Sigma (or $\sum$) Notation

In many branches of Mathematics, especially applied Mathematics and/or Statistics, it routinely occurs that one wants to talk about sums of large numbers of measurements of some quantity. For example, one might one to find the average value of a set of readings, or one might wish to know how a set of readings deviates from its average.

Instead of writing out an expression containing thousands, or millions, or billions, of summands, scientists have developed a shorthand notation:

Instead of $10 + 20 + 30 + 40 + \cdots + 5270 + 5280$ we write $\displaystyle\sum_{i=1}^{528} 10i$, In general, instead of

$$a_m + a_{m+1} + a_{m+2} + a_{m+3} + \cdots + a_{n-3} + a_{n-2} + a_{n-1} + a_n,$$

we write

$$\sum_{i=m}^{i=n} a_i \text{ or, even more briefly, } \sum_{i=m}^{n} a_i;$$

instead of

$$x_m + x_{m+1} + x_{m+2} + x_{m+3} + \cdots + x_{n-3} + x_{n-2} + x_{n-1} + x_n,$$

we write $\sum\limits_{j=m}^{j=n} x_j$ or $\sum\limits_{j=m}^{n} x_j$.

Instead of

$f(m) + f(m+1) + f(m+2) + f(m+3) + \cdots + f(n-3) + f(n-2) + f(n-1) + f(n)$, we write

$$\sum_{k=m}^{k=n} f(k) \text{ or } \sum_{k=m}^{n} f(k).$$

The notation $\sum\limits_{k=m}^{k=n} f(k)$ consists of four components:

first , there is the Σ which tells us that a sum is to be taken over a range of values,

second , there is the $k = m$ under the Σ which tells us that the **summation index** is k and that k starts out at the **lower summation limit** m ,

third , there is the $k = m$, or just m over the Σ which tells us that k stops at the **upper summation limit** n ,

fourth , there is the $f(k)$ which gives us a formula for the summands in terms of the summation index k.

It is required that the summation limits be integers and that the summation index increases by 1 as it runs through all integer values between the lower and upper summation limits.

The **number of summands** is $n - m + 1$, and the **average value** of the summands is

$$\frac{\sum_{i=m}^{n} f(i)}{n - m + 1} =$$

$$\frac{f(m) + f(m+1) + f(m+2) + \cdots + f(n-2) + f(n-1) + f(n)}{n - m + 1}$$

Such averages are often denoted using the Greek letter μ (pronounced “mew”).

Another quantity which is often of interest is the **standard deviation** :

the deviation of a reading $f(i)$ from its mean μ is $|f(i) - \mu|$. The average of these deviations is

$$\frac{\sum_{i=m}^{n} |f(i) - \mu|}{n - m + 1}$$

For very good technical reasons, this is seldom used. Instead, the common practice is to use the square root of the average of the squares of the deviations:

$$\sigma = \sqrt{\frac{\sum_{i=m}^{n} |f(i) - \mu|^2}{n - m + 1}}$$

is called the **standard deviation** , or **root-mean square** of the values $f(i)$.

The lower case Greek letter σ is pronounced “sigma”.
The upper case Greek letter Σ is pronounced exactly the same, but it can, and often is, read as “the sum of … as i runs from m to n”.

Examples

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

$$\sum_{i=1}^{10} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 =$$

$$1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 = \frac{10(10+1)(21)}{6} = 365$$

$$\sum_{i=1}^{10} 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 10$$

Some Special Sums:

The sums of the powers of the first n integers are of special interest: we write:

$$S_n^k = \sum_{i=1}^{n} i^k,$$

so that

$$S_n^0 = \sum_{i=1}^{n} i^0 = 1^0 + 2^0 + 3^0 + \cdots + (n-1)^0 + n^0 = 1 + 1 + 1 + \cdots + 1 + 1 = n,$$

just tells us that the sum of n 1's is n, and

$$S_n^1 = \sum_{i=1}^{n} i^1 = 1^1 + 2^1 + 3^1 + \cdots + (n-1)^1 + n^1 = \frac{n(n+1)}{2},$$ is just a special case of the formula for an Arithmetic Progression

$$S_n^2 = \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6},$$

$$S_n^3 = \sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

Also, it is useful to know the basic formula for a

Geometric Progression :

$$\sum_{i=0}^{n} x^i = x^0 + x^1 + x^2 + x^3 + \cdots + x^{n-2} + x^{n-1} + x^n = \frac{x^{n+1} - 1}{x - 1},$$

or

$$\sum_{i=0}^{n} x^i = 1 + x + x^2 + x^3 + \cdots + x^{n-2} + x^{n-1} + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

This is easily derived by computing the product of

$x - 1$ and $1 + x + x^2 + x^3 + \cdots + x^{n-2} + x^{n-1} + x^n$:

$$(x-1)(1 + x + x^2 + x^3 + \cdots + x^{n-2} + x^{n-1} + x^n) =$$

$$\begin{aligned}&x(1 + x + x^2 + x^3 + \cdots + x^{n-2} + x^{n-1} + x^n)\\ &-(1 + x + x^2 + x^3 + \cdots + x^{n-2} + x^{n-1} + x^n) =\end{aligned}$$

$$\begin{array}{rl} x + x^2 + x^3 + \cdots + x^{n-2} + x^{n-1} + x^n + x^{n+1} & \\ -1 - x - x^2 - x^3 - \cdots - x^{n-2} - x^{n-1} - x^n \qquad & = x^{n+1} - 1 \end{array}$$

Properties

$$\sum_{i=m}^{n}(a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

$$\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$$

$$\sum_{i=m}^{n} a_i = \sum_{i=m}^{p} a_i + \sum_{i=p}^{n} a_i \text{ if } m \le p \le n$$

Note that the ranges of summation in first two equations are identical. If they are different much extra care must be taken.

Examples:

Sums are usually evaluated by reducing them to one or more of the above forms by algebraic manipulation:

$$\sum_{i=0}^{6} 2^i = 1 + 2 + 4 + 8 + 16 + 32 + 64 = \frac{2^{6+1} - 1}{2 - 1} = \frac{128 - 1}{1} = 127$$

$$\sum_{i=0}^{30} 5^{i+2} = \sum_{i=1}^{30} 5^i 5^2 = 25 \sum_{i=1}^{30} 5^i = 25 \frac{5^{30+1} - 1}{5 - 1} = \frac{25}{4}(5^{31} - 1)$$

$$\sum_{i=0}^{40} 4^{-i+2} = \sum_{i=1}^{40} 4^{-i} 4^2 = 16 \sum_{i=1}^{40} \left(\frac{1}{4}\right)^i = 16 \frac{\left(\frac{1}{4}\right)^{40+1} - 1}{\left(\frac{1}{4}\right) - 1} = 16 \frac{4^{-41} - 1}{-\frac{3}{4}} = \frac{64}{3}(1 - 4^{-41})$$

$$\sum_{i=10}^{20} 2^i = \sum_{i=0}^{10} 2^{i+10} = 2^{10} \sum_{i=0}^{10} 2^i =$$

$$1024(1+2+4+8+16+32+64+128+256+512+1024) = 1024\frac{2^{10+1}-1}{2-1} =$$

$$1024\frac{2048-1}{1} = 1024 \times 2047 = 2,096,128$$

$$\sum_{i=10}^{20} i = \sum_{i=1}^{20} i - \sum_{i=1}^{9} i = \frac{20(20+1)}{2} - \frac{9(9+1)}{2} = \frac{20(20+1)}{2} - \frac{9(9+1)}{2} =$$

$$210 - 45 =$$

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$$\sum_{i=20}^{40} i^2 = \sum_{i=1}^{40} i^2 - \sum_{i=1}^{19} i^2 = \frac{40(40+1)(2(40)+1)}{6} - \frac{19(19+1)(2(19)+1)}{6} = \frac{40(41)(81)}{6} - \frac{19(20)(39)}{6} = 20(41)(27) - 19(10)(13) = 10[2(41)(27) - 19(13)] =$$

$$10[2214 - 247] = 10(1967) = 19,670$$

Telescoping Sums

It often happens that sums can be collapsed nicely, for example:

$$\sum_{k=1}^{n} (k+1)^m - k^m =$$

$$(2^m - 1^m) + (3^m - 2^m) + (4^m - 3^m) + \cdots + (n^m - (n-1)^m) + ((n+1)^m - n^m) =$$

$$(n+1)^m - 1$$

In general, we have a sum of the form

$$\sum_{i=1}^{n} (a_{i+1} - a_1) = (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) + (a_{n+1} - a_n) = \qquad a_{n+1} - a_1$$

A Very Useful Technique

It is often necessary to work with two sums whose ranges of summation are slightly different. If the two sums have a differing number of terms, we must split off a few terms from the one with the largest number of terms so as to have a fit:

For example:

$$\sum_{i=1}^{i=n} i^2 + \sum_{i=1}^{n+2} i^2 = \sum_{i=1}^{i=n} i^2 + \sum_{i=1}^{n} i^2 + (n+1)^2 + (n+2)^2 = 2\sum_{i=1}^{i=n} i^2 + (n+1)^2 + (n+2)^2 = 2\frac{n(n+1)(2n+1)}{6} + (n+1)^2 + (n+2)^2$$

Another possibility is that although the two sums have the same number of terms, the ranges are offset. For example, we might wish to evaluate

$$\sum_{i=1}^{i=n} a_i + \sum_{i=3}^{i=n+2} a_i.$$

Note that both sums have the same number of terms .

We have to adjust one of the sums so that its summation range equals that of the other. We will choose to work on $\sum_{i=3}^{n+2} a_i$ and we will introduce a new summation index $k = i - 2$, so that $k = 1$ when $i = 3$, and $k = n$ when $i = n + 2$.

We of course can solve for i in terms of k: $i = k + 2$, so we replace every occurrence of i in $\sum_{i=3}^{n+2} a_i$ with $k + 2$:

$$\sum_{i=3}^{i=n+2} a_i = \sum_{k+2=3}^{k+2=n+2} a_{k+2} = \sum_{k=1}^{k=n} a_{k+2}.$$

Now it doesn't matter what symbol we use for our summation index, so we write:

$$\sum_{k=1}^{k=n} a_{k+2} = \sum_{i=1}^{i=n} a_{i+2}.$$

Using this in our original expression, we get

$$\sum_{i=1}^{i=n} a_i + \sum_{i=3}^{i=n+2} a_i = \sum_{i=1}^{i=n} a_i + \sum_{i=1}^{i=n} a_{i+2} = \sum_{i=1}^{i=n} (a_i + a_{i+2}).$$

Example:

$$\sum_{i=1}^{i=n} i^2 + \sum_{i=3}^{i=n+2} i^2 = \sum_{i=1}^{i=n} i^2 + (i+2)^2 = \sum_{i=1}^{i=n} i^2 + i^2 + 2i + 1 =$$

$$2\sum_{i=1}^{i=n} i^2 + 2\sum_{i=1}^{i=n} i + \sum_{i=1}^{i=n} 1 = 2\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} + n =$$

$$n\frac{2n^2+6n+7}{3}$$