Integration By Parts

There is NO formula for $\int f(x)g(x)dx$.

It almost never happens that $\int f(x)g(x)dx = \left(\int f(x)dx\right)\left(\int g(x)dx\right)$

Notice that $\int df = f(x) + C$. We often shorten this to $\int df = f$ to indicate that the integral and differential operators "cancel" each other.

The Product Rule for derivatives, $$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + \frac{d}{dx}(u)v$$

has no simple counterpart for antiderivatives. It can be restated in terms of differentials as $d(uv) = udv + vdu$, and if we apply indefinite integral signs, we get $\int d(uv) = \int udv + \int vdu$, or $uv = \int udv + \int vdu$.

We usually use the equivalent formula $\int udv = uv - \int vdu$.

Example: Evaluate $\int x \sin x dx$

Solution: Use integration by parts, with $u = x$, and $dv = \sin x dx$.
Then $du = dx$, and $v = -\cos x$, so

$$\int x \sin x dx = \left(\int u dv = uv - \int v du\right) = x(-\cos x) - \int(-\cos x)dx =$$

$$-x \cos x + \int \cos x dx = -x \cos x + \sin x + C$$

We can also use this technique with definite integrals:

Evaluate $\int_0^{\pi} x \sin x dx$

Solution: $\int_0^{\pi} x \sin x dx = \left(\int u dv = uv - \int v du\right) =$

$x(-\cos x)|_0^{\pi} - \int_0^{\pi} (-\cos x) dx =$

$-x \cos x|_0^{\pi} + \int_0^{\pi} \cos x dx = -x \cos x|_0^{\pi} + \sin x|_0^{\pi} =$

$-\pi \cos \pi - (-0 \cos 0) + \sin \pi - \sin 0 = -\pi(-1) = \pi$

Question: How do I know what u and dv should be?

Students first encountering the technique of using the equation $\int u dv = uv - \int v du$ have trouble knowing what to take for u and what to take for dv.

Answer: Get lots of experience. This is an area where we learn a lot from experience. The Integration by Parts technique is characterized by the need to select u from a number of possibilities. Once u has been chosen, dv is determined, and we hope for the best.

The basic idea underlying Integration by Parts is that we hope that in going from $\displaystyle\int u dv$ to $\displaystyle\int v du$ we will end up with a simpler integral to work with. In the example we have just seen, we were lucky.

Let's try it again, the unlucky way:

Example: Evaluate $\int (\sin x) x \, dx$

Solution: Use integration by parts, with $u = \sin x$, and $dv = x \, dx$.
Then $du = \cos x \, dx$, and $v = \frac{x^2}{2}$, so

$$\int (\sin x) x \, dx = \left(\int u \, dv = uv - \int v \, du \right) = \sin x \frac{x^2}{2} - \int \frac{x^2}{2} \cos x \, dx =$$

$$\frac{x^2}{2} \sin x - \frac{1}{2} \int x^2 \cos x \, dx$$

which involves a tougher looking integral than we started with.

As a rule of thumb, one-third of the possible choices will lead to an easier integral, one-third will lead to a harder one, and one-third will lead to one of equal difficulty.

It is also possible to spin your wheels, and go around in circles, as we shall soon see.

Let's try a strategic approach to our example: u has to be selected so that $udv = x\sin x dx$, so we look at the possible choices for u: x, $\sin x$, or $x\sin x$. Once u is selected, we have $dv = \frac{x\sin x dx}{u}$, and all we have to is find du(easy) and $v = \int dv$ (possibly very hard or impossible). Then we try to decide if we can get anywhere with $\int v du$. Thus, we can let u be any factor of $f(t)$, including 1, and the corresponding dv is determined. (Of course, if we let $u = 1$, the problem of finding v is just our original integration problem, so we will omit it.) If we cannot then find v we know we have a non-viable selection of the pair u and dv.

We shall illustrate the rather inefficient technique of examining all the possibilities and discarding the non-viable ones in the following examples. Organizing our information in a table is helpful:

u	$dv = \frac{x\sin x dx}{u}$	$v = \int dv$	du	$v du$	$\int v du$	Better?
x	$\frac{x\sin x dx}{x} = \sin x dx$	$\int \sin x dx = -\cos x$	dx	$-\cos x dx$	$\int -\cos x dx = -\sin x$	YES!
$\sin x$	$\frac{x\sin x dx}{\sin x} = x dx$	$\int x dx = \frac{x^2}{2}$	$\cos x dx$	$\frac{x^2}{2}\cos x dx$	$\int \frac{x^2}{2}\cos x dx$	NO!
$x\sin x$	$\frac{x\sin x dx}{x\sin x} = dx$	$\int dx = x$	$\sin x + x\cos x$	$x\sin x + x^2\cos x$	$\int x(\sin x + x\cos x)dx$	NO!

There are some simple integrals where little choice is available: knowing which of a large number techniques to use is crucial.

Example: $\int \ln x dx$ obviously requires

$u = \ln x$, $dv = dx$, so that $v = x$ and $du = \dfrac{dx}{x}$.

$$\int \ln x dx = \left(\int u dv = uv - \int v du \right) = (\ln x)x - \int x \frac{dx}{x} = x \ln x - \int dx =$$

$x \ln x - x + C$

Example: $\int \arctan x dx$ also obviously requires

$u = \arctan x$, $dv = dx$, so that $v = x$ and $du = \dfrac{dx}{1+x^2}$

$$\int \arctan x dx = \left(\int u dv = uv - \int v du \right) = (\arctan x)x - \int x \frac{dx}{1+x^2} = x \arctan x - \frac{1}{2} \int \frac{2x dx}{1+x^2} = x \arctan x - \frac{1}{2} \ln(1+x^2) + C =$$

$x \arctan x - \ln \sqrt{1+x^2} + C$

Indefinite Integration of e^{kt} times another factor

We now look at a family of integrals which show up in LSD, (Linear Systems Design), particularly in the calculation of Laplace Transforms.

We define $\mathcal{G}(G) = \int e^{kt}G(t)dt$, where G is any function. The types of *continuous* function G that arise in practical situations are often sums and products of polynomials, exponential functions, and sinusoidal functions. Fortunately, we know how to evaluate these using the technique of integration by parts.

Examples:

(1) $G(t) = c$, a constant. Integration by parts is not needed here. Then $\mathcal{G}(c) = \displaystyle\int e^{kt}G(t)dt = \frac{c}{k}e^{kt} + C$

(2) $G(t) = t$. Then $\mathcal{G}(t) = \int e^{kt}t\,dt$.
Here $f(t) = e^{kt}t$ has four different possible factorizations:

u	$dv = \frac{f(t)dt}{u}$	v	du	vdu	Viable?
t	$e^{kt}dt$	$\frac{1}{k}e^{kt}$	dt	$\frac{1}{k}e^{kt}dt$	Yes!
e^{kt}	tdt	t^2	$ke^{kt}dt$	$kt^2e^{kt}dt$	No!
$e^{kt}t$	dt	t	$(t+k)e^{kt}dt$	$t(t+k)e^{kt}dt$	No!

We use the one viable factorization, $u = t$, $v = e^{kt}dt$:
$\int t(e^{kt}dt) = \int udv = uv - \int vdu =$
$t\frac{1}{k}e^{kt} - \int \frac{1}{k}e^{kt}dt = \frac{t}{k}e^{kt} - \frac{1}{k^2}e^{kt} + C$
$= \left(\frac{t}{k} - \frac{1}{k^2}\right)e^{kt} + C$

(3) $G(t) = t^2$. Then $\mathcal{G}(t^2) = \int e^{kt}t^2dt$, so $f(t) = e^{kt}t^2$. This has four possible factorizations:

u	$dv = \frac{f(t)dt}{u}$	v	du	vdu	Viable?
t	$e^{kt}tdt$	$\left(\frac{t}{k} - \frac{1}{k^2}\right)e^{kt}$	dt	$\left(\frac{t}{k} - \frac{1}{k^2}\right)e^{kt}dt$	Yes, but messy
t^2	$e^{kt}dt$	$\frac{1}{k}e^{kt}$	$2tdt$	$\frac{2}{k}te^{kt}dt$	Yes!
e^{kt}	t^2dt	$\frac{1}{3}t^3$	$ke^{kt}dt$	$\frac{k}{3}t^3e^{kt}dt$	No!
$e^{kt}t$	tdt	$\frac{1}{2}t^2$	$(t+k)e^{kt}dt$	$\frac{1}{2}t^2(t+k)e^{kt}dt$	No!
$e^{kt}t^2$	dt	t	$(kt^2+2t)e^{kt}dt$	$t(kt^2+2t)e^{kt}dt$	No!

We see that if we use the factorization $u = t^2$, $dv = e^{kt}dt$, we get $\int t^2e^{kt}dt = \int udv = uv - \int vdu =$
$\frac{t^2}{k}e^{kt} - \frac{2}{k}\int te^{kt}dt =$
$\frac{t^2}{k}e^{kt} - \frac{2}{k}\left(\frac{t^2}{k} - \frac{2}{k}\right)e^{kt} =$
$\left(\frac{t^2}{k} - \frac{2t}{k^2} + \frac{2}{k^3}\right)e^{kt} + C$

(4) We notice how the problem of evaluating $\int t^2e^{kt}$ was reduced to the evaluation of $\int e^{kt}tdt$, which had just been done. We suspect the existence of a *reduction formula* for $\int e^{kt}t^ndt$. Having been successful in taking $dv = e^{kt}dt$ in the two preceding examples, we decide to do this again, and we have $u = \frac{e^{kt}t^ndt}{e^{kt}dt} = t^n$. We calculate $v = \frac{1}{k}e^{kt}$ and $du = nt^{n-1}dt$, so that:
$\int e^{kt}t^ndt = \int udv = uv - \int vdu = \frac{1}{k}t^ne^{kt} - \frac{n-1}{k}\int e^{kt}t^{n-1}dt.$

Thus we have $\mathcal{G}(t^n) = \frac{1}{k}t^ne^{kt} - \frac{n-1}{k}\mathcal{G}(t^{n-1})$.

(5) $G(t) = \sin at$. We have $f(t) = e^{kt}\sin at$, so there are just three choices for u:

u	$dv = \frac{f(t)dt}{u}$	v	du	vdu	Viable?
$\sin at$	$e^{kt}dt$	$\frac{1}{k}e^{kt}$	$a\cos atdt$	$\frac{a}{k}e^{kt}\cos atdt$	Yes
$e^{kt}dt$	$\sin atdt$	$-\frac{1}{a}\cos at$	$ke^{kt}dt$	$-\frac{k}{a}e^{kt}\cos atdt$	Yes
$e^{kt}\sin at$	dt	t	$(k\sin at + a\cos at)e^{kt}dt$	$t(k\sin at + a\cos at)e^{kt}dt$	No!

The first two choices are both viable, and we see that they both lead to the evaluation of $\int e^{kt}\cos at\,dt$. We will examine both cases closely:

First Choice: $u_1 = \sin at$, $dv_1 = e^{kt}dt$
We get: $\mathcal{G}(\sin at) = \int e^{kt}\sin at\,dt = \int u_1 dv_1 = u_1 v_1 - \int v_1 du_1 = \sin at\frac{1}{k}e^{kt} - \int \frac{a}{k}e^{kt}\cos at\,dt =$
$\sin at\frac{1}{k}e^{kt} - \frac{a}{k}\int e^{kt}\cos at\,dt =$
$\sin at\frac{1}{k}e^{kt} - \frac{a}{k}\mathcal{G}(\cos at)$
or

$$\mathcal{G}(\sin at) = \sin at\frac{1}{k}e^{kt} - \frac{a}{k}\mathcal{G}(\cos at)$$

In evaluating $\int e^{kt}\cos at\,dt$ we again have three choices, two of which are viable:

u	$dv = \frac{f(t)dt}{u}$	v	du	$v\,du$	Viable?
$\cos at$	$e^{kt}dt$	$\frac{1}{k}e^{kt}$	$-a\sin at\,dt$	$-\frac{a}{k}e^{kt}\sin at\,dt$	Yes
$e^{kt}dt$	$\cos at\,dt$	$\frac{1}{a}\sin at$	$ke^{kt}dt$	$\frac{k}{a}e^{kt}\sin at\,dt$	Yes
$e^{kt}\cos at$	dt	t	$(k\cos at - a\sin at)e^{kt}dt$	$t(k\sin at + a\cos at)e^{kt}dt$	No!

Again we have two choices. First and Best Choice:
We will let $U_1 = \cos at$, $dV_1 = e^{kt}dt$, and we get:
$\mathcal{G}(\cos at) = \int \cos at e^{kt}dt = \int U_1 dV_1 = U_1V_1 - \int V_1 dU_1 =$
$\cos at\frac{1}{k}e^{kt} - \int \frac{1}{k}e^{kt}(-a\sin at\,dt) =$
$\frac{1}{k}\cos at e^{kt} + \frac{a}{k}\int e^{kt}\sin at\,dt =$
$\frac{1}{k}\cos at e^{kt} + \frac{a}{k}\mathcal{G}(\sin at)$,
or

$$\mathcal{G}(\cos at) = \frac{1}{k}\cos at e^{kt} + \frac{a}{k}\mathcal{G}(\sin at)$$

so we are back where we started! However, if we substitute this into the equation

$$\mathcal{G}(\sin at) = \sin at\frac{1}{k}e^{kt} - \frac{a}{k}\mathcal{G}(\cos at)$$

we get

$$\mathcal{G}(\sin at) = \sin at\frac{1}{k}e^{kt} - \frac{a}{k}\left(\frac{1}{k}\cos ate^{kt} + \frac{a}{k}\mathcal{G}(\sin at)\right) = \sin at\frac{1}{k}e^{kt} - \frac{a}{k^2}\cos ate^{kt} - \frac{a^2}{k^2}\mathcal{G}(\sin at)$$

which can be solved for $\mathcal{G}(\sin at)$:

$$(1+\frac{a^2}{k^2})\mathcal{G}(\sin at) = \frac{a^2+k^2}{k^2}\mathcal{G}(\sin at) = \sin at\frac{1}{k}e^{kt} - \frac{a}{k^2}\cos ate^{kt} = e^{kt}(\sin at\frac{k}{k^2} - \frac{a}{k^2}\cos at) = \frac{e^{kt}}{k^2}(k\sin at - a\cos$$

so

$$\mathcal{G}(\sin at) = \frac{k\sin at - a\cos at}{a^2+k^2}e^{kt}$$

Last and Worst Choice: We will now let $U_2 = e^{kt}$, $dV_2 = \cos at$, and we get:

$$\mathcal{G}(\cos at) = e^{kt}(\frac{1}{a}\sin at) - \int\frac{1}{a}\sin at(e^{kt}kdt) = \frac{1}{a}e^{kt}\sin at - \frac{k}{a}\int e^{kt}\sin atdt = \frac{1}{a}e^{kt}\sin at - \frac{k}{a}\mathcal{G}(\sin at)$$

This time, however, if we substitute this into the equation

$$\mathcal{G}(\sin at) = \sin at\frac{1}{k}e^{kt} - \frac{a}{k}\mathcal{G}(\cos at)$$

we get

$$\mathcal{G}(\sin at) = \sin at\frac{1}{k}e^{kt} - \frac{a}{k}\left(\frac{1}{a}e^{kt}\sin at - \frac{k}{a}\mathcal{G}(\sin at)\right) = \mathcal{G}(\sin at)$$

so there is no information gained.

Second Choice: $u_2 = e^{kt}$, $dv_2 = \sin atdt$
We get: $\mathcal{G}(\sin at) = \int e^{kt}\sin atdt = \int u_2dv_2 = u_2v_2 - \int v_2du_2 = e^{kt}(-\frac{1}{a}\cos at) - \int\frac{k}{a}e^{kt}\cos atdt = -\frac{1}{a}e^{kt}\cos at - \frac{k}{a}\mathcal{G}(\cos at)$

or

$$\mathcal{G}(\sin at) = -\frac{1}{a}e^{kt}\cos at - \frac{k}{a}\mathcal{G}(\cos at)$$

Using, from above,

$$\mathcal{G}(\cos at) = \frac{1}{k}\cos ate^{kt} + \frac{a}{k}\mathcal{G}(\sin at)$$

we get

$$\mathcal{G}(\sin at) = -\frac{1}{a}e^{kt}\cos at - \frac{k}{a}\left(\frac{1}{k}\cos ate^{kt} + \frac{a}{k}\mathcal{G}(\sin at)\right) = \mathcal{G}(\sin at)$$

so there is no new information. On the other hand, if we use

$$\mathcal{G}(\cos at) = \frac{1}{a}e^{kt}\sin at - \frac{k}{a}\mathcal{G}(\sin at)$$

we get the same value as before.

(6)$\mathcal{G}(\cos at)$. This is easily derived from the calculations of the previous example.

$$\mathcal{G}(\cos at) = \frac{1}{k}\cos ate^{kt}+\frac{a}{k}\mathcal{G}(\sin at) = \frac{1}{k}\cos ate^{kt}+\frac{a}{k}\left(\frac{k\sin at - a\cos at}{a^2+k^2}e^{kt}\right) = \frac{a\sin at + k\cos at}{a^2+k^2}e^{kt}$$

so

$$\mathcal{G}(\cos at) = \frac{a\sin at + k\cos at}{a^2+k^2}e^{kt}$$

(7)$\mathcal{G}(t^n \sin at) = \int t^n \sin ate^{kt}dt$
Let $u = t^n \sin at$, $dv = e^{kt}dt$, so that $du = (nt^{n-1}\sin at + at^n \cos at)dt$, and $v = \frac{1}{k}e^{kt}$. Then
$\mathcal{G}(t^n \sin at) = \int t^n \sin ate^{kt}dt = \int udv = uv - \int vdu =$
$t^n \sin at(\frac{1}{k}e^{kt}) - \int \frac{1}{k}e^{kt}((nt^{n-1}\sin at + at^n \cos at)dt) =$
$\frac{1}{k}t^n \sin ate^{kt} - \frac{n}{k}\mathcal{G}(t^{n-1}\sin at) - \frac{a}{k}\mathcal{G}(t^n \cos at)$
or

$$\mathcal{G}(t^n \sin at) = \frac{1}{k}t^n \sin ate^{kt} - \frac{n}{k}\mathcal{G}(t^{n-1}\sin at) - \frac{a}{k}\mathcal{G}(t^n \cos at)$$

(8)$\mathcal{G}(t^n \cos at) = \int t^n \cos ate^{kt}dt$. Let $U = t^n \cos at$ and $dV = e^{kt}dt$,
so that $dU = (nt^{n-1}\cos at - at^n \sin at)dt$ and $V = \frac{1}{k}e^{kt}$. Then

$\mathcal{G}(t^n \cos at) = \int t^n \cos at e^{kt} dt = \int U dV = UV - \int V du =$
$t^n \cos at \frac{1}{k} e^{kt} - \int e^{kt}((nt^{n-1} \cos at - at^n \sin at)dt) =$
$\frac{1}{k} t^n \cos at e^{kt} - \frac{n}{k} \mathcal{G}(t^{n-1} \cos at) + \frac{a}{k} \mathcal{G}(t^n \sin at).$

We can now solve for $\mathcal{G}(t^n \sin at)$:

$$\mathcal{G}(t^n \sin at) = \frac{1}{a^2 + k^2} \left[(k \sin at - a \cos at) t^n e^{kt} - n(\mathcal{G}(t^{n-1}(k \sin at - a \cos at))) \right]$$

and hence

$$\mathcal{G}(t^n \cos at) = \frac{1}{a^2 + k^2} \left[(k \cos at - a \sin at) t^n e^{kt} + n(\mathcal{G}(t^{n-1}(k \cos at - a \sin at))) \right]$$