Differential Equations

Definition: A differential equation is any equation containing a (possibly unknown) function and one or more of its derivatives.

Examples: $y' = \sin x$

$y'''' + y' + y = 3$

$(y'')^3 + y' = e^{2x}$

The **order** of a differential equation is the order of the highest derivative appearing in it.

We shall usually be looking for functions which satisfy a given differential equation and other natural conditions. These are called **solutions** of the differential equation (d.e. for short). In this course we will only study very simple first order d.e.'s.

Separable Equations

Definition: A first order d.e. is said to be **separable** if it can be written in the form

$$y' = g(x)h(y)$$

Such equations can be manipulated so as to have the variables x and y **separated**:

$\dfrac{dy}{dx} = g(x)h(y)$ is equivalent to

$\dfrac{dy}{h(y)} = g(x)dx,$

and a general solution is given by

$\displaystyle\int \frac{dy}{h(y)} = \int g(x)dx$ and will look like $H(y) = G(x) + C$

Example: $y' = \dfrac{x}{y}$ is separable, because it can be re-written as

$y' = x\dfrac{1}{y}$ and then can be put into the form

$\dfrac{dy}{dx} = x\dfrac{1}{y}$ or $y\,dy = x\,dx$. Integration gives

$\displaystyle\int y\,dy = \int x\,dx$ or $\dfrac{y^2}{2} = \dfrac{x^2}{2} + C$

This is equivalent to saying that all solutions of $y' = \dfrac{x}{y}$ must satisfy

$x^2 - y^2 = -2C$ for some constant C.

Graphically, we see that the family of hyperbolas $x^2 - y^2 = -2C$, $C \neq 0$, plus the pair of lines $y = x$ and $y = -x$, constitutes the set of solutions of the given differential equation. Specifying a point (x_0, y_0) through which such a hyperbola must pass is equivalent to what is called an **initial condition**.

For example, if the solution is to pass through the point (1,2), we must take $C = \dfrac{1^2 - 2^2}{-2} = \dfrac{3}{2}$.

The most important differential equation we have so far encountered is $y' - ky = 0$ or $\dfrac{y'}{y} = k$,

that is, situations where the relative rate of change is a constant k. This is easily solved by separating variables:

$\dfrac{\frac{dy}{dt}}{y} = k$ becomes $\dfrac{dy}{y} = kdt$, which we integrate:

$\displaystyle\int \frac{dy}{y} = \int kdt$ results in $\ln|y| = kt + C$.

Taking exponentials of both sides, we get

$|y| = e^{kt+C} = e^C e^{kt}$ or $y = \pm e^C e^{kt}$.

We usually write this in the form $y(t) = y(0)e^{kt}$.

Another inportant and related differential equation is the **logistic growth equation** used to model biological populations:

$y' = ky(M-y)$ where $k > 0$ is a growth rate factor and M is the **carrying capacity** of the ecosystem.

Separation of variables gives:

$\displaystyle\int \frac{dy}{y(M-y)} = \int kdt = kt + C$, and we have

$$\int \frac{dy}{y(M-y)} = \int \left(\frac{1}{M}\frac{1}{y} + \frac{1}{M}\frac{1}{M-y}\right) dy = \frac{1}{M}\int \frac{dy}{y} + \frac{1}{M}\int \frac{dy}{M-y} =$$

$$\frac{1}{M}\ln|y|\frac{1}{M}\ln|M-y| = \frac{1}{M}\ln\left|\frac{y}{M-y}\right|$$

Thus we have $\ln\left|\dfrac{y}{M-y}\right| = M(kt+C)$

Taking exponentials, we get:

$$\left|\frac{y}{M-y}\right| = e^{M(kt+C)} = e^{Mkt}e^{MC}$$

Denoting $y(0)$ by y_0, we get:

$\left|\frac{y_0}{M-y_0}\right| = e^{Mk(0)}e^{MC} = e^{MC}$, so we have

$$\left|\frac{y}{M-y}\right| = \left|\frac{y_0}{M-y_0}\right| e^{Mkt}$$

Since y represents a population, we must have $y > 0$. It is not necessary to assume that $y < M$ in order for this model to work well, but we shall do so for the purposes of this example.

We then have
$\frac{y}{M-y} = \frac{y_0}{M-y_0}e^{Mkt}$, which may be solved for y:

$$y = \frac{y_0 M}{y_0 + (M-y_0)e^{-kMT}}$$

Arc Length

The length of the curve $(x(t), y(t))$ as t varies from t_0 to t_1 is given by

$$L = \int_{t=t_0}^{t=t_1} \sqrt{(x'(t))^2 + (y'(t))^2}dt$$

If we wish to find the length of a curve which is the graph of a function $y = f(x)$, as x runs from a to b,

we let $x(t) = t$, $y(t) = f(x(t)) = f(x)$ and we get $x'(t) = 1$, and $y'(t) = f'(x(t))x'(t) = f'(x)$, so we have a simple formula for the length:

$$L = \int_{x=a}^{x=b} \sqrt{1 + (f'(x))^2}dx = \int_a^b \sqrt{1 + (f'(x))^2}dx = \int_a^b \sqrt{1 + (y')^2}dx$$

Similarly, if we have a curve $x = g(y)$ with y running from c to d we get

$$L = \int_{y=c}^{y=d} \sqrt{1 + (g'(y))^2}dy = \int_c^d \sqrt{1 + (g'(y))^2}dy = \int_c^d \sqrt{1 + (x')^2}dy$$

Example: Consider the curve given by

$x(t) = \cos t$, $y(t) = \sin t$ $, 0 \leq t \leq \pi$.

Its length is

$$L = \int_{t=0}^{t=\pi} \sqrt{x'(t)^2 + y'(t)^2}dt = \int_{t=0}^{t=\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}dt =$$

$$\int_{t=0}^{t=\pi} 1dt = t|_0^{\pi} = \pi$$

Example: Find the length of the curve $y = x^{\frac{3}{2}}$, $0 \le a \le x \le b$

We have $L = \int_a^b \sqrt{1 + (y')^2}dx = \int_a^b \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}dx = \int_a^b \sqrt{1 + \frac{9}{4}x}dx =$

$$\frac{4}{9}\left.\frac{(1+\frac{9}{4}x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_a^b = \frac{8}{27}\left.\left(1 + \frac{9}{4}x\right)^{\frac{3}{2}}\right|_a^b = \frac{8}{27}\left.\left(\frac{4+9x}{4}\right)^{\frac{3}{2}}\right|_a^b = \frac{1}{27}\left.(4+9x)^{\frac{3}{2}}\right|_a^b =$$

$$\frac{1}{27}\left[(4+9b)^{\frac{3}{2}} - (4+9a)^{\frac{3}{2}}\right]$$

Example: Find the length of the curve $x = y^2$, $0 \le c \le y \le b$

We have $L = \int_{y=c}^{y=d} \sqrt{1 + (x')^2}dy = \int_{y=c}^{y=d} \sqrt{1 + (2y)^2}dy = \int_{y=c}^{y=d} \sqrt{1 + 4y^2}dy$

Making the substitution $y = \frac{1}{2}\tan\theta$, we have
$dy = \frac{1}{2}\sec^2\theta d\theta$, $1 + 4y^2 = 1 + \tan^2\theta = \sec^2\theta$,
$\theta_c = \arctan 2c$ when $y = c$ and $\theta_d = \arctan 2d$ when $y = d$, so

$$L = \int_{\theta=\theta_c}^{\theta=\theta_d} \sqrt{\sec^2\theta}\frac{1}{2}\sec^2\theta d\theta = \frac{1}{2}\int_{\theta=\theta_c}^{\theta=\theta_d} \sec^3\theta d\theta =$$

$$\frac{1}{2}\left.\left[\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\right]\right|_{\theta=\theta_c}^{\theta=\theta_d} =$$

$$\frac{1}{4}\left[\sec\theta_d\tan\theta_d + \ln|\sec\theta_d + \tan\theta_d|\right] - \frac{1}{4}\left[\sec\theta_c\tan\theta_c + \ln|\sec\theta_c + \tan\theta_c|\right] =$$

$$\frac{1}{4}\left[\sqrt{1+4d^2}2d + \ln|\sqrt{1+4d^2} + 2d|\right] - \frac{1}{4}\left[\sqrt{1+4c^2}2c + \ln|\sqrt{1+4c^2} + 2c|\right] =$$

$$\frac{d\sqrt{1+4d^2} - c\sqrt{1+4c^2}}{2} + \frac{1}{4}\ln\frac{\sqrt{1+4d^2} + 2d}{\sqrt{1+4c^2} + 2c}$$

Note that if we let $c = 0$, we get the formula for the distance along the parabola to the point (d^2, d):

$$L = \frac{d\sqrt{1+4d^2}}{2} + \frac{1}{4}\ln(\sqrt{1+4d^2} + 2d)$$

and if we let $b = d^2$, we get the (equivalent) formula for the distance from (0,0) to $(b, \sqrt{b})$:

$$L = \frac{\sqrt{b}\sqrt{1+4b}}{2} + \frac{1}{4}\ln\left(\sqrt{1+4b} + 2\sqrt{b}\right)$$

Areas of Surfaces of Revolution

Let a cone have height h and radius r, so that the circumference of the base is $2\pi r$. If the cone is cut along a straight line (of length $\ell = \sqrt{r^2+h^2}$) from the vertex to the base and flattened out, we obtain a sector of the circle whose radius is ℓ. The area of the circle of radius ℓ is $\pi\ell^2 = \pi(r^2+h^2)$, so the ratio of the area of the sector to that of the circle is $\frac{2\pi r}{2\pi\ell} = \frac{r}{\ell}$, so the area of the sector is $\frac{r}{\ell}\pi\ell^2 = \pi r\sqrt{r^2+h^2}$.

If we now take two cones, with one being a subset of the other, we can calculate the area of the region between the bases of the two cones. This region is called a **frustum.**

Let the larger and smaller cones have heights and radii h_2 and r_2 and h_1 and r_1.
Let $\ell_2 = \sqrt{r_2^2+h_2^2}$ and $\ell_1 = \sqrt{r_1^2+h_1^2}$, so that the areas of the larger and smaller cones are

$A_2 = \pi r_2\sqrt{r_2^2+h_2^2}$ and $A_1 = \pi r_1\sqrt{r_1^2+h_1^2}$

The area of the frustum is thus

$$A = A_2 - A_1 = \pi r_2\sqrt{r_2^2+h_2^2} - \pi r_1\sqrt{r_1^2+h_1^2} =$$

$$\pi\left[r_2\sqrt{r_2^2+h_2^2} - r_1\sqrt{r_1^2+h_1^2}\right] = \pi(r_2\ell_2 - r_1\ell_1)$$

Writing $\ell = \ell_2 - \ell_1$ and $r = \frac{r_1+r_2}{2}$, and using similar triangles, we derive

$A = 2\pi r\ell$.

We can then use this formula to derive a formula for the area of the surface obtained by rotating the curve $(x(t), y(t))$, $t_1 \le t \le t_2$ about the x- and $y-$axes respectively:

$S_x = \int_{t_1}^{t_2} 2\pi y(t)\sqrt{(x'(t))^2+(y'(t))^2}dt$ and

$$S_y = \int_{t_1}^{t_2} 2\pi x(t)\sqrt{(x'(t))^2+(y'(t))^2}dt$$

If the curve is the graph of a function $y = f(x)$, $a \le x \le b$, then the area of the surface obtained by revolving the curve about the x-axis is

$$S_x = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}dx$$

and the area of the surface obtained by revolving the curve about the y-axis is

$$S_y = \int_a^b 2\pi x\sqrt{1+(f'(x))^2}dx$$

If the curve is the graph of a function $x = g(y)$, $c \leq x \leq d$, then the area of the surface obtained by revolving the curve about the x-axis is

$$S_x = \int_c^d 2\pi y\sqrt{1+(g'(y))^2}dy$$

and the area of the surface obtained by revolving the curve about the y-axis is

$$S_y = \int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}dy$$

Example: Let $y = \sqrt{r^2 - x^2}$, $-r \leq x \leq r$ be rotated about the x-axis. The surface obtained is a sphere of radius r. Its area is(using $y' = \frac{-x}{\sqrt{r^2-x^2}}$)

$$S_x = \int_{-r}^{r} 2\pi y\sqrt{1+(y')^2}dx = 2\int_0^r 2\pi\sqrt{r^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}dx =$$

$$4\pi\int_0^r \sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}dx = 4\pi\int_0^r \sqrt{r^2-x^2}\sqrt{\frac{r^2}{r^2-x^2}}dx =$$

$$4\pi \textstyle\int_0^r r\,dx = 4\pi r^2$$

Example: Let $y = \sqrt{r^2 - x^2}$, $a \leq x \leq a+h$ be rotated about the x-axis. The surface obtained is a region of the sphere of radius r lying between two vertical and parallel planes. Its area is (again using $y' = \frac{-x}{\sqrt{r^2-x^2}}$)

$$S_x = \int_a^{a+h} 2\pi y\sqrt{1+(y')^2}dx = \int_a^{a+h} 2\pi\sqrt{r^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}dx =$$

$$2\pi\int_a^{a+h} 2\pi\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}dx = 2\pi\int_a^{a+h} \sqrt{r^2-x^2}\sqrt{\frac{r^2}{r^2-x^2}}dx =$$

$$2\pi\int_a^{a+h} r\,dx = 2\pi rh$$

Thus we have the interesting fact that the area of the region obtained by slicing a sphere with two parallel planes a constant distance apart does not depend on the distance of the planes from the centre of the sphere.

Example:(#2, p.524) Find the area of the surface obtained by rotating the curve $y^2 = 4x+4, 0 \leq x \leq 8$, about the x-axis.

Solution 1: We have $y = 2(x+1)^{\frac{1}{2}}$, so $y' = 2\frac{1}{2}(x+1)^{-\frac{1}{2}} = (x+1)^{-\frac{1}{2}}$,

and $S_x = \int_0^8 2\pi(2)(x+1)^{\frac{1}{2}}\sqrt{1+\left((x+1)^{-\frac{1}{2}}\right)^2}\,dx =$

$$4\pi\int_0^8 (x+1)^{\frac{1}{2}}\sqrt{1+\frac{1}{x+1}}dx = 4\pi\int_0^8 (x+2)^{\frac{1}{2}}dx = 4\pi\left.\frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^8 = \frac{8\pi}{3}\left.(x+2)^{\frac{3}{2}}\right|_0^8 = \frac{8\pi}{3}\left[(8+2)^{\frac{3}{2}} - (0+2)^{\frac{3}{2}}\right.$$

$$\frac{16\pi}{3}\left(5\sqrt{5}-1\right)\sqrt{2}$$

Solution 2: We have $x = \frac{y^2-4}{4} = \frac{1}{4}y^2 - 1$, so $x' = \frac{y}{2}$,

and $S_x = \int_{y=2}^{y=6} 2\pi y\sqrt{1+(x')^2}dy = 2\pi\int_{y=2}^{y=6} y\sqrt{1+\left(\frac{y}{2}\right)^2}dy =$

$$\pi\int_{y=2}^{y=6} y\sqrt{4+y^2}dy = (\text{letting } y = 2\tan\theta) =$$

$$\pi\int_{\theta=\arctan 1}^{\theta=\arctan 3} 2\tan\theta\sqrt{4+4\tan^2\theta^2}2\sec^2\theta d\theta = \pi\int_{\theta=\arctan 1}^{\theta=\arctan 3} 2\tan\theta\sqrt{4\sec^2\theta}2\sec^2\theta d\theta =$$

$$\pi\int_{\theta=\arctan 1}^{\theta=\arctan 3} 2\tan\theta(2\sec\theta)2\sec^2\theta d\theta = 8\pi\int_{\theta=frac\pi 4}^{\theta=\arctan 3}\sec^2\theta\sec\theta\tan\theta d\theta =$$

$(\text{ letting } u = \sec\theta) =$

$$8\pi\int_{u=\sqrt{2}}^{u=\sqrt{10}} u^2du = 8\pi\left.\frac{u^3}{3}\right|_{u=\sqrt{2}}^{u=\sqrt{10}} = \frac{8}{3}\pi\left(10\sqrt{10}-2\sqrt{2}\right)$$

$$\frac{16\pi}{3}\left(5\sqrt{5}-1\right)\sqrt{2}$$

Example:(#4, p.524) Find the area of the surface obtained by rotating the curve $y = \frac{x^2}{4} - \frac{\ln x}{2}$, $1 \leq x \leq 4$, about the x-axis.

Solution: We have $y' = \frac{x}{2} - \frac{1}{2x}$, and $S_x = \int_1^4 2\pi \left(\frac{x^2}{4} - \frac{\ln x}{2}\right)\sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}\,dx =$

$2\pi \int_1^4 \left(\frac{x^2}{4} - \frac{\ln x}{2}\right)\sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\,dx =$

$2\pi \int_1^4 \left(\frac{x^2}{4} - \frac{\ln x}{2}\right)\left(\frac{x}{2} + \frac{1}{2x}\right) dx =$, etc....

Example:(#6, p.524) Find the area of the surface obtained by rotating the curve $y = \cos x, 0 \le x \le \frac{\pi}{3}$, about the x-axis.

Solution: We have $y' = -\sin x$,

and $S_x = \int_{x=0}^{x=\frac{\pi}{3}} 2\pi \cos x\sqrt{1 + (-\sin x)^2}dx = (\text{ letting } u = \sin x) =$

$2\pi \int_0^{\frac{\sqrt{3}}{2}} \sqrt{1 + u^2}du = (\text{ letting } u = \tan\theta) = 2\pi \int_0^{\arctan\frac{\sqrt{3}}{2}} \sec^3\theta d\theta =$

$2\pi \left(\frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2}\right)\Bigg|_0^{\arctan\frac{\sqrt{3}}{2}} =$

$\pi\ (\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)|_0^{\arctan\frac{\sqrt{3}}{2}} =$

$\pi\left(\frac{\sqrt{7}}{2}\frac{\sqrt{3}}{2} + \ln\left|\frac{\sqrt{7}}{2} + \frac{\sqrt{3}}{2}\right|\right) = \pi\left(\frac{\sqrt{21}}{4} + \ln\frac{\sqrt{7}+\sqrt{3}}{2}\right)$

Example:(#8, p.524) Find the area of the surface obtained by rotating the curve $y = \frac{3}{2}x^{\frac{2}{3}}, 1 \le x \le 8$, about the x-axis.

Solution: We have $y' = x^{-\frac{1}{3}}$,

and $S_x = \int_1^8 2\pi\frac{3}{2}x^{\frac{2}{3}}\sqrt{1 + \left(x^{-\frac{1}{3}}\right)^2}dx = 3\pi\int_1^8 x^{\frac{2}{3}}\sqrt{1 + x^{-\frac{2}{3}}}dx =$

$3\pi\int_1^8 x^{\frac{2}{3}}\sqrt{\frac{1 + x^{\frac{2}{3}}}{x^{\frac{2}{3}}}}dx = 3\pi\int_{x=1}^{x=8} x^{\frac{1}{3}}\sqrt{1 + x^{\frac{2}{3}}}dx = (\text{ letting } u = x^{\frac{1}{3}})$

$9\pi\int_{u=1}^{u=2} u^3\sqrt{u^2 + 1}du = \cdots = \frac{3\pi}{5}\left(50\sqrt{5} - 2\sqrt{2}\right)$

Example:(#10, p.524) Find the area of the surface obtained by rotating the curve $x = 1 + 2y^2$,

$1 \le y \le 2$, about the x-axis.

Solution 1: We have $y = \sqrt{\frac{x-1}{2}} = \left(\frac{x-1}{2}\right)^{\frac{1}{2}}$, so $y' = \frac{1}{4}\left(\frac{x-1}{2}\right)^{-\frac{1}{2}}$,

and $S_x = \int_3^9 2\pi \left(\frac{x-1}{2}\right)^{\frac{1}{2}} \sqrt{1 + \left(\frac{1}{4}\left(\frac{x-1}{2}\right)^{-\frac{1}{2}}\right)^2}\, dx =$

$2\pi \int_3^9 \left(\frac{x-1}{2}\right)^{\frac{1}{2}} \sqrt{1 + \frac{1}{8(x-1)}} dx =$

$2\pi \int_3^9 \left(\frac{x-1}{2}\right)^{\frac{1}{2}} \sqrt{\frac{8(x-1)+1}{8(x-1)}} dx = \frac{\pi}{2} \int_3^9 \sqrt{8x-7} dx =$

$\frac{\pi}{2}\frac{1}{8} \left.\frac{(8x-7)^{\frac{3}{2}}}{\frac{3}{2}}\right|_3^9 = \frac{\pi}{24}\left.(8x-7)^{\frac{3}{2}}\right|_3^9 = \frac{\pi}{24}(8(9)-7)^{\frac{3}{2}} - \frac{\pi}{24}(8(3)-7)^{\frac{3}{2}} =$

$\frac{\pi}{24}\left(65\sqrt{65} - 17\sqrt{17}\right)$

Moments and Centres of Mass

Let $\mathcal{R} = \{(x,y) | a \le x \le b, 0 \le y \le f(x)\}$ be a region in the plane.

Definition: The **moment** M_y about the y-axis of a lamina of uniform density ρ occupying the region $\mathcal{R}$ is

$$M_y = \rho \int_a^b x f(x) dx$$

and the **moment** M_x about the x-axis of the lamina is

$$M_x = \rho \int_a^b \frac{1}{2} (f(x))^2 \, dx$$

The **mass** of the lamina is $M = \rho \int_a^b f(x) dx$ and the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right)$$

Let $\mathcal{R} = \{(x,y) | a \le x \le b, g(x) \le y \le f(x)\}$ be a region in the plane.

Then the moment M_y about the y-axis of a lamina of uniform density ρ occupying the region $\mathcal{R}$ is

$$M_y = \rho \int_a^b x \left[f(x) - g(x) \right] dx$$

and the **moment** M_x about the x-axis of the lamina is

$$M_x = \rho \int_a^b \frac{1}{2} \left[(f(x))^2 - (g(x))^2 \right] dx = \rho \int_a^b \frac{1}{2} \left[f(x) + g(x) \right] \left[f(x) - g(x) \right] dx$$

The **mass** of the lamina is $M = \rho \int_a^b \left[f(x) - g(x) \right] dx = \rho A$, where A is the area of $\mathcal{R}$. The center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) =$$

$$\left(\frac{\int_a^b x \left[f(x) - g(x) \right] dx}{\int_a^b \left[f(x) - g(x) \right] dx}, \frac{\int_a^b \frac{1}{2} \left[(f(x))^2 - (g(x))^2 \right] dx}{\int_a^b \left[f(x) - g(x) \right] dx} \right)$$

$(\bar{x}, \bar{y})$ is called the **centroid** of $\mathcal{R}$.

Example: Find the centroid of the triangle with vertices (0,0), $(b, 0)$, and (b, h).

We assume $\rho = 1$, so that $M = A$.

We have $A = \int_0^b \frac{h}{b} x dx = \frac{h}{b} \frac{x^2}{2} \Big|_0^b = \frac{h}{b}\frac{b^2}{2} = \frac{1}{2}bh$

$M_y = \int_0^b x \left(\frac{h}{b}x\right) dx = \frac{h}{b}\frac{x^3}{3}\Big|_0^b = \frac{h}{b}\frac{b^3}{3} = \frac{1}{3}b^2h$

$M_x = \int_0^b \frac{1}{2}\left(\frac{h}{b}x\right)^2 dx = \frac{h^2}{2b^2}\frac{x^3}{3}\Big|_0^b = \frac{h^2}{2b^2}\frac{b^3}{3} = \frac{1}{6}bh^2$

Thus $(\bar{x}, \bar{y}) = \left(\frac{\frac{1}{3}b^2h}{\frac{1}{2}bh}, \frac{\frac{1}{6}bh^2}{\frac{1}{2}bh}\right) = \left(\frac{2}{3}b, \frac{1}{6}h\right)$

Example: Find the centroid of the semicircle

$\mathcal{R} = \left\{(x, y) | -r \le x \le r, 0 \le y \le \sqrt{r^2 - x^2}\right\}$.

We assume $\rho = 1$, so that $M = A$.

We have $A = \int_{-r}^{r} \sqrt{r^2 - x^2} dx = \frac{\pi}{2}r^2$

$M_y = \int_{-r}^{r} x\sqrt{r^2 - x^2} dx = 0$, since $\mathcal{R}$ is symmetric about the y-axis

$M_x = \int_{-r}^{r} \frac{1}{2}\left(\sqrt{r^2 - x^2}\right)^2 dx = \int_{-r}^{r} \frac{1}{2}\left(r^2 - x^2\right) dx = \frac{1}{2}\left(r^2x - \frac{x^3}{3}\right)\Big|_{-r}^{r} =$

$\frac{1}{2}\left(r^2r - \frac{r^3}{3}\right) - \frac{1}{2}\left(r^2(-r) - \frac{(-r)^3}{3}\right) = \frac{2}{3}r^3$

Thus $(\bar{x}, \bar{y}) = \left(0, \frac{\frac{2}{3}r^3}{\frac{\pi}{2}r^2}\right) = \left(0, \frac{4}{3\pi}r\right)$

Example: If $0 \le h \le r$, Find the centroid of the subset of the semicircle:

$\mathcal{R} = \left\{(x, y) | -r \le x \le r, h \le y \le \sqrt{r^2 - x^2}\right\}$.

We have $A = 2\int_0^{\sqrt{r^2-h^2}} \left(\sqrt{r^2 - x^2} - h\right) dx =$

$2\int_0^{\sqrt{r^2-h^2}} \left(r^2 - x^2\right)^{\frac{1}{2}} dx - 2h\int_0^{\sqrt{r^2-h^2}} dx =$(letting $x = r\sin\theta$)

$$2\int_0^{\theta=\arcsin\frac{\sqrt{r^2-h^2}}{r}} \left(r^2-r^2\sin^2\theta\right)^{\frac{1}{2}} r\cos\theta d\theta - 2h\sqrt{r^2-h^2} =$$

$$2\int_0^{\theta=\arcsin\frac{\sqrt{r^2-h^2}}{r}} r\cos\theta r\cos\theta d\theta - 2h\sqrt{r^2-h^2} =$$

$$2r^2\int_0^{\theta=\arcsin\frac{\sqrt{r^2-h^2}}{r}} \cos^2\theta d\theta - 2h\sqrt{r^2-h^2} =$$

$$2r^2\int_0^{\theta=\arcsin\frac{\sqrt{r^2-h^2}}{r}} \frac{1+\cos 2\theta}{2} d\theta - 2h\sqrt{r^2-h^2} =$$

$$r^2\int_0^{\theta=\arcsin\frac{\sqrt{r^2-h^2}}{r}} 1+\cos 2\theta d\theta - 2h\sqrt{r^2-h^2} =$$

$$r^2\left(\theta+\frac{\sin 2\theta}{2}\right)\Bigg|_0^{\theta=\arcsin\frac{\sqrt{r^2-h^2}}{r}} - 2h\sqrt{r^2-h^2} =$$

$$r^2\left(\theta+\sin\theta\cos\theta\right)\Big|_0^{\theta=\arcsin\frac{\sqrt{r^2-h^2}}{r}} - 2h\sqrt{r^2-h^2} =$$

$$r^2\left[\arcsin\frac{\sqrt{r^2-h^2}}{r} + \sin\left(\arcsin\frac{\sqrt{r^2-h^2}}{r}\right)\cos\left(\arcsin\frac{\sqrt{r^2-h^2}}{r}\right)\right] - 2h\sqrt{r^2-h^2} =$$

$$r^2\left[\arcsin\frac{\sqrt{r^2-h^2}}{r} + \frac{\sqrt{r^2-h^2}}{r}\sqrt{1-\left(\frac{\sqrt{r^2-h^2}}{r}\right)^2}\right] - 2h\sqrt{r^2-h^2} =$$

$$r^2\left[\arcsin\frac{\sqrt{r^2-h^2}}{r} + \frac{\sqrt{r^2-h^2}}{r}\sqrt{1-\left(\frac{r^2-h^2}{r^2}\right)}\right] - 2h\sqrt{r^2-h^2} =$$

$$r^2\left[\arcsin\frac{\sqrt{r^2-h^2}}{r} + \frac{\sqrt{r^2-h^2}}{r}\sqrt{\frac{h^2}{r^2}}\right] - 2h\sqrt{r^2-h^2} =$$

$$r^2\left[\arcsin\frac{\sqrt{r^2-h^2}}{r} + \frac{\sqrt{r^2-h^2}}{r}\frac{h}{r}\right] - 2h\sqrt{r^2-h^2} =$$

$$r^2\arcsin\frac{\sqrt{r^2-h^2}}{r} + h\sqrt{r^2-h^2} - 2h\sqrt{r^2-h^2} =$$

$$r^2\arcsin\frac{\sqrt{r^2-h^2}}{r} - h\sqrt{r^2-h^2}$$

$M_y = \int_{-\sqrt{r^2-h^2}}^{\sqrt{r^2-h^2}} x\left[\sqrt{r^2-x^2}-h\right]dx = 0$, since $\mathcal{R}$ is symmetric about the y-axis

$$M_x = \int_{-\sqrt{r^2-h^2}}^{\sqrt{r^2-h^2}} \frac{1}{2}\left[\left(\sqrt{r^2-x^2}\right)^2 - h^2\right] dx = 2\frac{1}{2}\int_0^{\sqrt{r^2-h^2}} r^2 - x^2 dx =$$

$$(r^2-h^2)x - \frac{x^3}{3}\Big|_0^{\sqrt{r^2-h^2}} = \frac{2}{3}(r^2-h^2)\sqrt{r^2-h^2}$$

Thus $(\bar{x}, \bar{y}) = \left(0, \frac{2(r^2-h^2)\sqrt{r^2-h^2}}{3\left(r^2 \arcsin\frac{\sqrt{r^2-h^2}}{r} - h\sqrt{r^2-h^2}\right)}\right)$

Hydrostatic Pressure

The gravitational force exerted by a 1 cubic metre of water is 9.8×1000 Newtons. The gravitational force exerted by a rectangular column of water with square base of area 1 meter^2 and height d is therefore $9800d$ N.

If we have a horizontal region $\mathcal{R}$ of area A(expressed in square metres), the force exerted by a column of water d metres high is gAd, where g is the gravitational coefficient $9.8\frac{\text{metre}}{\text{sec}^2}$.

Pressure is measured in **pascals**: 1 Pascal=$1\dfrac{\text{Newton}}{\text{square metre}}$, or in

kilopascals(kPa): 1 kilopascal=1000 pascals=$1000\dfrac{\text{Newtons}}{\text{square metre}}$

Let $\mathcal{R} = \{(x, y) | c \le y \le d, g(y) \le x \le f(x)\}$ be a region in the plane.

Definition: The **hydrostatic force** exerted against $\mathcal{R}$ by water whose surface is at the level $y = d$ is

$$\int_c^d (d-y)9.8\,[g(y) - f(y)]\,dy$$

Theorem: This force equals $g\,(d - \bar{y})\,A$, where A and $\bar{y}$ are the area and the y-coordinate of $\mathcal{R}$.

Thus the mathematical theory of hydrostatic force is equivalent to that of the theory of centres of mass.
