-2x)$, so

$f'(1) = \dfrac{1}{1+(1-1^2)^2}(-2(1)) = -2$

(6) $f(x) = 2\int_{\sqrt{x}}^{0} \sqrt{1-t^2}dt$ at $x = \frac{1}{2}$.

Solution: $f(x) = -2\int -0^{\sqrt{x}}\sqrt{1-t^2}dt$, so

$f'(x) = -2\sqrt{1-(\sqrt{x})^2}(\sqrt{x})' = -2\sqrt{1-|x|}\frac{1}{2\sqrt{x}} - \sqrt{1-|x|}\frac{1}{\sqrt{x}}$, so

$f'(\frac{1}{2}) = -\sqrt{1-\left|\frac{1}{2}\right|}\frac{1}{\sqrt{\frac{1}{2}}} = -\sqrt{\frac{1}{2}}\frac{1}{\sqrt{\frac{1}{2}}} = -1$

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(7) A mould is placed in a culturing vat at 12:00 noon. This particular mould grows in a cylindrical shape which is perfectly circular and is of uniform thickness and density. Its volume is known to grow according to the constant proportional growth rate law. If the diameter of the mould culture is measured to be 6cm at 3:00 p.m and 9cm at 6:00 p.m., what was it (in cm) at noon?

Solution: Let $V(t)$ be the volume and $D(t)$ be the diameter t hours after 12:00 noon. Let h be the constant thickness. Then $V(t) = \pi\frac{D(t)^2}{4}h$, so $D(t) = \frac{4V(t)}{h}$ and thus $D'(t) = \frac{4V'(t)}{h}$.

We have $\frac{D'(t)}{D(t)} = \frac{\frac{4V'(t)}{h}}{\frac{4V(t)}{h}} = \frac{V'(t)}{V(t)} = k$ for some constant k, so $D(t) = D(0)e^{kt}$.

Since $D(3) = 6$ and $D(6) = 9$, we have $6 = D(0)e^{3k}$ and $9 = D(0)e^{6k}$.

Dividing the second of these two equations by the first, we get $e^{3k} = \frac{3}{2}$.

Inserting this in the first equation, we get $6 = D(0)\frac{3}{2}$, so $D(0) = \frac{2}{3}6 = 4$.

The possible answers to the question 8 are in **ANSWER SET V:**

(A) $\frac{1}{4}$	(B) $\frac{3}{8}$	(C) $\frac{1}{2}$	(D) $\frac{5}{8}$	(E) $\frac{3}{4}$	(F) $\frac{7}{8}$	(G) 1	(H) $\frac{9}{8}$	(I) $\frac{5}{4}$	(J) $\frac{3}{2}$

(8) A FORCE of 20N is required to hold a spring that has been stretched from it's natural length of 10cm to 20cm. How much <u>work</u> (in Joules) is done in stretching the spring from 15 cm to 20 cm?

Solution: We have $40N = k(10 \text{ cm }) = k(0.10 \text{ m })$, so the spring constant $k = 400\frac{\text{N}}{\text{m}}$

The work done in stretching the spring from 15 cm to 20 cm is therefore

$$W = \int_{0.05}^{0.10} 400x\,dx = 400\frac{x^2}{2}\Bigg|_{0.05}^{0.10} = 200\left((0.10)^2 - (0.05)^2\right) = \frac{3}{4}$$

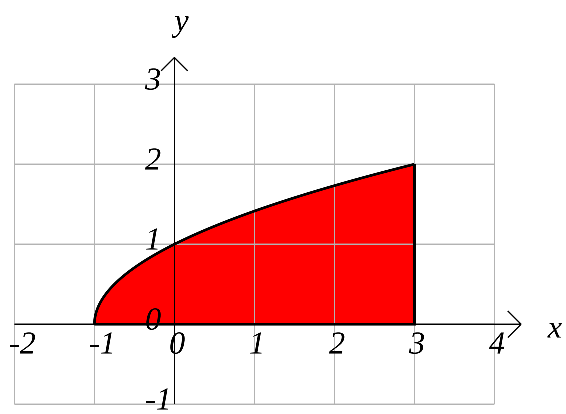
Answer: E

The possible answers to the questions 9,10,11,12 and 13 are in **ANSWER SET VI:**

(A) $(4-t^2)^2$	(B) $(6-2t)\sqrt{t+1}$	(C) t^2-1	(D) $\sqrt{t+1}$	(E) $2\sqrt{t+1}$
(F) $8t-2t^3$	(G) $t+1$	(H) $4-t^2$	(I) $4t-t^3$	(J) $(3-t)\sqrt{t+1}$

Let $\mathcal{R}$ be the region in the x-y plane bounded by $y=\sqrt{x+1}$, $y=0$, $x=-1$, and $x=3$ with

$$I=\int_{-1}^{3}F(x)dx=\int_0^2 G(y)dy$$



If I is to be the area of the region $\mathcal{R}$, then **(9)** $F(t)=$

Solution: The area is $\int_{-1}^{3}\sqrt{x+1}dx$, so $F(t)=\sqrt{t+1}$ Answer: D

and **(10)** $G(t)=$

Solution: Solve for $x=y^2-1$, so the area is $\int_0^2\left[3-(y^2-1)\right]dy$ so $G(t)=4-t^2$ Answer: H

(11) If I is to be the volume of the solid obtained by revolving $\mathcal{R}$ about the x-axis divided by π, then $F(t)=$

Solution: The volume is $V=\int_{-1}^{3}\pi\left(\sqrt{x+1}\right)^2dx=\pi\int_{-1}^{3}x+1dx$, so
$\dfrac{V}{\pi}=\int_{-1}^{3}x+1dx$, so $F(t)=t+1$, Answer: G

(12) If I is to be the volume of the solid obtained by revolving $\mathcal{R}$ about the x-axis divided by 2π, then $G(t)=$

Solution: The volume is $V=\int_0^2 2\pi y\left[3-(y^2-1)\right]dy=2\pi\int_0^2 4y-y^3dy$, so
$\dfrac{V}{2\pi}=\int_0^2 4y-y^3dy$, so $G(t)=4t-t^3$, Answer: I

(13) If I is to be the volume of the solid obtained by revolving $\mathcal{R}$ about the line $x=3$ divided by π, then $G(t)=$ Answer: A

Solution: The volume is $V=\int_0^2\pi\left[3-(y^2-1)\right]^2dy=\pi\int_0^2\left[4y-y^2\right]^2dy$, so

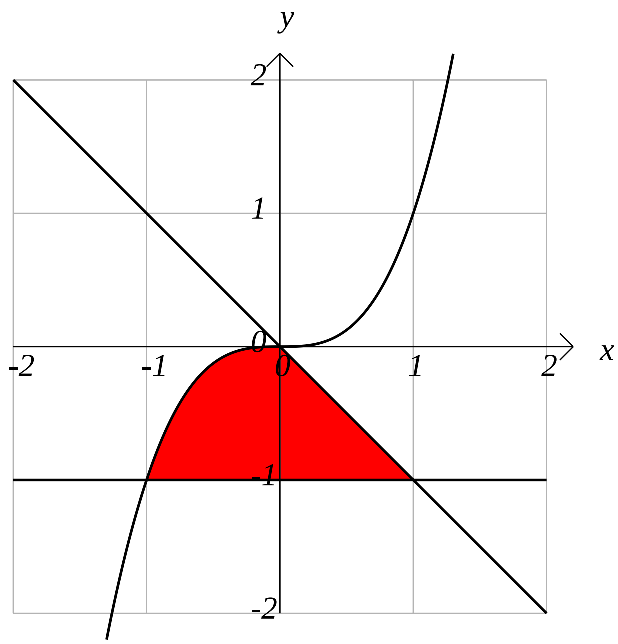
$\frac{V}{\pi} = \int_0^2 \left[4y - y^2\right]^2 dy$, so $G(t) = \left[4t - t^2\right]^2$,

Answer: I

The possible answers to question 14 are in **ANSWER SET VII:**

(A) $y + y^{\frac{1}{3}}$	(B) $-(y + y^{\frac{1}{3}})$	(C) $y^{\frac{1}{3}}$	(D)$y^{\frac{4}{3}}$	(E) $y^{\frac{1}{3}} - 1$
(F) $y^{\frac{1}{3}} + 1$	(G)$y^{\frac{1}{3}} - y$	(H)$y - y^{\frac{1}{3}}$	(I) $y^2 + y^{\frac{4}{3}}$	(J) $-(y^2 + y^{\frac{4}{3}})$

Let $\mathcal{R}$ be the region in the x-y plane bounded by $y = x^3$, $x + y = 0$, and $y = -1$ with $I = \int_{-1}^{0} G(y)dy$.



(14) If I is to be the area of $\mathcal{R}$, then $G(y) =$

Solution: The area is $A = \int_{-1}^{0} \left(-y - y^{\frac{1}{3}}\right) dy = \int_{-1}^{0} -\left(y + y^{\frac{1}{3}}\right) dy$, so $G(t) = -\left(t + t^{\frac{1}{3}}\right)$

Answer: B

The possible answers to question 15 are in **ANSWER SET VIII:**

(A) $-\infty$	(B) $-e$	(C) -2	(D) -1	(E) 0	(F) 1	(G) 2	(H) e	(I) 3	(J) $+\infty$

(15) $\lim_{x\to+\infty}\left\{\ln\left[\left(\frac{2x}{2x+1}\right)^{4x}\right]\right\} =$

Solution: $\lim_{x\to+\infty}\left\{\ln\left[\left(\frac{2x}{2x+1}\right)^{4x}\right]\right\} = \lim_{x\to+\infty} 4x\ln\left(\frac{2x}{2x+1}\right) = 4\lim_{x\to+\infty} x\left[\ln 2x - \ln(2x+1)\right] =$

$4\lim_{x\to+\infty}\frac{\ln 2x - \ln(2x+1)}{\frac{1}{x}} =$ (by L'Hopital) $4\lim_{x\to+\infty}\frac{\frac{1}{x}-\frac{2}{2x+1}}{-\frac{1}{x^2}} =$

$-4\lim_{x\to+\infty} x^2\left(\frac{1}{x}-\frac{2}{2x+1}\right) = -4\lim_{x\to+\infty} x^2\left(\frac{2x+1-2x}{x(2x+1)}\right) = -4\lim_{x\to+\infty} x\left(\frac{1}{2x+1}\right) =$

$-4\lim_{x\to+\infty}\left(\frac{x}{2x+1}\right) = -4\frac{1}{2} = -2$ Answer: C

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