Volumes of Solid Objects

Let ℓ be a straight coordinate line in space with origin **O**. We denote by $\mathcal{P}_t$ the plane perpendicular to ℓ at a distance t from **O**.

Suppose we have a solid object S which lies between $\mathcal{P}_a$ and $\mathcal{P}_b$ ($a \leq b$, and the area of the cross-section $S \cap \mathcal{P}_t$ of S with $\mathcal{P}_t$ is given by $A(t)$.

Let $V(x)$ be the volume of the solid lying between $\mathcal{P}_a$ and $\mathcal{P}_x$.

We calculate the derivative of $V(x)$:

$$V'(x) = \lim_{h\to 0} \frac{V(x+h) - V(x)}{h} = \lim_{h\to 0} \frac{A(x^*(h))h}{h} = \lim_{h\to 0} \frac{A(x^*(h))h}{h} =$$

$$\lim_{h\to 0} A(x^*(h)) = A(x)$$

where $x^*(h)$ is defined to be the least number $x$in $[x, x+h]$ for which $V(x+h) - V(x) = A(x)h$.

Thus $V(x)$ is an antiderivative of $A(x)$.

From this we get a basic formula derived from what is called the **Method of Disks** or **Slicing**:

$$V = \int_a^b A(x)dx$$

We usually can count on the cross-sections being simple geometrical figures, such as circles, annulli, squares or standard triangles.

Example: Find the volume of the pyramid S which has 4 faces, all of which are equilateral triangles with side of length 1.

Note: the area of an equilateral triangle whose side is of length a is $\sqrt{3}\frac{a^2}{4}$.

Solution: Let **O** be one of the vertices, and let ℓ be the line through **O** which is perpendicular to the opposite face. Then $\mathcal{P}_x$ is an equilateral triangle with side $\sqrt{\frac{3}{2}x}$ and area $A(x) = \sqrt{3}\frac{3x^2}{8}$.

The height of the pyramid is $\sqrt{\frac{2}{3}}$, so we evaluate the definite integral

$$V = \int_0^{\sqrt{\frac{2}{3}}} \sqrt{3}\frac{3x^2}{8}dx = \frac{3\sqrt{3}}{8}\int_0^{\sqrt{\frac{2}{3}}} x^2dx = \frac{3\sqrt{3}}{8}\left.\frac{x^3}{3}\right|_0^{\sqrt{\frac{2}{3}}} = \frac{\sqrt{3}}{8}\left(\sqrt{\frac{2}{3}}\right)^3 =$$

$$\frac{\sqrt{3}}{8}\frac{2}{3}\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{12}$$

which we note is equal to one-third the area $\frac{\sqrt{3}}{4}$ of the base times the height $\sqrt{\frac{2}{3}}$.

Annular Cross-Sections

Definition: An **annulus** is the refion lying between two concentric circles. If the radii of the outer and inner circles are $\mathcal{O}$ and $\mathcal{I}$, then the area of the annulus is

$$A = \pi\mathcal{O}^2 - \pi\mathcal{I}^2 = \pi(\mathcal{O}^2 - \mathcal{I}^2) = \pi(\mathcal{O} + \mathcal{I})(\mathcal{O} - \mathcal{I})$$

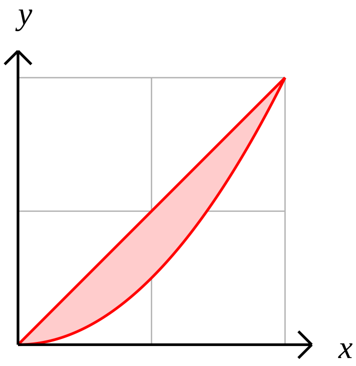
It is frequently necessary to compute the volume of an object that has been turned on a lathe. We take ℓ

to be the axis of the lathe, and let the outer and inner diameters at a distance x from the chuck be $\mathcal{O}(x)$ and $\mathcal{I}(x)$.

Then the volume of the turned object from $x = a$ to $x = b$ is

$$V = \int_a^b \pi\left[\mathcal{O}(x)^2 - \mathcal{I}(x)^2\right]dx = \pi\int_a^b [\mathcal{O}(x) + \mathcal{I}(x)]\,[\mathcal{O}(x) - \mathcal{I}(x)]dx$$

Example: The region $\mathcal{R} = \left\{(x, y) | 0 \leq x \leq 1, \text{ and } x^2 \leq y \leq x\right\}$ is to be rotated about the line $y = -5$. What is the volume of the resulting solid?



Solution: We have $\mathcal{O}(x) = x-(-5) = x+5$, and $\mathcal{I}(x) = x^2-(-5) = x^2+5$, so

$$V = \pi \int_0^1 \left[(x+5) + (x^2+5)\right]\left[(x+5) - (x^2+5)\right] dx =$$

$$-\pi \int_0^1 \left[x^2+x+10\right]\left[x^2-x\right] dx = -\pi \int_0^1 x^4 - x^2 + 10x^2 - 10x dx =$$

$$-\pi \int_0^1 x^4 + 9x^2 - 10x dx = -\pi \left(\frac{x^5}{5} + 9\frac{x^3}{3} - 10\frac{x^2}{2}\right)\Bigg|_0^1 = \frac{9\pi}{5}$$

Example: The same region is to be rotated about the line $x = -5$. Find the volume generated.

We have to be very careful with this one, because we are dealing with a vertical axis of rotation.

We have to rewrite the region in terms of functions of y:

$\mathcal{R} = \{(x, y) | 0 \le y \le 1, \text{ and } y \le x \le \sqrt{y}\}$.

We write the outer and inner diameters as functions of y:

$\mathcal{O}(y) = \sqrt{y} - (-5) = y^{\frac{1}{2}} + 5, \mathcal{I}(y) = y - (-5) = y + 5.$

We have $V = \pi \int_0^1 [\mathcal{O}(y) + \mathcal{I}(y)][\mathcal{O}(y) - \mathcal{I}(y)]dy =$

$\pi \int_0^1 [(y^{\frac{1}{2}} + 5) + (y + 5)][(y^{\frac{1}{2}} + 5) - (y + 5)]dy =$

$\pi \int_0^1 [y^{\frac{1}{2}} + y + 10][y^{\frac{1}{2}} - y]dy = \pi \int_0^1 y - y^2 + 10[y^{\frac{1}{2}} - y]dy =$

$\pi \int_0^1 -y^2 - 9y + 10y^{\frac{1}{2}}dy = \pi \left(-\frac{y^3}{3} - 9\frac{y^2}{2} + 10\frac{y^{\frac{3}{2}}}{\frac{3}{2}} \right)\Bigg|_0^1 = \pi \left(-\frac{1}{3} - \frac{9}{2} + \frac{20}{3} \right) =$

$\pi \left(\frac{38}{6} - \frac{27}{6} \right) = \frac{11\pi}{6}$

We note that the difference between the two volumes just computed is $\frac{11\pi}{6} - \frac{9\pi}{5} = \pi\frac{55 - 54}{30} = \frac{\pi}{30}$, which is small but possibly important! It's about 2%.