Work Done by a Variable Force

We know from basic Science the the work done by a constant force F exerted over a distance d is $W = Fd$. Suppose the force is variable: we might be looking at the displacement of a spring over a distance d, the sending of a rocket into space, or the winding of a cable onto a drum.

If the interval $[a, b]$ is partitioned by a sequence of numbers $x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n$ where $a = x_0$ and $x_n = b$ with $\Delta x_j = x_j - x_{j-1}$ of I_n, and we have a tagset $\mathsf{T} = \{t_1 \leq t_2 \leq \cdots \leq t_n\}$, then the work W done in exerting the force $F(x)$ as x increases from a to b can be approximated by a Riemann sum: $W \doteq \sum_{i=1}^{n} F(t_i)\Delta x_i$

If we take the limit as the mesh of the partition approaches 0, we get a definite integral, which we define to be the work W:

$$W = \lim_{||\mathcal{P}|| \to 0} \sum_{i=1}^{n} F(t_i)\Delta x_i = \int_a^b F(x)dx$$

Hooke's Law

The force exerted by a spring is directly proportional to its displacement x from its natural length.

$$F(x) = kx\frac{\text{Newton}}{\text{cm}} = kx\frac{\text{N}}{\text{cm}},$$

where $k > 0$ is called the **spring constant** and x is the displacement from the natural length.

Example: A spring has natural length 30 cm, and a force of 50 Newtons(N) is required to extend it to 35 cm. what is the spring constant? How much work is required to extend it from 32 to 36 cm?

Solution: We have $F(35-30) = 50N = k(35-30)\ \text{cm}\frac{\text{N}}{\text{cm}} = k5\text{cm}$, so $k = \frac{50}{5} = 10$

The work required to extend the spring from 32 to 36 cm is

$$W = \int_{(32-30)\text{cm}}^{(36-30)\text{cm}} 10x\frac{\text{N}}{\text{cm}}dx = 10\frac{\text{N}}{\text{cm}}\int_{2\text{cm}}^{6\text{cm}} x\,dx = 10\frac{\text{N}}{\text{cm}}\left(\frac{x^2}{2}\right)\Bigg|_{2\text{cm}}^{6\text{cm}} =$$

$$5\frac{\text{N}}{\text{cm}}\left((6\text{cm})^2 - (2\text{cm})^2\right) = 160\text{N}\cdot\text{cm}$$

Example 4: A conical water tank of height 8m and diameter 6m at the base must have all of its fluid contents pumped to the top of the tank. If it is full to a depth of 4m, how much work will this require?

Solution: Let the interval $[0, 4]$ be partitioned by $0 = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = 4$, and a tagset $\mathsf{T} = \{t_1 \leq t_2 \leq \cdots \leq t_n\}$ given.

Then the work W_i required to pump the water which lies between level x_{i-1} and x_i approximately equals the force F_i required to lift it times the distance $d_i = 8 - t_i$ that it must be lifted.

The volume V_i of the water lying between level x_{i-1} and x_i is approximately that of the disk of radius $r_i = (8 - t_i)\frac{3}{8}$ and thickness

$\Delta x_i = x_i - x_{i-1}$, so $V_i \doteq \pi r_i^2 \Delta x_i = \frac{9\pi}{64}(8 - t_i)^2 \Delta x_i$.

The mass M_i of this volume of water is

$$M_i = 1000 \text{ kg } \times V_i \doteq 1000 \times \frac{9\pi}{64}(8 - t_i)^2 \Delta x_i \text{kg},$$

so the force F_i needed to lift it is

$$F_i = 9.8 M_i \doteq 9800 \frac{9\pi}{64} \left((8 - t_i)\right)^2 \Delta x_i \text{ N}.$$

and therefore the work W_i needed to lift this mass to the top of the tank is about

$$F_i d_i = 9800 \frac{9\pi}{64} \left((8 - t_i)\right)^2 \Delta x_i \text{ N}(8 - t_i) = 9800\pi (8 - t_i)^3 \frac{9}{64} \Delta x_i$$

Adding them all up, we get an estimate for the total amount of work needed:

$$W \doteq \sum_{i=1}^{n} F_i d_i = \sum_{i=1}^{n} 9800 \frac{9\pi}{64} (8 - t_i)^3 \Delta x_i$$

Taking the limit as the mesh of the partition approaches 0, we get

$$W = \lim_{||\mathcal{P}||\to 0} \sum_{i=1}^{n} F_i d_i = \sum_{i=1}^{n} 9800\frac{9\pi}{64}(8 - t_i)^3 \Delta x_i =$$

$$9800\frac{9\pi}{64}\int_{x=0}^{x=4}(8 - x)^3 dx$$

We use the substitution $u = 8 - x$ to evaluate the definite integral: $dx = -du$, $u = 8$ when $x = 0$, and $u = 4$ when $x = 4$, so

$$9800\frac{9\pi}{64}\int_{x=0}^{x=4}(8-x)^3 dx = -9800\frac{9\pi}{64}\int_{u=8}^{u=4} u^3 du = -9800\frac{9\pi}{64}\left.\frac{u^4}{4}\right|_{u=8}^{u=4} =$$

$$-9800\frac{9\pi}{4(64)}4^4\left(1^4 - 4^4\right) = -9800(9\pi)(-255) \doteq 70.8 \times 10^6\text{J}$$

Example: A 40 metre long cable that weighs 5 kg/metre is hanging from the roof of a very tall building. How much work is required to lift it all to the roof level?

Solution: Suppose that x metres of cable has already been lifted to roof level, so that $40 - x$ metres weighing $5(40 - x)$ kg and requiring a force of $9.8 \times 5(40 - x) = 49(40 - x)$ N is left hanging. The work required to lift this remaining cable a distance Δx is therefore

$49(40-x)\Delta x$ J. Summing over a partition of the interval [0,40] gives the Riemann sum

$W \doteq \sum_{i=1}^{n} 49(40-x)\Delta x \to \int_0^{40} 49(40-x)dx$ as the mesh of the partition tends to 0.

Evaluating the definite integral, we get

$$W = \int_0^{40} 49(40-x)dxJ = 49\left(40x - \frac{x^2}{2}\right)\Bigg|_0^{40} J = 49\left(40(40) - \frac{(40)^2}{2}\right)J =$$

$49(800)J = 39,200J$