Natural Logarithms

Since $f(x) = \dfrac{1}{x}$ is continuous on the interval $(0, \infty)$, the function $F(x)$ which we define to be equal to the definite integral

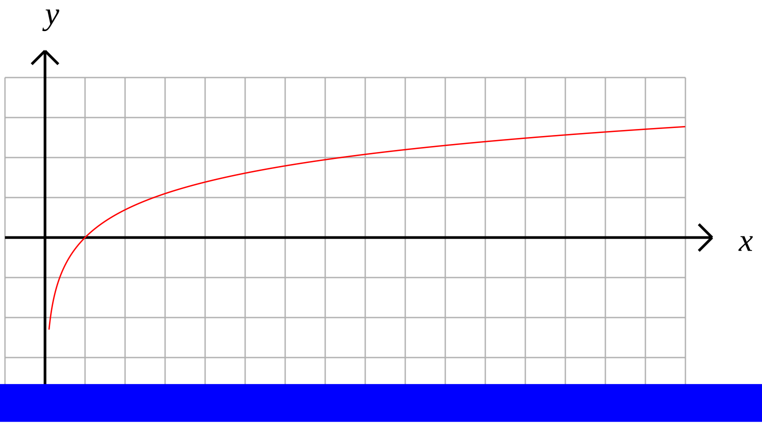
$F(x) = \int_1^x \frac{1}{t} dt$ is differentiable for all $x > 0$.

Its derivative is $F'(x) = \dfrac{1}{x} > 0$, so the graph of F is strictly monotone increasing, and therefore one-to-one.
Its second derivative is $F'(x) = -\dfrac{1}{x^2} < 0$, so the graph is concave down.

We also have $F(1) = 0$.

For reasons that will soon become clear, $F(x)$ is usually denoted by $\ln x$ and is called the **natural logarithm** function.

We know that the graph looks like:



Let $h(x) = F(ax)$, where $a > 0$ is a constant.

Then $h'(x) = F'(ax)\dfrac{d}{dx}(ax) = \dfrac{1}{ax}a = \dfrac{1}{x}$,

so $F(ax)$ and $F(x)$ have the same derivative, and therefore $F(ax) - F(x)$ has derivative 0 and is thus a constant function.

Letting $x = 1$, we get $F(a(1)) - F(1) = F(a) - 0 = F(a)$,

so we must have $F(ax) - F(x) = F(a)$, or $F(ax) = F(a) + F(x)$ for all $x > 0$.

Using the $\ln x$ notaation for $F(x)$, we get

$\ln(ax) = \ln a + \ln x$, or, replacing x by b, we get the familiar identity for logarithms:

$$\ln(ab) = \ln a + \ln b$$

Similarly we can show that

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b \text{ and } \ln\left(a^b\right) = b\ln a$$

We can use the last property to show that

$$\lim_{x\to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x\to\infty} \ln x = \infty :$$

We know that $\ln 2 > 0$, so $\ln 2^x = x\ln 2$. But this appoaches ∞ as x approaches ∞, and approaches $-\infty$ as x approaches $-\infty$.

When we differentiate the logarithm of a function $f(x)$, something very important happens: we get the **relative rate of change** of the function:

$$\frac{d}{dx}(\ln(f(x))) = \frac{1}{f(x)}f'(x) = \frac{f'(x)}{f(x)}$$

Logarithmic Differentiation

One of the most important uses of the natural logarithm function is in the computation of derivatives of functions which are made up of products, quotients and powers of more elementary functions. We use the three basic arithmetic properties of the logarithm to simplify the function.

Example: Find y' if $y = \dfrac{x^4\pi^x\sin^5(3x)}{(x-1)^3(x+2)^{7.5}}$

Taking logarithms of both sides of the equation, we get

$\ln y = \ln\left(\dfrac{x^4\pi^x\sin^5(3x)}{(x-1)^3(x+2)^{7.5}}\right)$ or

$$\ln y = 4\ln x + x\ln\pi + 5\ln\sin(3x) - 3\ln(x-1) - 7.5\ln(x+2)$$

which we now differentiate:

$$\frac{y'}{y} = 4\frac{1}{x} + \ln\pi + 5\frac{3\cos(3x)}{\sin(3x)} - 3\frac{1}{x-1} - 7.5\frac{1}{x+2}$$

which we need only simplify slightly to get y' in a usable form:

$$y' = y\left[\frac{4}{x} + \ln\pi + 15\cot(3x) - \frac{3}{x-1} - \frac{7.5}{x+2}\right]$$

Negative x

There will be occasions when we wish to apply logarithms and deal with negative values of the variables concerned.

Of course, $\ln x$ is undefined if $x \le 0$. However, $\ln|x|$ *is* defined if $x < 0$.

Let us then find the derivative of $\ln|x|$ for non-zero x.

If $x > 0$, it is of course $\frac{1}{x}$.

If $x < 0$, then $|x| = -x$, so $\ln|x| = \ln(-x)$, and we can apply the Chain Rule:

$$\frac{d}{dx}(\ln(-x)) = \frac{1}{-x}\frac{d}{dx}(-x) = \frac{1}{-x}(-1) = \frac{1}{x},$$

so we have the important formula

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \text{ if } x \neq 0$$