Slope and Concavity

To find the slope and concavity of a curve in the plane described by parametric equations is easy:

We have $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\dot{y}}{\dot{x}}$ and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}} = \frac{\frac{\frac{dx}{dt}\frac{d}{dt}\left(\frac{dy}{dt}\right)-\frac{dy}{dt}\frac{d}{dt}\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}}{\frac{dx}{dt}} =$$

$$\frac{\frac{dx}{dt}\left(\frac{d^2y}{dt^2}\right) - \frac{dy}{dt}\left(\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^3} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^3}$$

where we have used the dot notation for differentiation with respect to t.

Example:

Astroid: $x(t) = a\cos^3 t,\ y(t) = a\sin^3 t,$

$$\dot{x} = -3a\cos^2 t\sin t,\ \ddot{x} = -3a\cos t(-2\sin^2 t + \cos^2 t),$$

$$\dot{y} = 3a\sin^2 t\cos t,\ \ddot{y} = 3a\sin t(2\cos^2 t - \sin^2 t)$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\frac{\sin t}{\cos t} = -\tan t$$

$$\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^3} =$$

$$\frac{(-3a\cos^2 t\sin t)(3a\sin t(2\cos^2 t - \sin^2 t)) - (-3a\cos t(-2\sin^2 t + \cos^2 t))(3a}{(-3a\cos^2 t\sin t)^3}$$

$$\frac{-9a^2\sin^2 t\cos^2 t(2\cos^2 t - \sin^2 t) + 9a^2\sin^2 t\cos^2 t(-2\sin^2 t + \cos^2 t))}{-27a^3\cos^6 t\sin^6 t} =$$

$$\frac{(2\cos^2 t - \sin^2 t) - (-2\sin^2 t + \cos^2 t)}{3a\cos^4 t\sin^4 t} =$$

$$\frac{(2\cos^2 t - \sin^2 t) - (-2\sin^2 t + \cos^2 t)}{3a\cos^4 t\sin^4 t} = \frac{1}{3a\cos^4 t\sin^4 t} =$$

Areas under Parametric Curves

We can also easily find the area between the x axis and a parametric curve:

$$A = \int_{x=a}^{x=b} y\,dx == \int_{t=t_0}^{t=t_1} y(t)\dot{x}(t)dt$$

Example: Find the area of the **Astroid:** $x(t) = a\cos^3 t$, $y(t) = a\sin^3 t$.

Solution: Note that this is 4 times the area of the astroid lying in the first quadrant, so we have

$$A = 4\int_{x=0}^{x=a} y\,dx = 4\int_{t=\frac{\pi}{2}}^{t=0} \left(a\sin^3 t\right)\left(-3a\cos^2 t\sin t\right)dt =$$

$$12a^2\int_{t=0}^{t=\frac{\pi}{2}} \sin^4 t\cos^2 t dt = 12a^2\int_{t=0}^{t=\frac{\pi}{2}} \left(\frac{1-\cos 2t}{2}\right)^2\left(\frac{1+\cos 2t}{2}\right)dt =$$

$$\frac{3}{2}a^2\int_{t=0}^{t=\frac{\pi}{2}} (1-\cos 2t)(1-\cos 2t)(1+\cos 2t)\,dt =$$

$$\frac{3}{2}a^2\int_{t=0}^{t=\frac{\pi}{2}} (1-\cos 2t)\left(1-\cos^2 2t\right)dt = \frac{3}{2}a^2\int_{t=0}^{t=\frac{\pi}{2}} (1-\cos 2t)\sin^2 2t dt =$$

$$\frac{3}{2}a^2\int_{t=0}^{t=\frac{\pi}{2}}\sin^2 2t\,dt-\frac{3}{2}a^2\int_{t=0}^{t=\frac{\pi}{2}}\sin^2 2t\cos 2t\,dt=$$

$$\frac{3}{2}a^2\int_{t=0}^{t=\frac{\pi}{2}}\frac{1-\cos 4t}{2}dt-\left(\frac{3}{2}a^2\,\frac{1}{2}\frac{\sin^3 2t}{3}\Bigg|_{t=0}^{t=\frac{\pi}{2}}\right)=$$

$$\frac{3}{4}a^2\int_{t=0}^{t=\frac{\pi}{2}}1-\cos 4t\,dt=\frac{3}{4}a^2\;t-\frac{1}{4}\sin 4t\,\Bigg|_{t=0}^{t=\frac{\pi}{2}}=\frac{3\pi}{8}a^2$$

Arc Length of Parametric Curves

As mentioned previously, the formula for the length of a paremetric curve is

$$L = \int_{t=t_0}^{t=t_1} \sqrt{\dot{x}^2 + \dot{y}^2} dt$$

We compute the length of the perimeter of the astroid:

$$L = 4 \int_{t=0}^{t=\frac{\pi}{2}} \sqrt{(-3a\cos^2 t \sin t)^2 + \left(3a \sin^2 t \cos t\right)^2} dt =$$

$$L = 4 \int_{t=0}^{t=\frac{\pi}{2}} 3a\sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t} dt =$$

$$L = 12a \int_{t=0}^{t=\frac{\pi}{2}} \sqrt{(\sin^2 t + \cos^2 t)\sin^2 t \cos^2 t} dt =$$

$$L = 12a \int_{t=0}^{t=\frac{\pi}{2}} \sin t \cos t dt =$$

$$L = 6a \int_{t=0}^{t=\frac{\pi}{2}} \sin 2t dt = -3a \cos 2t|_{t=0}^{t=\frac{\pi}{2}} = -3a\left(\cos 2\frac{\pi}{2} - \cos 2(0)\right) =$$

$-3a(-1-1) = 6a$

Surface Areas

As mentioned previously, a formula for the surface area obtained by rotating a paremetric curve about the x-axis is

$$S = \int_{t=t_0}^{t=t_1} 2\pi y\sqrt{\dot{x}^2 + \dot{y}^2}dt$$

We compute the surface area obtained by rotating the astroid about the x-axis:

$$S = 2\int_{t=0}^{t=\frac{\pi}{2}} 2\pi a\sin^3 t\sqrt{(-3a\cos^2 t\sin t)^2 + \left(3a\sin^2 t\cos t\right)^2}dt =$$

$$4\pi a\int_{t=0}^{t=\frac{\pi}{2}} \sin^3 t 3a\sqrt{\cos^4 t\sin^2 t + \sin^4 t\cos^2 t}dt =$$

$$12\pi a^2\int_{t=0}^{t=\frac{\pi}{2}} \sin^3 t\sqrt{(\sin^2 t + \cos^2 t)\sin^2 t\cos^2 t}dt =$$

$$12\pi a^2\int_{t=0}^{t=\frac{\pi}{2}} \sin^3 t\sin t\cos t dt = 12\pi a^2\int_{t=0}^{t=\frac{\pi}{2}} \sin^4 t\cos t dt =$$

$$12\pi a^2\left.\frac{\sin^5 t}{5}\right|_{t=0}^{t=\frac{\pi}{2}} = \frac{12}{5}\pi a^2$$