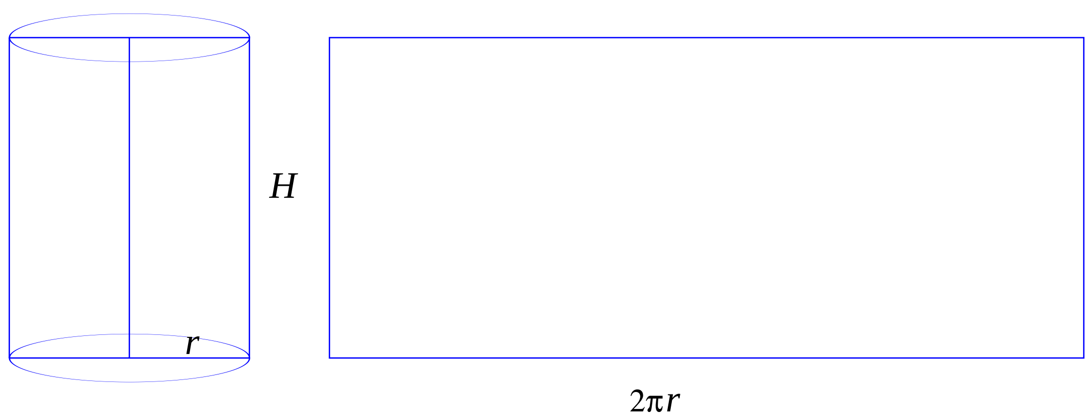
Volumes by Cylindrical Shells

A **cylindrical shell** is a region contained between two cylinders of the same height with the same central axis. We usually denote the height of the cylinders by H, the radius of the inner cylinder by r, and the thickness of the shell by t, so that the radius of the larger cylinder is $r + t$. The volume of the shell is then the difference between the volumes of the two cylinders:

$$V = \pi(r + t)^2H - \pi r^2H = \pi[(r + t)^2 - r^2]H = \pi(2r + t)tH$$



Notice that if t is very small, then $V = 2\pi r t H + \pi t^2 H \doteq 2\pi r t H$

Problem: Suppose we rotate a region
$\mathcal{R} = \{(x, y) | a \leq x \leq b, \text{ and } g(x) \leq y \leq f(x)\}$
about the line $x = k$ which lies to the left of $\mathcal{R}$. What is the volume of the resulting solid?

Solution: Let $\mathcal{R}_c$ be the region $\mathcal{R}_c = \{(x, y) | a \leq x \leq c, \text{ and } g(x) \leq y \leq f(x)\}$

Then the solid obtained by rotating $\mathcal{R}_{c+h} - \mathcal{R}_c$ about the line $x = k$ is a "cylindrical shell with uneven top and bottom lips" with inner radius $c-k$, outer radius $c+h-k$, thickness h, and height about $f(c)-g(c)$ (if f and g are continous at c.) Thus its volume is about $2\pi(c-k)(f(c)-g(c))h$: think of it as sliced along a vertical seam and flattened into a rectangular solid of thickness h, length equal to the circumference $2\pi(c-k)$ of the shell, and height $f(c)-g(c)$ the same as that of the shell.

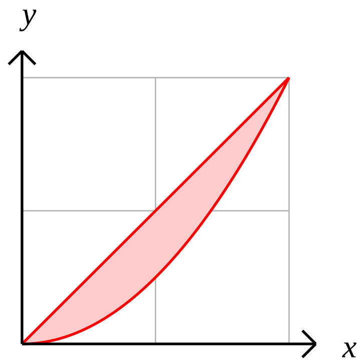
Let $V(x)$ be the volume of the solid obtained by rotating $\mathcal{R}_x$ about the line $x = k$.

We calculate the derivative of $V(x)$:
$V'(x) = \lim_{h\to 0} \frac{V(x+h) - V(x)}{h} = \lim_{h\to 0} \frac{2\pi(x^*(h) - k)[f(x^*(h)) - g(x^*(h))]h}{h} =$
$\lim_{h\to 0} 2\pi(x^*(h) - k)[f(x^*(h)) - g(x^*(h))] = 2\pi(x-k)[f(x) - g(x)]$
where $x^*(h)$ is defined to be the least number $x$in $[x, x+h]$ for which
$V(x+h) - V(x) = 2\pi(x-k)[f(x) - g(x)]h$.
Thus $V(x)$ is an antiderivative of $2\pi(x-k)[f(x) - g(x)]$.

From this we get a basic formula derived from what is called the **Method of Cylindrical Shells**:

$$V = 2\pi \int_a^b (x-k)[f(x) - g(x)]dx \qquad \text{(Vertical Axis Formula)}$$

Example: The region $\mathcal{R} = \left\{(x, y) | 0 \leq x \leq 1, \text{ and } x^2 \leq y \leq x\right\}$ is to be rotated about the line $x = -5$. What is the volume of the resulting solid?



Solution: We have $k = -5$, $f(x) = x$, and $g(x) = x^2$, so

$$V = 2\pi \int_0^1 (x - (-5))(x - x^2)dx = 2\pi \int_0^1 (x + 5)(x - x^2)dx =$$
$$2\pi \int_0^1 x^2 - x^3 + 5x - 5x^2 dx = 2\pi \int_0^1 -x^3 - 4x^2 + 5x dx =$$
$$2\pi \left(-\frac{x^4}{4} - 4\frac{x^3}{3} + 5\frac{x^2}{2} \right) \Bigg|_0^1 = 2\pi \left(-\frac{1}{4} - \frac{4}{3} + \frac{5}{2} \right) = \frac{11\pi}{6},$$

which agrees with our previous calculation.

Horizontal Axis of Rotation

Problem: Suppose the region is described in terms of functions of y:
$\mathcal{R} = \{(x, y) | c \leq y \leq d, \text{ and } g(y) \leq x \leq f(y)\}$
and the axis of rotation is the line $y = k$ which lies below $\mathcal{R}$. The volume of the resulting solid is given by

$$V = 2\pi \int_c^d (y - k)[f(y) - g(y)]dy \qquad \text{(Horizontal Axis Formula)}$$

Example: The same region is to be rotated about the line $y = -5$. Find the volume generated.

We again rewrite the region in terms of functions of y:

$$\mathcal{R} = \left\{(x, y)|0 \le y \le 1, \text{ and } y \le x \le y^{\frac{1}{2}}\right\}.$$

We have $V = 2\pi \int_0^1 (y+5)(y^{\frac{1}{2}} - y)dy =$

$$2\pi \int_0^1 -y^2 + y^{\frac{3}{2}} - 5y + 5y^{\frac{1}{2}}dy =$$

$$2\pi \left(-\frac{y^3}{3} + \frac{y^{\frac{5}{2}}}{\frac{5}{2}} - 5\frac{y^2}{2} + 5\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_0^1 = 2\pi \left(-\frac{1}{3} + \frac{1}{\frac{5}{2}} - 5\frac{1}{2} + 5\frac{1}{\frac{3}{2}}\right) =$$

$$2\pi \left(3 + \frac{4}{10} - \frac{25}{10}\right) = 2\pi \left(3 - \frac{21}{10}\right) = \frac{9\pi}{5}$$

which again agrees with our previous calculation.