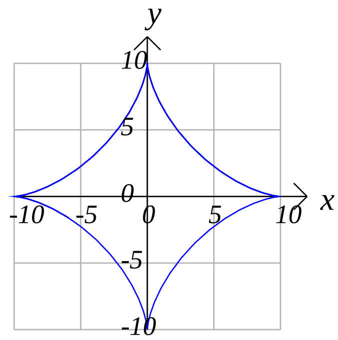
Parametric Equations

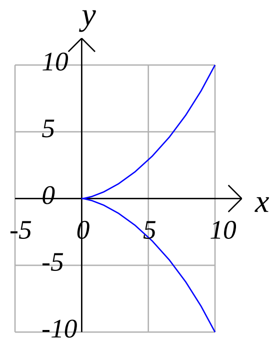
Curves in the plane are often described by giving the x-coordinates and y-coordinates as functions of some parameter such as t.

Examples:

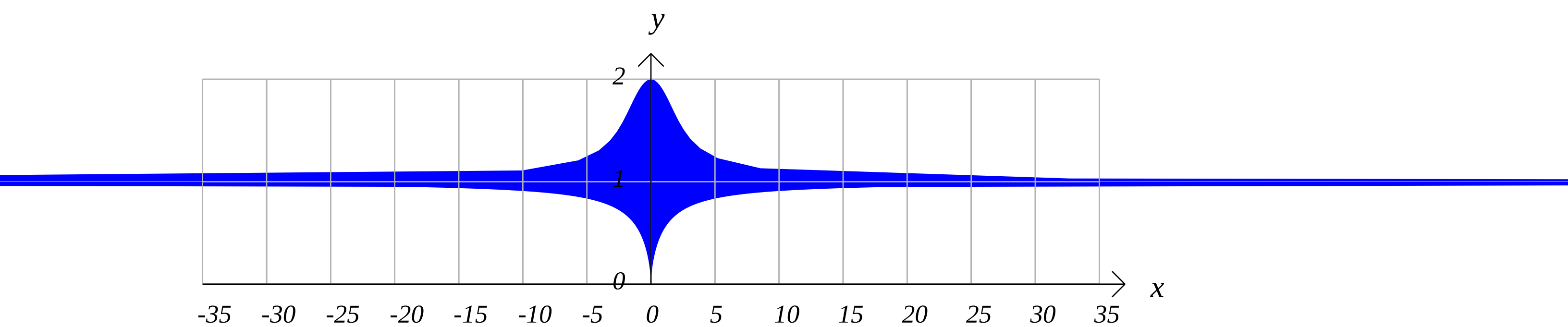
Astroid: $x(t) = a\cos^3 t,\ y(t) = a\sin^3 t$



Cissoid: $x(t) = 2a\dfrac{t^2}{1+t^2},\ y(t) = 2a\dfrac{t^3}{1+t^2}$



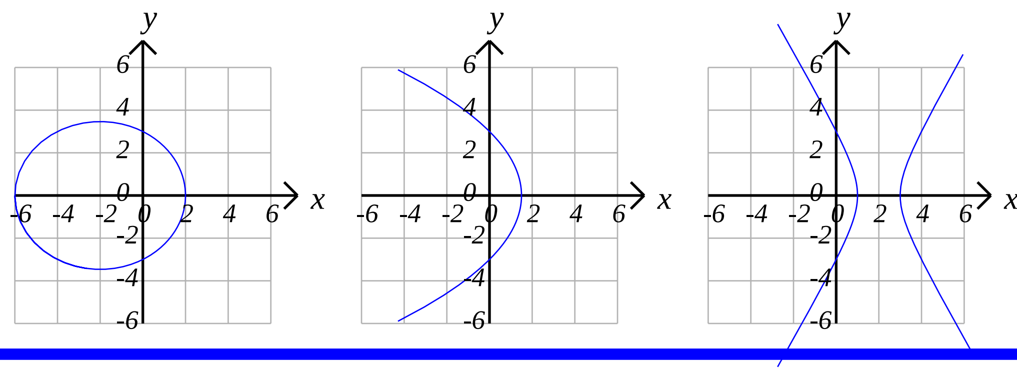
Conchoid: $x(t) = a\tan t + \sin t,\ y(t) = a + \cos t$



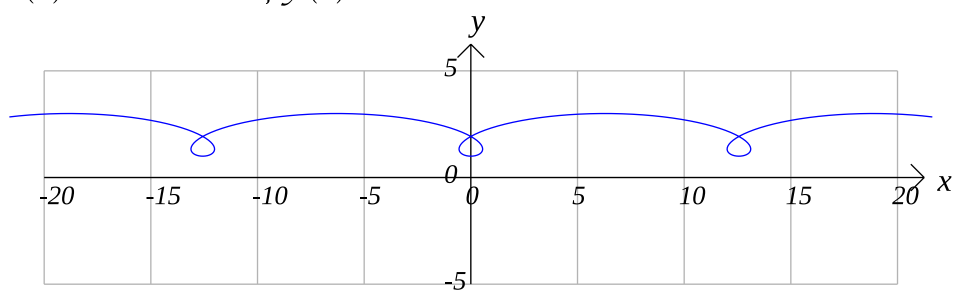
Conic Sections:

The parametric equations of a conic whose axis makes an angle α with the x-axis are, where d is the semi-latus rectum and e is the eccentricity:

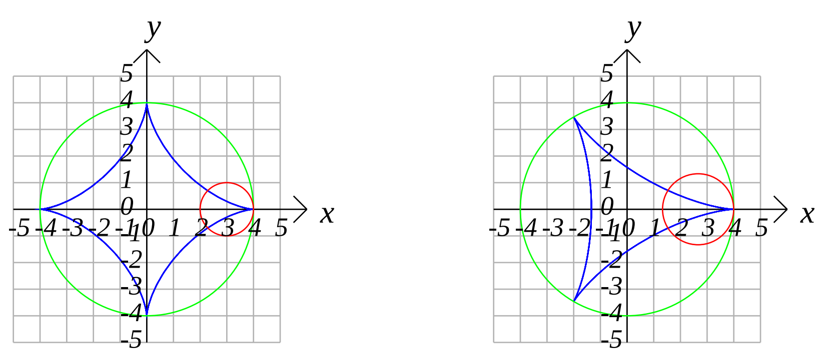
$$x(t) = \frac{d\cos t}{1 + e\cos(t - \alpha)},\ y(t) = \frac{d\sin t}{1 + e\cos(t - \alpha)}$$



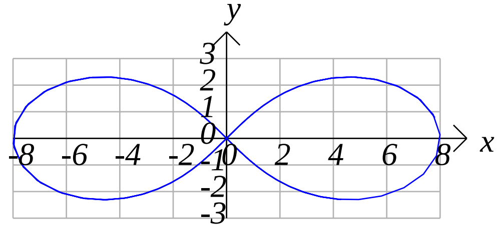
Cycloid: $x(t) = at - b\sin t,\ y(t) = a - b\cos t$



Hypocycloid: $x(t) = (a - b)\cos t + b\cos\frac{b - a}{b}t,\ y(t) = (a - b)\sin t - b\sin\frac{b - a}{b}t$



Lemniscate: $x(t) = 2a\frac{\cos t}{b - \cos 2t}, y(t) = a\frac{\sin 2t}{b - \cos 2t}$



Limacon: $x(t) = (1 + c\sin t)\cos t, y(t) = (1 + c\sin t)\sin t$

Lissajous: $x(t) = a\sin dt, y(t) = b\cos t$

Tractrix: $x(t) = a\sin t, y(t) = \cos t + \log\tan\frac{t}{2}$

Witch of Agnesi: $x(t) = 2a\cot t, y(t) = 2a\sin^2 t$

Those who wish to see much more comprehensive sets of curves should look at:

$http://www-groups.dcs.st-and.ac.uk/\sim history/Java/$

$http://www.best.com/\sim xah/SpecialPlaneCurve\_dir/specialPlaneCurves.html$

$http://www.astro.virginia.edu/\sim eww6n/math/math0.html$

Slope and Concavity

To find the slope and concavity of a curve in the plane described by parametric equations is easy:

We have $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\dot{y}}{\dot{x}}$ and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}} = \frac{\frac{\frac{dx}{dt}\frac{d}{dt}\left(\frac{dy}{dt}\right) - \frac{dy}{dt}\frac{d}{dt}\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}}{\frac{dx}{dt}} =$$

$$\frac{\frac{dx}{dt}\left(\frac{d^2y}{dt^2}\right) - \frac{dy}{dt}\left(\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^3} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^3} = \frac{\begin{vmatrix} \dot{x} & \dot{y} \\ \ddot{x} & \ddot{y} \end{vmatrix}}{\dot{x}^3}$$

where we have used the dot notation for differentiation with respect to t.

Example:

Astroid: $x(t) = a\cos^3 t$, $y(t) = a\sin^3 t$,

$\dot{x} = -3a\cos^2 t\sin t$, $\ddot{x} = -3a\cos t(-2\sin^2 t + \cos^2 t)$,

$\dot{y} = 3a\sin^2 t\cos t$, $\ddot{y} = 3a\sin t(2\cos^2 t - \sin^2 t)$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\frac{\sin t}{\cos t} = -\tan t$$

$$\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^3} =$$

$$\frac{(-3a\cos^2 t\sin t)(3a\sin t(2\cos^2 t - \sin^2 t)) - (-3a\cos t(-2\sin^2 t + \cos^2 t))(3a\sin^2 t\cos t)}{(-3a\cos^2 t\sin t)^3} =$$

$$\frac{-9a^2\sin^2 t\cos^2 t(2\cos^2 t - \sin^2 t) + 9a^2\sin^2 t\cos^2 t(-2\sin^2 t + \cos^2 t))}{-27a^3\cos^6 t\sin^6 t} =$$

$$\frac{(2\cos^2 t - \sin^2 t) - (-2\sin^2 t + \cos^2 t)}{3a\cos^4 t\sin^4 t} = \frac{1}{3a\cos^4 t\sin^4 t}$$

Areas Under Parametric Curves

We can also easily find the area between the x axis and a parametric curve:

$$A = \int_{x=a}^{x=b} y\,dx == \int_{t=t_0}^{t=t_1} y(t)\dot{x}(t)dt$$

Example: Find the area of the **Astroid:** $x(t) = a\cos^3 t$, $y(t) = a\sin^3 t$.

Solution: Note that this is 4 times the area of the astroid lying in the first quadrant, so we have

$$A = 4\int_{x=0}^{x=a} y\,dx = 4\int_{t=\frac{\pi}{2}}^{t=0} \left(a\sin^3 t\right)\left(-3a\cos^2 t\sin t\right) dt =$$

$$12a^2 \int_{t=0}^{t=\frac{\pi}{2}} \sin^4 t\cos^2 t\,dt = 12a^2 \int_{t=0}^{t=\frac{\pi}{2}} \left(\frac{1-\cos 2t}{2}\right)^2 \left(\frac{1+\cos 2t}{2}\right) dt =$$

$$\frac{3}{2}a^2 \int_{t=0}^{t=\frac{\pi}{2}} (1-\cos 2t)(1-\cos 2t)(1+\cos 2t)\,dt =$$

$$\frac{3}{2}a^2 \int_{t=0}^{t=\frac{\pi}{2}} (1-\cos 2t)\left(1-\cos^2 2t\right) dt = \frac{3}{2}a^2 \int_{t=0}^{t=\frac{\pi}{2}} (1-\cos 2t)\sin^2 2t\,dt =$$

$$\frac{3}{2}a^2 \int_{t=0}^{t=\frac{\pi}{2}} \sin^2 2t\,dt - \frac{3}{2}a^2 \int_{t=0}^{t=\frac{\pi}{2}} \sin^2 2t\cos 2t\,dt =$$

$$\frac{3}{2}a^2 \int_{t=0}^{t=\frac{\pi}{2}} \frac{1-\cos 4t}{2} dt - \left(\frac{3}{2}a^2\, \frac{1}{2}\frac{\sin^3 2t}{3}\Bigg|_{t=0}^{t=\frac{\pi}{2}}\right) =$$

$$\frac{3}{4}a^2 \int_{t=0}^{t=\frac{\pi}{2}} 1-\cos 4t\,dt = \frac{3}{4}a^2\; t - \frac{1}{4}\sin 4t\Bigg|_{t=0}^{t=\frac{\pi}{2}} = \frac{3}{8}\pi a^2$$

Arc Length of Parametric Curves

As mentioned previously, the formula for the length of a parametric curve is

$$L = \int_{t=t_0}^{t=t_1} \sqrt{\dot{x}^2 + \dot{y}^2}dt$$

We compute the length of the perimeter of the astroid:

$$L = 4\int_{t=0}^{t=\frac{\pi}{2}} \sqrt{(-3a\cos^2 t\sin t)^2 + \left(3a\sin^2 t\cos t\right)^2}dt =$$

$$L = 4\int_{t=0}^{t=\frac{\pi}{2}} 3a\sqrt{\cos^4 t\sin^2 t + \sin^4 t\cos^2 t}dt =$$

$$L = 12a\int_{t=0}^{t=\frac{\pi}{2}} \sqrt{(\sin^2 t + \cos^2 t)\sin^2 t\cos^2 t}dt =$$

$$L = 12a\int_{t=0}^{t=\frac{\pi}{2}} \sin t\cos t dt =$$

$$L = 6a\int_{t=0}^{t=\frac{\pi}{2}} \sin 2t dt = -3a\cos 2t|_{t=0}^{t=\frac{\pi}{2}} = -3a\left(\cos 2\frac{\pi}{2} - \cos 2(0)\right) =$$

$$-3a(-1-1) = 6a$$

Volumes of Solids of Revolution

Similarly, we can compute volumes of solid of revolution of parametric curves:

For example, if we revolve a curve about the x-axis we have:

$$V = \int_{x=a}^{x=b} \pi y^2 dx$$

Let us do this for the astroid:

$$V = 2\int_{x=0}^{x=a} \pi(a\sin^3 t)^2(-3a\cos^2 t\sin t dt) = -6\pi a^3\int_{t=\frac{\pi}{2}}^{t=0} \left(\sin^2 t\right)^3\cos^2 t\sin t dt =$$

$$6\pi a^3\int_{t=0}^{t=\frac{\pi}{2}} \left(1-\cos^2 t\right)^3\cos^2 t\sin t dt = (\text{ letting } u = \cos t\text{ })$$

$$6\pi a^3\int_{u=1}^{u=0} \left(1-u^2\right)^3 u^2(-du) = 6\pi a^3\int_{u=0}^{u=1} \left(1-3u^2+3u^4-u^6\right)u^2 du =$$

$$6\pi a^3 \int_{u=0}^{u=1} u^2 - 3u^4 + 3u^6 - u^8 du = 6\pi a^3 \left[\frac{u^3}{3} - 3\frac{u^5}{5} + 3\frac{u^7}{7} - \frac{u^9}{9} \Big|_{u=0}^{u=1} \right] =$$

$$6\pi a^3 \left[\frac{1}{3} - 3\frac{1}{5} + 3\frac{1}{7} - \frac{1}{9} \right] = 6\pi a^3 \left[\frac{2}{9} + 3\left(\frac{1}{7} - \frac{1}{5}\right) \right] = 6\pi a^3 \left[\frac{2}{9} + 3\frac{-2}{35} \right] =$$

$$6\pi a^3 \left[\frac{70 - 54}{315} \right] = \frac{32}{105}\pi a^3$$

Surface Areas

As mentioned previously, a formula for the surface area obtained by rotating a parametric curve about the x-axis is

$$S = \int_{t=t_0}^{t=t_1} 2\pi y \sqrt{\dot{x}^2 + \dot{y}^2} dt$$

We compute the surface area obtained by rotating the astroid about the x-axis:

$$S = 2\int_{t=0}^{t=\frac{\pi}{2}} 2\pi \left(a \sin^3 t\right) \sqrt{(-3a\cos^2 t \sin t)^2 + \left(3a \sin^2 t \cos t\right)^2} dt =$$

$$4\pi a \int_{t=0}^{t=\frac{\pi}{2}} \sin^3 t 3a \sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t} dt =$$

$$12\pi a^2 \int_{t=0}^{t=\frac{\pi}{2}} \sin^3 t \sqrt{(\sin^2 t + \cos^2 t)\sin^2 t \cos^2 t} dt =$$

$$12\pi a^2 \int_{t=0}^{t=\frac{\pi}{2}} \sin^3 t \sin t \cos t dt = 12\pi a^2 \int_{t=0}^{t=\frac{\pi}{2}} \sin^4 t \cos t dt =$$

$$12\pi a^2 \frac{\sin^5 t}{5} \Big|_{t=0}^{t=\frac{\pi}{2}} = \frac{12}{5}\pi a^2$$

Elimination of the Parameter

It is sometimes possible to use the parametric equations of a curve to find an equation for the curve. If the equation can be seen to be that of a familiar curve, this gives us useful information.

Example 1: $x(t) = \cos t$, $y(t) = \sin t$ clearly satisfies $x^2 + y^2 = 1$, so the curve is just the unit circle.

Example 2: $x(t) = 2\cos t$, $y(t) = 3\sin t$: we have $\dfrac{x(t)}{2} = \cos t$ and $\dfrac{y(t)}{3} = \sin t$, so $1 \cos^2 t + \sin^2 t = \left(\dfrac{x(t)}{2}\right)^2 + \left(\dfrac{y(t)}{3}\right)$,

or

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$$

so the curve is an ellipse with centre $(0,0)$, with horizontal semi-minor axis of length 2 and vertical semi-major axis of length 3.

Example 3: $x(t) = 1 + 2\cos t$, $y(t) = -1 + 3\sin t$: we have $\dfrac{x(t)-1}{2} = \cos t$ and $\dfrac{y(t)+1}{3} = \sin t$, so $1 \cos^2 t + \sin^2 t = \left(\dfrac{x(t)-1}{2}\right)^2 + \left(\dfrac{y(t)+1}{3}\right)$,

or

$$\frac{(x-1)^2}{2^2} + \frac{(y+1)^2}{3^2} = 1$$

so the curve is an ellipse with centre $(1,-1)$, with horizontal semi-minor axis of length 2 and vertical semi-major axis of length 3.

Example 4: $x(t) = 1 + 2t$, $y(t) = 3t^2 - 2$: we solve for t:

$t = \dfrac{x(t)-1}{2}$ and $t^2 = \dfrac{y(t)+2}{3}$, so

$\left(\dfrac{x(t)-1}{2}\right)^2 = \dfrac{y(t)+2}{3}$

so the curve has the equation of a parabola:

$\dfrac{(x-1)^2}{4} = \dfrac{y+2}{3}$ or $y = -\dfrac{2}{3} + \dfrac{3}{4}(x-1)^2$