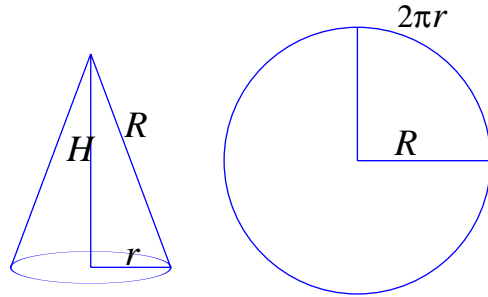
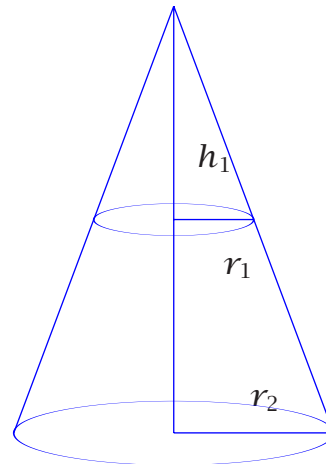


# Areas of Surfaces of Revolution

Let a cone have height  $h$  and radius  $r$ , so that the circumference of the base is  $2\pi r$ . If the cone is cut along a straight line (of length  $R = \sqrt{r^2 + h^2}$ ) from the vertex to the base and flattened out, we obtain a sector of the circle whose radius is  $R$ . The area of the circle of radius  $R$  is  $\pi R^2 = \pi(r^2 + h^2)$ , so the ratio of the area of the sector to that of the circle is  $\frac{2\pi r}{2\pi R} = \frac{r}{R}$ , so the area of the sector is  $\frac{r}{R}\pi R^2 = \pi r\sqrt{r^2 + h^2}$ .



If we now take two cones, with one being a subset of the other, we can calculate the area of the region between the bases of the two cones. This region is called a **frustum**.



Let the larger and smaller cones have heights and radii  $h_2$  and  $r_2$  and  $h_1$  and  $r_1$ . Let  $R_2 = \sqrt{r_2^2 + h_2^2}$  and  $R_1 = \sqrt{r_1^2 + h_1^2}$ , so that the areas of the larger and smaller cones are

$$A_2 = \pi r_2 \sqrt{r_2^2 + h_2^2} \text{ and } A_1 = \pi r_1 \sqrt{r_1^2 + h_1^2}$$



The area of the frustum is thus

$$A = A_2 - A_1 = \pi r_2 \sqrt{r_2^2 + h_2^2} - \pi r_1 \sqrt{r_1^2 + h_1^2} =$$

$$\pi \left[ r_2 \sqrt{r_2^2 + h_2^2} - r_1 \sqrt{r_1^2 + h_1^2} \right] = \pi (r_2 R_2 - r_1 R_1)$$

Writing  $R = R_2 - R_1$  and  $r = \frac{r_1 + r_2}{2}$ , and using similar triangles, we derive

$$A = 2\pi r R.$$

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We can then use this formula to derive a formula for the area of the surface obtained by rotating the curve  $(x(t), y(t))$ ,  $t_1 \leq t \leq t_2$  about the  $x$ - and  $y$ -axes respectively:

$$S_x = \int_{t_1}^{t_2} 2\pi y(t) \sqrt{(x'(t))^2 + (y'(t))^2} dt \quad \text{and}$$

$$S_y = \int_{t_1}^{t_2} 2\pi x(t) \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

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If the curve is the graph of a function  $y = f(x)$ ,  $a \leq x \leq b$ , then the area of the surface obtained by revolving the curve about the  $x$ -axis is

$$S_x = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

and the area of the surface obtained by revolving the curve about the  $y$ -axis is

$$S_y = \int_a^b 2\pi x \sqrt{1 + (f'(x))^2} dx$$

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If the curve is the graph of a function  $x = g(y)$ ,  $c \leq y \leq d$ , then the area of the surface obtained by revolving the curve about the  $x$ -axis is

$$S_x = \int_c^d 2\pi y \sqrt{1 + (g'(y))^2} dy$$

and the area of the surface obtained by revolving the curve about the  $y$ -axis is

$$S_y = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} dy$$

**Example:** Let  $y = \sqrt{r^2 - x^2}$ ,  $-r \leq x \leq r$  be rotated about the  $x$ -axis. The surface obtained is a sphere of radius  $r$ . Its area is (using  $y' = \frac{-x}{\sqrt{r^2 - x^2}}$ )

$$\begin{aligned} S_x &= \int_{-r}^r 2\pi y \sqrt{1 + (y')^2} dx = 2 \int_0^r 2\pi \sqrt{r^2 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2} dx = \\ &4\pi \int_0^r \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx = 4\pi \int_0^r \sqrt{r^2 - x^2} \sqrt{\frac{r^2}{r^2 - x^2}} dx = \\ &4\pi \int_0^r r dx = \mathbf{4\pi r^2} \end{aligned}$$

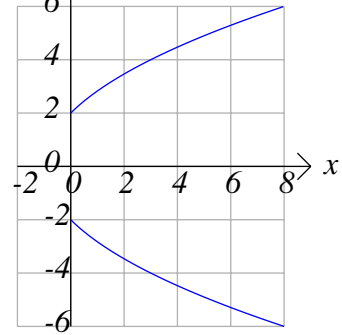
**Example:** Let  $y = \sqrt{r^2 - x^2}$ ,  $a \leq x \leq a + h$  be rotated about the  $x$ -axis. The surface obtained is a region of the sphere of radius  $r$  lying between two vertical and parallel planes. Its area is (again using  $y' = \frac{-x}{\sqrt{r^2 - x^2}}$ )

$$\begin{aligned} S_x &= \int_a^{a+h} 2\pi y \sqrt{1 + (y')^2} dx = \int_a^{a+h} 2\pi \sqrt{r^2 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2} dx = \\ &2\pi \int_a^{a+h} 2\pi \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx = 2\pi \int_a^{a+h} \sqrt{r^2 - x^2} \sqrt{\frac{r^2}{r^2 - x^2}} dx = \\ &2\pi \int_a^{a+h} r dx = \mathbf{2\pi rh} \end{aligned}$$

Thus we have the interesting fact that the area of the region obtained by slicing a sphere with two parallel planes a constant distance apart does not depend on the distance of the planes from the centre of the sphere.

### Example:(#6, p.552, green Stewart)

Find the area of the surface obtained by rotating the curve  $y^2 = 4x + 4$ ,  $0 \leq x \leq 8$ , about the  $x$ -axis.



**Solution 1:** We have  $y = 2(x + 1)^{\frac{1}{2}}$ , so  $y' = 2 \cdot \frac{1}{2}(x + 1)^{-\frac{1}{2}} = (x + 1)^{-\frac{1}{2}}$ ,

$$\text{and } S_x = \int_0^8 2\pi(2)(x + 1)^{\frac{1}{2}} \sqrt{1 + \left((x + 1)^{-\frac{1}{2}}\right)^2} dx =$$

$$4\pi \int_0^8 (x + 1)^{\frac{1}{2}} \sqrt{1 + \frac{1}{x + 1}} dx = 4\pi \int_0^8 (x + 2)^{\frac{1}{2}} dx = 4\pi \left. \frac{(x + 2)^{\frac{3}{2}}}{\frac{3}{2}} \right|_0^8 = \frac{8\pi}{3} (x + 2)^{\frac{3}{2}} \Big|_0^8 = \frac{8\pi}{3} \left[ (8 + 2)^{\frac{3}{2}} - (0 + 2)^{\frac{3}{2}} \right] = \frac{16\pi}{3} (5\sqrt{5} - 1) \sqrt{2}$$

**Solution 2:** We have  $x = \frac{y^2 - 4}{4} = \frac{1}{4}y^2 - 1$ , so  $x' = \frac{y}{2}$ ,

$$\text{and } S_x = \int_{y=2}^{y=6} 2\pi y \sqrt{1 + (x')^2} dy = 2\pi \int_{y=2}^{y=6} y \sqrt{1 + \left(\frac{y}{2}\right)^2} dy =$$

$$\pi \int_{y=2}^{y=6} y \sqrt{4 + y^2} dy = (\text{letting } y = 2 \tan \theta) =$$

$$\pi \int_{\theta=\arctan 1}^{\theta=\arctan 3} 2 \tan \theta \sqrt{4 + 4 \tan^2 \theta} 2 \sec^2 \theta d\theta =$$

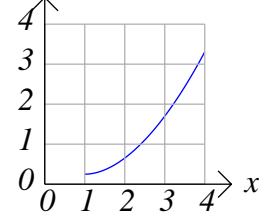
$$\pi \int_{\theta=\arctan 1}^{\theta=\arctan 3} 2 \tan \theta \sqrt{4 \sec^2 \theta} 2 \sec^2 \theta d\theta =$$

$$\pi \int_{\theta=\arctan 1}^{\theta=\arctan 3} 2 \tan \theta (2 \sec \theta) 2 \sec^2 \theta d\theta =$$

$$8\pi \int_{\theta=\frac{\pi}{4}}^{\theta=\arctan 3} \sec^2 \theta \sec \theta \tan \theta d\theta = (\text{letting } u = \sec \theta) =$$

$$8\pi \int_{u=\sqrt{2}}^{u=\sqrt{10}} u^2 du = 8\pi \left. \frac{u^3}{3} \right|_{u=\sqrt{2}}^{u=\sqrt{10}} = \frac{8}{3}\pi (10\sqrt{10} - 2\sqrt{2}) = \frac{16\pi}{3} (5\sqrt{5} - 1) \sqrt{2}$$

### Example:(#8, p.552), green Stewart)



Find the area of the surface obtained by rotating the curve  $y = \frac{x^2}{4} - \frac{\ln x}{2}$ ,  $1 \leq x \leq 4$ , about the  $x$ -axis.

**Solution:** We have  $y' = \frac{x}{2} - \frac{1}{2x}$ ,

$$\text{and } S_x = \int_1^4 2\pi \left( \frac{x^2}{4} - \frac{\ln x}{2} \right) \sqrt{1 + \left( \frac{x}{2} - \frac{1}{2x} \right)^2} dx =$$

$$2\pi \int_1^4 \left( \frac{x^2}{4} - \frac{\ln x}{2} \right) \sqrt{\left( \frac{x}{2} + \frac{1}{2x} \right)^2} dx =$$

$$2\pi \int_1^4 \left( \frac{x^2}{4} - \frac{\ln x}{2} \right) \left( \frac{x}{2} + \frac{1}{2x} \right) dx =, \text{ etc....}$$

**Example:(#6, p.524), brown Stewart** Find the area of the surface obtained by rotating the curve  $y = \cos x$ ,  $0 \leq x \leq \frac{\pi}{3}$ , about the  $x$ -axis.

**Solution:** We have  $y' = -\sin x$ ,

$$\text{and } S_x = \int_{x=0}^{x=\frac{\pi}{3}} 2\pi \cos x \sqrt{1 + (-\sin x)^2} dx = (\text{letting } u = \sin x) =$$

$$2\pi \int_0^{\frac{\sqrt{3}}{2}} \sqrt{1 + u^2} du = (\text{letting } u = \tan \theta) = 2\pi \int_0^{\arctan \frac{\sqrt{3}}{2}} \sec^3 \theta d\theta =$$

$$2\pi \left( \frac{\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|}{2} \right) \Big|_0^{\arctan \frac{\sqrt{3}}{2}} =$$

$$\pi (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \Big|_0^{\arctan \frac{\sqrt{3}}{2}} =$$

$$\pi \left( \frac{\sqrt{7}}{2} \frac{\sqrt{3}}{2} + \ln \left| \frac{\sqrt{7}}{2} + \frac{\sqrt{3}}{2} \right| \right) = \pi \left( \frac{\sqrt{21}}{4} + \ln \frac{\sqrt{7} + \sqrt{3}}{2} \right)$$

**Example:(#8, p.524), brown Stewart** Find the area of the surface obtained by rotating the curve  $y = \frac{3}{2}x^{\frac{2}{3}}$ ,  $1 \leq x \leq 8$ , about the  $x$ -axis.

**Solution:** We have  $y' = x^{-\frac{1}{3}}$ ,

$$\text{and } S_x = \int_1^8 2\pi \frac{3}{2} x^{\frac{2}{3}} \sqrt{1 + (x^{-\frac{1}{3}})^2} dx = 3\pi \int_1^8 x^{\frac{2}{3}} \sqrt{1 + x^{-\frac{2}{3}}} dx =$$

$$3\pi \int_1^8 x^{\frac{2}{3}} \sqrt{\frac{1 + x^{\frac{2}{3}}}{x^{\frac{2}{3}}}} dx = 3\pi \int_{x=1}^{x=8} x^{\frac{1}{3}} \sqrt{1 + x^{\frac{2}{3}}} dx = (\text{letting } u = x^{\frac{1}{3}})$$

$$9\pi \int_{u=1}^{u=2} u^3 \sqrt{u^2 + 1} du = \dots = \frac{3\pi}{5} (50\sqrt{5} - 2\sqrt{2})$$

**Example:(#10, p.524), brown Stewart)** Find the area of the surface obtained by rotating the curve  $x = 1 + 2y^2$ ,  $1 \leq y \leq 2$ , about the  $x$ -axis.

**Solution 1:** We have  $y = \sqrt{\frac{x-1}{2}} = \left(\frac{x-1}{2}\right)^{\frac{1}{2}}$ , so  $y' = \frac{1}{4} \left(\frac{x-1}{2}\right)^{-\frac{1}{2}}$ ,

$$\text{and } S_x = \int_3^9 2\pi \left(\frac{x-1}{2}\right)^{\frac{1}{2}} \sqrt{1 + \left(\frac{1}{4} \left(\frac{x-1}{2}\right)^{-\frac{1}{2}}\right)^2} dx =$$

$$2\pi \int_3^9 \left(\frac{x-1}{2}\right)^{\frac{1}{2}} \sqrt{1 + \frac{1}{8(x-1)}} dx =$$

$$2\pi \int_3^9 \left(\frac{x-1}{2}\right)^{\frac{1}{2}} \sqrt{\frac{8(x-1)+1}{8(x-1)}} dx = \frac{\pi}{2} \int_3^9 \sqrt{8x-7} dx =$$

$$\frac{\pi}{2} \frac{1}{8} \frac{(8x-7)^{\frac{3}{2}}}{\frac{3}{2}} \Big|_3^9 = \frac{\pi}{24} (8x-7)^{\frac{3}{2}} \Big|_3^9 = \frac{\pi}{24} (8(9)-7)^{\frac{3}{2}} - \frac{\pi}{24} (8(3)-7)^{\frac{3}{2}} =$$

$$\frac{\pi}{24} (65\sqrt{65} - 17\sqrt{17})$$