Trigonometric Integrals

It is often necessary to evaluate integrals of the form

$$\int (\sin x)^m (\cos x)^n dx, \text{ where } m \text{ and } n \text{ are integers.}$$

If one of the exponents, either m or n is and odd, there is a straight-forward simplification.

Case 1: $m = 2M + 1$ is odd.

Then $(\sin x)^m = (\sin x)^{2M+1} = (\sin^2 x)^M \sin x = (1 - \cos^2 x)^M \sin x$,

so our integral becomes

$$\int (\sin x)^m (\cos x)^n dx = \int (1 - \cos^2 x)^M \sin x (\cos x)^n dx$$

and we may make the substitution $u = \cos x$ with $du = -\sin x dx$ to get

$$\int (\sin x)^m (\cos x)^n dx = \int (1 - \cos^2 x)^M \sin x (\cos x)^n dx = -\int (1 - u^2)^M u^n du$$

Example: $\int (\sin x)^{11} (\cos x)^{10} dx = -\int (1 - u^2)^5 u^{10} du =$

$$-\int \left[1 - 5u^2 + 10(u^2)^2 - 10(u^2)^3 + 5(u^2)^4 - (u^2)^5\right] u^{10} du =$$

$$-\int u^{10} - 5u^{12} + 10u^{14} - 10u^{16} + 5u^{18} - u^{20} du =$$

$$-\left(\frac{u^{11}}{11} - 5\frac{u^{13}}{13} + 10\frac{u^{15}}{15} - 10\frac{u^{17}}{17} + 5\frac{u^{19}}{19} - \frac{u^{21}}{21}\right) + C =$$

$$-\frac{\cos^{11} x}{11} + \frac{5\cos^{13} x}{13} - \frac{2\cos^{15} x}{5} + \frac{10\cos^{17} x}{17} - \frac{5\cos 19x}{19} + \frac{\cos^{21} x}{21} + C$$

Example: $\int (\sin x)^{11}(\cos x)^{-10}dx = -\int (1-u^2)^5 u^{-10}du =$

$-\int \left[1 - 5u^2 + 10(u^2)^2 - 10(u^2)^3 + 5(u^2)^4 - (u^2)^5\right] u^{-10}du =$

$-\int u^{-10} - 5u^{-8} + 10u^{-6} - 10u^{-4} + 5u^{-2} - u^0 du =$

$-\left(\frac{u^{-9}}{-9} - 5\frac{u^{-7}}{-7} + 10\frac{u^{-5}}{-5} - 10\frac{u^{-3}}{-3} + 5\frac{u^{-1}}{-1} - u\right) + C =$

$-\left(-\frac{\cos^{-9} x}{9} + \frac{5\cos^{7} x}{7} - 2\cos^{-5} x + \frac{10\cos^{-3} x}{3} - 5\cos^{-1} x - \cos x\right) +$
$C =$

$\frac{1}{9}\sec^9 x - \frac{5}{7}\sec^7 x + 2\sec^5 x - \frac{10}{3}\sec^3 x + 5\sec x + \cos x + C$

Example: $\int (\sin x)^{-11}(\cos x)^{-10}dx = \int (\sin x)^{-12}(\cos x)^{-10}\sin x dx = -\int (1-u^2)^{-6}u^{-10}du$

can be done, but requires the method of Partial Fractions, which we shall see later.

The situation is similar when the power of $\cos x$ is odd.

Example: $\int (\sin x)^{10}(\cos x)^{11}dx = \int (\sin x)^{10}(\cos x)^{10}\cos x dx =$

$$\int (\sin x)^{10}(\cos^2 x)^5 \cos x dx = \int (\sin x)^{10}(1-\sin^2 x)^5 \cos x dx =$$

(letting $u = \sin x$ and $du = \cos x dx$)

$$\int u^{10}(1-u^2)^5 du = \int u^{10}\left[1 - 5u^2 + 10(u^2)^2 - 10(u^2)^3 + 5(u^2)^4 - (u^2)^5\right]$$

$$\int u^{10} - 5u^{12} + 10u^{14} - 10u^{16} + 5u^{18} - u^{20}du =$$

$$\frac{u^{11}}{11} - 5\frac{u^{13}}{13} + 10\frac{u^{15}}{15} - 10\frac{u^{17}}{17} + 5\frac{u^{19}}{19} - \frac{u^{21}}{21} + C =$$

$$\frac{1}{11}\sin^{11} x - \frac{5}{13}\sin^{13} x + \frac{2}{5}\sin^{15} x - \frac{10}{17}\sin^{17} x + \frac{5}{19}\sin^{19} - \frac{1}{21}\sin^{21} x + C$$

When both powers are odd, it is easiest to select the function with the highest power for substitution:

Good Example: $\int (\sin x)^{11}(\cos x)^3 dx = \int (\sin x)^{11}(\cos x)^2 \cos x dx =$

$$\int (\sin x)^{11}(1 - \sin^2 x) \cos x dx =$$

(letting $u = \sin x$ and $du = \cos x dx$)

$$\int u^{11}(1 - u^2) du = \int u^{11} - u^{13} du = \frac{u^{11}}{11} - \frac{u^{14}}{14} + C =$$

$$\frac{1}{11}\sin^{11} x - \frac{1}{14}\sin^{13} x + C$$

Bad Example: $\int (\sin x)^{11}(\cos x)^3 dx = \int (\sin x)^{10}(\cos x)^3 \sin x dx =$

$\int (1-\cos^2 x)^5(\cos x)^3 \sin x dx =$

(letting $u = \cos x$ and $du = -\sin x dx$)

$\int (1-u^2)^5 u^3(-du) =$

$\int \left[-1+5u^2-10(u^2)^2+10(u^2)^3-5(u^2)^4+(u^2)^5\right] u^3 du =$

$\int \left[-u^3+5u^5-10u^7+10u^9-5u^{11}+u^{13}\right] du =$

$-\frac{u^4}{4}+5\frac{u^6}{6}-10\frac{u^8}{8}+10\frac{u^{10}}{10}-5\frac{u^{12}}{12}+\frac{u^{14}}{14}+C =$

$-\frac{1}{4}\cos^4 x+\frac{5}{6}\cos^6 x-\frac{5}{4}\cos^8 x+\cos^{10} x-\frac{5}{12}\cos^{12} x+\frac{1}{14}\cos^{14} x+C$

When neither power is odd, we need to use a double angle formula from trigonometry:

$$\cos 2x \equiv 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

can be solved for $\cos^2 x$ and $\sin^2 x$:

$\cos^2 x = \dfrac{1+\cos 2x}{2}$ and $\sin^2 x = \dfrac{1-\cos 2x}{2}$

Thus, if we want to integrate $\int (\sin x)^m (\cos x)^n dx$, where m and n are even integers, we write $m = 2M$ and $n = 2N$ and we have:

$$\int (\sin x)^m (\cos x)^n dx = \int (\sin x)^{2M} (\cos x)^{2N} dx = \int (\sin^2 x)^M (\cos^2 x)^N dx$$

$$\int \left(\frac{1-\cos 2x}{2}\right)^M \left(\frac{1+\cos 2x}{2}\right)^N dx =$$

$$2^{-(M+N)} \int (1-\cos 2x)^M \, (1+\cos 2x)^N \, dx$$

Example: $\int \cos^2 x dx = \int \frac{1+\cos 2x}{2} dx =$

$$\frac{1}{2}\int 1+\cos 2x dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C =$$

$$\frac{x}{2} + \frac{1}{4}\sin 2x + C = \frac{x}{2} + \frac{1}{4}2\sin x\cos x + C = \frac{x+\sin x\cos x}{2} + C$$

Example: $\int \sin^2 x dx = \int \frac{1-\cos 2x}{2} dx =$

$$\frac{1}{2}\int 1-\cos 2x dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C =$$

$$\frac{x}{2} - \frac{1}{4}\sin 2x + C = \frac{x}{2} - \frac{1}{4}2\sin x\cos x + C = \frac{x-\sin x\cos x}{2} + C$$

It should come as no surprise that $\int \left(\cos^2 x + \sin^2 x\right) dx = x + C$

Example: $\int \sin^2 x \cos^2 x dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2} dx\right) =$

$\frac{1}{4}\int 1-\cos^2 2x dx = \frac{1}{4}\int 1-\frac{1+\cos 4x}{2} dx = \frac{1}{8}\int 1-\cos 4x dx =$

$\frac{1}{8}\left(x-\frac{1}{4}\sin 4x\right)+C = \frac{x}{8}-\frac{\sin 4x}{32}+C$

Products of powers of secant and tangent functions

$\sec^m x \tan^n x$ can be expressed as a product of powers of $\sin x$ and $\cos x$, but it is often more convenient to use the identity $\tan^2 x + 1 = \sec^2 x$ and the differentials $d(\tan x) = \sec^2 x dx$ and $d(\sec x) = \sec x \tan x dx$.
We assume that m and n are non-negative.

Case 1: The power m of $\sec x$ is positive and even: then $m = 2M$ and we have:

$\int \sec^m x \tan^n x dx = \int (\sec^2 x)^M \tan^n x dx = \int (\sec^2 x)^{M-1} \tan^n x \sec^2 x dx =$
$\int (1 + \tan^2 x)^{M-1} \tan^n x \sec^2 x dx =$ and we can make the substitution $u = \tan x$, $du = \sec^2 x dx$, and we get

$$\int \left(1 + \tan^2 x\right)^{M-1} \tan^n x \sec^2 x dx = \int \left(1 + u^2\right)^{M-1} u^n du$$

Example: $\int \sec^{10}x \tan^5 x\,dx = \int \left(1+u^2\right)^4 u^5 du =$

$\int \left(1+4u^2+6u^4+4u^6+u^8\right) u^5 du = \int u^5 + 4u^7 + 6u^9 + 4u^{11} + u^{13} du =$

$$\frac{u^6}{6} + 4\frac{u^8}{8} + 6\frac{u^{10}}{10} + 4\frac{u^{12}}{12} + \frac{u^{14}}{14} + C =$$

$$\frac{1}{6}u^6 + \frac{1}{2}u^8 + \frac{3}{5}u^{10} + \frac{1}{3}u^{12} + \frac{1}{14}u^{14} + C =$$

$$\frac{1}{6}\tan^6 x + \frac{1}{2}\tan^8 x + \frac{3}{5}\tan^{10} x + \frac{1}{3}\tan^{12} x + \frac{1}{14}\tan^{14} x + C$$

If the power of $\sec x$ is not even, but the power of n of $\tan x$ is odd, so that $n = 2N + 1$, and m is positive, we can make the substitution $u = \sec x$, $du = \sec x \tan x dx$, and get

$$\int \sec^m x \tan^n x dx = \int \sec^{m-1} x \tan^{2n} x \sec x \tan x dx =$$

$$\int u^{m-1} \left(\tan^2 x\right)^N du = \int u^{m-1} \left(\sec^2 x - 1\right)^N du =$$

$$\int u^{m-1} \left(u^2 - 1\right)^N du$$

Example: $\int \sec^5 x \tan^5 x dx = \int u^4(u^2-1)^2 du = \int u^4(u^4 - 2u^2 + 1) du =$

$$\int u^8 - 2u^6 + u^4 du \frac{u^9}{9} - 2\frac{u^7}{7} + \frac{u^5}{5} + C =$$

$$\frac{1}{9}\sec^9 x - \frac{2}{7}\sec^7 x + \frac{1}{5}\sec^5 x + C$$

We are left with the cases where m is odd and n is even, all of which can be reduced to the problem of finding the antiderivative of an odd power of $\sec x$.

Example: $\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx =$

$$\int \left(\frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \right) dx = \int \frac{(\sec^2 x + \sec x \tan x) dx}{\sec x + \tan x} =$$

$$\int \frac{d(\sec x + \tan x)}{\sec x + \tan x} = \ln(\sec x + \tan x) + C$$

Example: $\mathcal{I} = \int \sec^3 x\,dx$

Using Integration by Parts, with
$u = \sec x$, $dv = \sec^2 x$, $v = \tan x$, $du = \sec x \tan x\,dx$, we get

$$\mathcal{I} = \int \sec^3 x\,dx = \left(\int u\,dv = uv - \int v\,du\right) =$$

$$\sec x \tan x - \int \tan x \sec x \tan x\,dx = \sec x \tan x - \int \tan^2 x \sec x\,dx =$$

$$\sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx =$$

$$\sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx = \sec x \tan x - \mathcal{I} + \int \sec x\,dx =$$
$\sec x \tan x + \ln|\sec x + \tan x| + C$ so

$2\mathcal{I} = \sec x \tan x + \ln|\sec x + \tan x| + C$ and

$$\int \sec^3 x\,dx = \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C$$

Example: $\int \tan^2 x \sec x \, dx$ appears in the previous calculation, and is one of the simpler cases left:

$\mathcal{I} = \sec x \tan x - \int \tan^2 x \sec x \, dx$ gives us

$\int \tan^2 x \sec x \, dx = \sec x \tan x - \mathcal{I} =$

$$\sec x \tan x - \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C =$$

$$\frac{\sec x \tan x - \ln|\sec x + \tan x|}{2} + C$$

Integrals of products of sine and cosine functions

with different arguments

The identities:

$$\cos x \cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$$

$$\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

$\sin x \cos y = \frac{1}{2}\left(\sin(x-y) + \sin(x+y)\right)$ may be used:

Example: $\int \cos 5t \cos 7t\,dt = \int \frac{1}{2}\left(\cos(5t+7t) + \cos(5t-7t)\right) dt =$

$$\int \frac{1}{2}\left(\cos 12t + \cos(-2t)\right) dt = \frac{1}{2}\left(\frac{1}{12}\sin 12t + \frac{1}{2}\sin 2t\right) + C =$$

$$\frac{1}{24}\sin 12t + \frac{1}{4}\sin 2t + C$$