Moments and Centres of Mass

Let $\mathcal{R} = \{(x, y)|a \leq x \leq b, 0 \leq y \leq f(x)\}$ be a region in the plane.

Definition: The **moment** M_y about the y-axis of a lamina of uniform density ρ occupying the region $\mathcal{R}$ is

$$M_y = \rho \int_a^b x f(x) dx$$

and the **moment** M_x about the x-axis of the lamina is

$$M_x = \rho \int_a^b \frac{1}{2} (f(x))^2 dx$$

The **mass** of the lamina is $M = \rho \int_a^b f(x) dx$ and the

center of mass is $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right)$

Regions Between Two Curves

Let $\mathcal{R} = \{(x, y)|a \leq x \leq b, g(x) \leq y \leq f(x)\}$ be a region in the plane.

Then the **moment** M_y about the y-axis of a lamina of uniform density ρ occupying the region $\mathcal{R}$ is

$$M_y = \rho \int_a^b x [f(x) - g(x)] dx$$

and the **moment** M_x about the x-axis of the lamina is

$$M_x = \rho \int_a^b \frac{1}{2} \left[(f(x))^2 - (g(x))^2 \right] dx =$$

$$\rho \int_a^b \frac{1}{2} [f(x) + g(x)] [f(x) - g(x)] dx$$

The **mass** of the lamina is $M = \rho \int_a^b [f(x) - g(x)]\, dx = \rho A$,

where A is the area of $\mathcal{R}$.

The **center of mass** is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) =$$

$$\left(\frac{\int_a^b x\,[f(x) - g(x)]\, dx}{\int_a^b [f(x) - g(x)]\, dx}, \frac{\int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right] dx}{\int_a^b [f(x) - g(x)]\, dx}\right)$$

$(\bar{x}, \bar{y})$ is called the **centroid** of $\mathcal{R}$.

Example: Find the centroid of the triangle with vertices (0,0), $(b,0)$, and (b,h).

We assume $\rho = 1$, so that $M = A$.

We have $A = \int_0^b \frac{h}{b} x\,dx = \frac{h}{b} \left.\frac{x^2}{2}\right|_0^b = \frac{h}{b}\frac{b^2}{2} = \frac{1}{2}bh$

$$M_y = \int_0^b x\left(\frac{h}{b}x\right)dx = \frac{h}{b}\left.\frac{x^3}{3}\right|_0^b = \frac{h}{b}\frac{b^3}{3} = \frac{1}{3}b^2h$$

$$M_x = \int_0^b \frac{1}{2}\left(\frac{h}{b}x\right)^2 dx = \frac{h^2}{2b^2}\left.\frac{x^3}{3}\right|_0^b = \frac{h^2}{2b^2}\frac{b^3}{3} = \frac{1}{6}bh^2$$

Thus $(\bar{x}, \bar{y}) = \left(\frac{\frac{1}{3}b^2h}{\frac{1}{2}bh}, \frac{\frac{1}{6}bh^2}{\frac{1}{2}bh}\right) = \left(\frac{2}{3}b, \frac{1}{6}h\right)$

Example: Find the centroid of the semicircle

$\mathcal{R} = \left\{(x, y) \mid -r \le x \le r, 0 \le y \le \sqrt{r^2 - x^2}\right\}.$

We assume $\rho = 1$, so that $M = A$.

We have $A = \int_{-r}^{r} \sqrt{r^2 - x^2}dx = \frac{\pi}{2}r^2$

$M_y = \int_{-r}^{r} x\sqrt{r^2 - x^2}dx = 0$, since $\mathcal{R}$ is symmetric about the y-axis

$$M_x = \int_{-r}^{r} \frac{1}{2}\left(\sqrt{r^2 - x^2}\right)^2 dx = \int_{-r}^{r} \frac{1}{2}\left(r^2 - x^2\right) dx = \frac{1}{2}\left(r^2 x - \frac{x^3}{3}\right)\Bigg|_{-r}^{r} =$$

$$\frac{1}{2}\left(r^2 r - \frac{r^3}{3}\right) - \frac{1}{2}\left(r^2(-r) - \frac{(-r)^3}{3}\right) = \frac{2}{3}r^3$$

Thus $(\bar{x}, \bar{y}) = \left(0, \frac{\frac{2}{3}r^3}{\frac{\pi}{2}r^2}\right) = \left(0, \frac{4}{3\pi}r\right)$

Example: If $0 \le h \le r$, Find the centroid of the subset of the semicircle:

$$\mathcal{R} = \left\{(x, y) | -r \le x \le r, h \le y \le \sqrt{r^2 - x^2}\right\} =$$

$$\left\{(x, y) | -\sqrt{r^2 - h^2} \le x \le \sqrt{r^2 - h^2}, h \le y \le \sqrt{r^2 - x^2}\right\} =.$$

We have $A = 2\int_0^{\sqrt{r^2-h^2}} \left(\sqrt{r^2 - x^2} - h\right) dx =$

$$2\int_0^{\sqrt{r^2-h^2}} \left(r^2 - x^2\right)^{\frac{1}{2}} dx - 2h\int_0^{\sqrt{r^2-h^2}} dx = \text{(letting } x = r\sin\theta)$$

$$2\int_0^{\theta=\arcsin\frac{\sqrt{r^2-h^2}}{r}} \left(r^2 - r^2\sin^2\theta\right)^{\frac{1}{2}} r\cos\theta d\theta - 2h\sqrt{r^2 - h^2} =$$

$$2\int_0^{\theta=\arcsin\frac{\sqrt{r^2-h^2}}{r}} r\cos\theta r\cos\theta d\theta - 2h\sqrt{r^2 - h^2} =$$

$$2r^2\int_0^{\theta=\arcsin\frac{\sqrt{r^2-h^2}}{r}} \cos^2\theta d\theta - 2h\sqrt{r^2 - h^2} =$$

$$2r^2\int_0^{\theta=\arcsin\frac{\sqrt{r^2-h^2}}{r}} \frac{1 + \cos 2\theta}{2} d\theta - 2h\sqrt{r^2 - h^2} =$$

$$r^2\int_0^{\theta=\arcsin\frac{\sqrt{r^2-h^2}}{r}} 1 + \cos 2\theta d\theta - 2h\sqrt{r^2 - h^2} =$$

$$r^2\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_0^{\theta=\arcsin\frac{\sqrt{r^2-h^2}}{r}} - 2h\sqrt{r^2 - h^2} =$$

$$r^2\left(\theta + \sin\theta\cos\theta\right)\Big|_0^{\theta=\arcsin\frac{\sqrt{r^2-h^2}}{r}} - 2h\sqrt{r^2 - h^2} =$$

$$r^2\left[\arcsin\frac{\sqrt{r^2 - h^2}}{r} + \sin\left(\arcsin\frac{\sqrt{r^2 - h^2}}{r}\right)\cos\left(\arcsin\frac{\sqrt{r^2 - h^2}}{r}\right)\right] - 2h\sqrt{r^2 - h^2} =$$

$$r^2\left[\arcsin\frac{\sqrt{r^2 - h^2}}{r} + \frac{\sqrt{r^2 - h^2}}{r}\sqrt{1 - \left(\frac{\sqrt{r^2 - h^2}}{r}\right)^2}\right] - 2h\sqrt{r^2 - h^2} =$$

$$r^2\left[\arcsin\frac{\sqrt{r^2 - h^2}}{r} + \frac{\sqrt{r^2 - h^2}}{r}\sqrt{1 - \left(\frac{r^2 - h^2}{r^2}\right)}\right] - 2h\sqrt{r^2 - h^2} =$$

$$r^2\left[\arcsin\frac{\sqrt{r^2-h^2}}{r}+\frac{\sqrt{r^2-h^2}}{r}\sqrt{\frac{h^2}{r^2}}\right]-2h\sqrt{r^2-h^2}=$$

$$r^2\left[\arcsin\frac{\sqrt{r^2-h^2}}{r}+\frac{\sqrt{r^2-h^2}}{r}\frac{h}{r}\right]-2h\sqrt{r^2-h^2}=$$

$$r^2\arcsin\frac{\sqrt{r^2-h^2}}{r}+h\sqrt{r^2-h^2}-2h\sqrt{r^2-h^2}=$$

$$r^2\arcsin\frac{\sqrt{r^2-h^2}}{r}-h\sqrt{r^2-h^2}$$

$$M_y = \int_{-\sqrt{r^2-h^2}}^{\sqrt{r^2-h^2}} x\left[\sqrt{r^2-x^2}-h\right]dx = 0, \text{ since } \mathcal{R} \text{ is symmetric about the } y\text{-axis}$$

$$M_x = \int_{-\sqrt{r^2-h^2}}^{\sqrt{r^2-h^2}} \frac{1}{2}\left[\left(\sqrt{r^2-x^2}\right)^2-h^2\right]dx = 2\frac{1}{2}\int_0^{\sqrt{r^2-h^2}} r^2-x^2dx =$$

$$(r^2-h^2)x-\frac{x^3}{3}\bigg|_0^{\sqrt{r^2-h^2}} = \frac{2}{3}(r^2-h^2)\sqrt{r^2-h^2}$$

Thus $(\bar{x},\bar{y}) = \left(0, \dfrac{2(r^2-h^2)\sqrt{r^2-h^2}}{3\left(r^2\arcsin\frac{\sqrt{r^2-h^2}}{r}-h\sqrt{r^2-h^2}\right)}\right)$