The Substitution Rule

Now that we know why we want to be able to find antiderivatives, we realize that we must develop the techniques of finding them.

One of the most important techniques is called the “Method of Substitution”.

It is essentially the reverse of the Chain Rule.

The basic idea is that an antidifferentiation problem can be simplified by replacing a complicated expression by substituting a new variable for it.

Example: $\int 5x^2 \sqrt[3]{x^3+1}dx$

We see that the expression under the root symbol is a problem, so we let some new symbol reperesent it:

we let $u = x^3 + 1$. Then our problem is to evaluate

$$\int 5x^2 \sqrt[3]{x^3+1}dx = \int 5x^2 \sqrt[3]{u}dx = \int 5x^2 u^{\frac{1}{3}} dx$$

Now this looks simpler, but there is a problem: We have a mixture of variables.

The Umpth Commandment:
Thou shalt not mix thy variables when integrating!

We decided to replace a messy expression involving x, namely $x^3 + 1$ with the simpler expression u.

When we do this, we have to replace **every** other x-expression with the appropriate u-expression.

The most critical x-expression that must be looked after is the dx. In this particular example, we have $u = x^3 + 1$, so if we differentiate and use differentials, we have:

$du = 3x^2 dx$, or $dx = \frac{1}{3x^2} du$. Using this in our integral, we have:

$$\int 5x^2 \sqrt[3]{x^3 + 1}\, dx = \int 5x^2 u^{\frac{1}{3}} dx = \int 5x^2 u^{\frac{1}{3}} \frac{1}{3x^2} du = \frac{5}{3} \int u^{\frac{1}{3}} du =$$

$$\frac{5}{3}\left(\frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right) + C = \frac{5}{3}\left(\frac{u^{\frac{4}{3}}}{\frac{4}{3}}\right) + C = \frac{5}{3}\left(\frac{3}{4}u^{\frac{4}{3}}\right) + C = \frac{5}{4}u^{\frac{4}{3}} = \frac{5}{4}(x^3 + 1)^{\frac{4}{3}} + C$$

Having found what we think might be the antiderivative of $5x^2\sqrt[3]{x^3+1}$, we should double check our answer: we let $F(x) = \frac{5}{4}(x^3+1)^{\frac{4}{3}}$, and we compute

$F'(x)$:

$$F'(x) = \left(\frac{5}{4}(x^3+1)^{\frac{4}{3}}\right)' = \frac{5}{4}\frac{4}{3}\left((x^3+1)\right)^{\frac{4}{3}-1}(x^3+1)' = \frac{5}{3}\left((x^3+1)\right)^{\frac{1}{3}}3x^2 = 5\left((x^3+1)\right)^{\frac{1}{3}}x^2 = 5x^2\sqrt[3]{x^3+1}$$

which is just what we wanted!

The Method of Substitiution is essentially a book-keeping technique. The idea is to substitute one variable for a slightly complicated expression so as to simplify the function being antidifferentiated.

Substitution Rule for Definite Integrals

Now suppose we wish to calculate $\int_0^1 5x^2 \sqrt[3]{x^3+1}dx$

When dealing with substitutions in definite integrals, there is a convenient bookkeeping trick:

instead of starting with $\int_0^1 5x^2 \sqrt[3]{x^3+1}dx$, it is better to use

$$\int_{x=0}^{x=1} 5x^2 \sqrt[3]{x^3+1}dx$$

When we make the substitution $u = x^3 + 1$, we simply observe that $u = 0^3 + 1 = 1$ when $x = 0$, and $u = 1^3 + 1 = 2$ when $x = 0$,

so all we have to do is write

$$\int_{x=0}^{x=1} 5x^2 \sqrt[3]{x^3+1}dx = \int_{u=1}^{u=2} 5x^2 u^{\frac{1}{3}} \frac{1}{3x^2} du = \frac{5}{3}\int_{u=1}^{u=2} u^{\frac{1}{3}} du = \frac{5}{3}\frac{u^{\frac{4}{3}}}{\frac{4}{3}}\Bigg|_{u=1}^{u=2}$$

$$\frac{5}{4}u^{\frac{4}{3}}\Bigg|_{u=1}^{u=2} = \frac{5}{4}1^{\frac{4}{3}} - \frac{5}{4}0^{\frac{4}{3}} = \frac{5}{4}$$

Example 4.5.6(from Text) "Evaluate $\int_0^4 \sqrt{3x+4}dx$"

We let $u = 3x + 4$, and find $du = 3dx$, so $dx = \frac{1}{3}du$.

Also, $u = 3(0) + 4 = 4$ when $x = 0$, and $u = 3(4) + 4 = 16$ when $x = 4$.

Thus we have

$$\int_0^4 \sqrt{3x+4}dx = \int_{x=0}^{x=4} \sqrt{3x+4}dx = \int_{u=4}^{u=16} \sqrt{u}\left(\frac{1}{3}du\right) = \frac{1}{3}\int_{u=4}^{u=16} u^{\frac{1}{2}}du =$$

$$\frac{1}{3}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_{u=4}^{u=16} = \frac{2}{9}u^{\frac{3}{2}}\Bigg|_{u=4}^{u=16} = \frac{2}{9}(16)^{\frac{3}{2}} - \frac{2}{9}(4)^{\frac{3}{2}} = \frac{2}{9}\left(2^4\right)^{\frac{3}{2}} - \frac{2}{9}\left(2^2\right)^{\frac{3}{2}} =$$

$$\frac{2}{9}\left(2^{(4\frac{3}{2})}\right) - \frac{2}{9}\left(2^{(2\frac{3}{2})}\right) = \frac{2}{9}\left(2^6\right) - \frac{2}{9}\left(2^3\right) = \frac{2}{9}2^3\left(2^3 - 1\right) = \frac{16}{9}7 = \frac{112}{9}$$