Inverse Trigonometric Functions

The trigonometric functions are not one-to-one. By restricting their domains, we can construct one-to-one functions from them. For example, if we restrict the domain of $\sin x$ to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we have a one-to-one function which has an inverse denoted by $\arcsin x$ or $\sin^{-1} x$.

Similarly if we restrict the domain of $\cos x$ to the interval $[0, \pi]$ we have a one-to-one function which has an inverse denoted by $\arccos x$ or $\cos^{-1} x$

If we restrict the domain of $\tan x$ to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ we have a one-to-one function which has an inverse denoted by $\arctan x$ or $\tan^{-1} x$.

The inverses of the other three trigonometric functions are not often used, but are defined similarly.

We always have the Cancellation Laws, which *only* hold on the appropriate domains:

$\sin(\sin^{-1} x) = x$, $\sin^{-1}(\sin x) = x$

$\cos(\cos^{-1} x) = x$, $\cos^{-1}(\cos x) = x$

$\tan(\tan^{-1} x) = x$, $\tan^{-1}(\tan x) = x$

$\sec(\sec^{-1} x) = x$, $\sec^{-1}(\sec x) = x$

$\csc(\csc^{-1} x) = x$, $\csc^{-1}(\csc x) = x$

$\cot(\cot^{-1} x) = x$, $\cot^{-1}(\cot x) = x$

Derivatives of Inverse Trig Functions

By differentiating the first Cancellation Law for each trig function, and using trigonometric identities we get a differentiation rule for its inverse:

For example:

$$\frac{d\left(\sin\left(\sin^{-1}x\right)\right)}{dx} = \frac{d(x)}{dx} \text{ so:}$$

$$\cos\left(\sin^{-1}x\right)\frac{d\left(\sin^{-1}x\right)}{dx} = 1 \text{ and therefore}$$

$$\frac{d\left(\sin^{-1}x\right)}{dx} = \frac{1}{\cos\left(\sin^{-1}x\right)} = \frac{1}{\sqrt{1-x^2}}$$

(Remember that $\cos\left(\sin^{-1}x\right) = \sqrt{1-\left(\sin\left(\sin^{-1}x\right)\right)^2} = \sqrt{1-x^2}$)

We list the standard differentiation rules for the six inverse trig functions. The first three should be memorized, and the student should practice deriving them all from first principles as done above.

$$\frac{d\left(\sin^{-1} x\right)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d\left(\cos^{-1} x\right)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d\left(\tan^{-1} x\right)}{dx} = \frac{1}{1+x^2}$$

$$\frac{d\left(\sec^{-1} x\right)}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d\left(\csc^{-1} x\right)}{dx} = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d\left(\cot^{-1} x\right)}{dx} = -\frac{1}{1+x^2}$$

Remember that these formulas are only valid when the domains are as in the definition of the inverse.

The differentiation rules for the inverse trig functions give us a whole new class of integration formulas, which need not be memorized, because we will soon get into the technique of trigonometric substitution which can be used to easily derive these formulas:

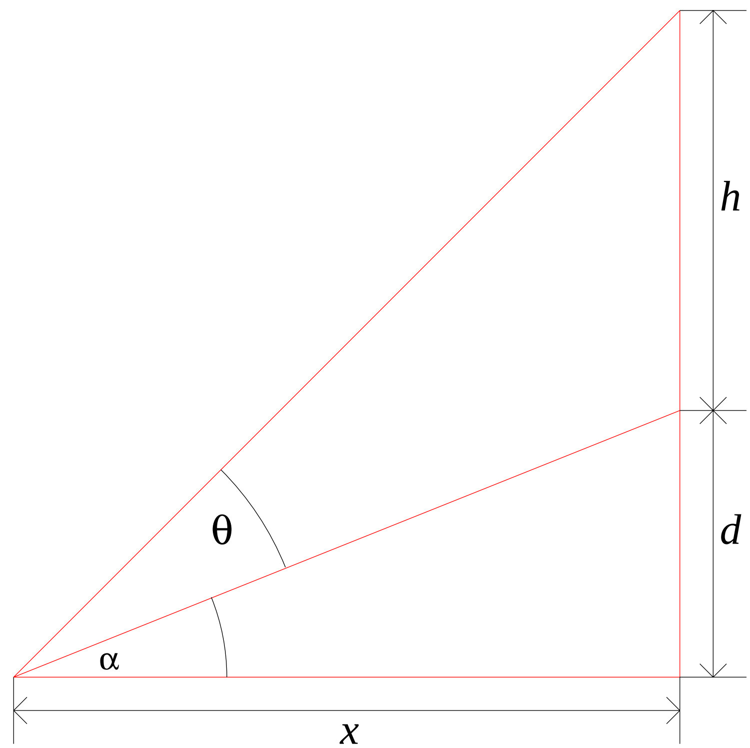
$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C = -\cos^{-1} x + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C = \cot^{-1} x + C$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C = -\csc x + C =$$

Example: Problem 6.7-66(p.412 of the Brown *Stewart*

A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer. How far from the wall should the observer stand so as to maximize the angle θ subtended at his eye by the painting?



Solution 1: (Using Inverse Trig Functions)

Variables:

x = observer's distance from the wall

α = angle between the horizontal and the bottom of the painting

θ = angle between the top and bottom of the painting

Relations:

$$\alpha = \arctan\frac{d}{x}$$

$$\theta + \alpha = \arctan\frac{d+h}{x}$$

$$\theta(x) = \arctan\frac{d+h}{x} - \arctan\frac{d}{x}$$

We have to find the value of x that will make θ as large as possible, so we differentiate:

$$\theta'(x) = \frac{1}{1+\left(\frac{d+h}{x}\right)^2}\left(-\frac{d+h}{x^2}\right) - \frac{1}{1+\left(\frac{d}{x}\right)^2}\left(-\frac{d}{x^2}\right) =$$

$$-\frac{d+h}{x^2+(d+h)^2} + \frac{d}{x^2+d^2} = -\frac{-(d+h)(x^2+d^2)+d\left(x^2+(d+h)^2\right)}{\left(x^2+(d+h)^2\right)(x^2+d^2)} =$$

$$\frac{-dx^2-d^3-hx^2-hd^2+dx^2+d^3+2d^2h+dh^2}{\left(x^2+(d+h)^2\right)(x^2+d^2)} =$$

$$\frac{h(d^2 + dh - x^2)}{\left(x^2 + (d+h)^2\right)(x^2 + d^2)} = 0 \text{ if } x = \sqrt{d(d+h)}$$

Solution 2: (Not Using Inverse Trig Functions)

We use the same variables, but different relations:

Relations:

$$\tan\alpha = \frac{d}{x}$$

$$\tan(\theta + \alpha) = \frac{d+h}{x}$$

Then we have

$$\tan(\theta+\alpha) = \frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta} = \frac{\frac{d}{x} + \tan\theta}{1 - \frac{d}{x}\tan\theta} = \frac{d + x\tan\theta}{x - d\tan\theta} = \frac{h+d}{x} \text{ or}$$

$$x(d + x\tan\theta) = (h+d)(x - d\tan\theta) \text{ or}$$

$$xd + x^2\tan\theta = (h+d)x - (h+d)d\tan\theta \text{ or}$$

$$x^2\tan\theta + (h+d)d\tan\theta = (h+d)x - xd \text{ or}$$

$$\tan\theta\left[x^2 + (h+d)d\right] = xh \text{ or}$$

$$\tan\theta = h\frac{x}{x^2 + (h+d)d}$$

We must find the value of x which will make $\tan\theta$ a maximum. Let $f(x) = \dfrac{x}{x^2 + (h+d)d}$, so that $\tan\theta = hf(x)$

Then

$$f'(x) = \left(\frac{x}{x^2 + (h+d)d}\right)' = \frac{(x^2 + (h+d)d)(x)' - x(x^2 + (h+d)d)'}{(x^2 + (h+d)d)^2} =$$

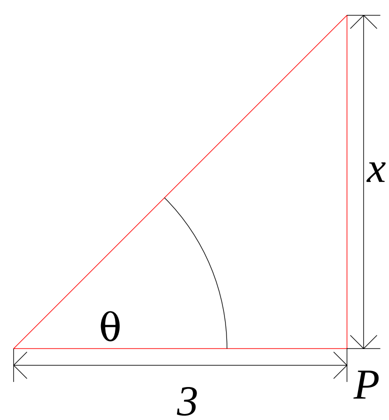
$$\frac{(x^2 + (h+d)d) - x(2x)}{(x^2 + (h+d)d)^2} = \frac{(h+d)d - x^2}{(x^2 + (h+d)d)^2} = 0 \text{ if}$$

$$(h+d)d - x^2 = 0 \text{ or } x = \sqrt{(h+d)d}$$

This value is known as the *geometric mean of d and $h+d$*.

Example: Problem 6.7-68(p.412 of the Brown *Stewart*, or 3.6-66(p.234) of the blue *Stewart*

A lighthouse is on a small island 3 km away from the nearest point **P** on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from **P**?



Solution 1: (Using Inverse Trig Functions)

Variables:

x = beam's distance from **P**

θ = angle between beam of light and line through the lighthouse and **P**

Relations:

$\theta = \arctan \dfrac{x}{3}$

Differentiating, we get

$$\theta' = \frac{1}{1 + \left(\frac{x}{3}\right)^2} \frac{x'}{3} = \frac{3}{9 + x^2} x', \text{ so}$$

$$x' = \frac{9 + x^2}{3} \theta' = \frac{9 + 1^2}{3} \theta' = \frac{10}{3} \theta' = \frac{10}{3} 8\pi = \frac{80}{3} \pi.$$

Solution 2: (Not Using Inverse Trig Functions)

We use the relation:

$\tan\theta(t) = \dfrac{x(t)}{3}$, so differentiation gives
$\sec^2\theta(t)\theta'(t) = \dfrac{x'(t)}{3}$.

We have $\theta'(t) = 4(2\pi)\dfrac{\text{radians}}{\text{min}} = 8\pi\dfrac{\text{radians}}{\text{min}}$, so
$x'(t) = 3\sec^2\theta(t)\theta'(t) = 24\pi\sec^2\theta(t)\dfrac{\text{km}}{\text{min}}$.
When $x = 1$, $\tan\theta(t) = \frac{1}{3}$, and since $\sec^2\alpha \equiv \tan^2\alpha + 1$, we have $\sec^2\theta(t) = \frac{1}{9} + 1 = \frac{10}{9}$, so
$x'(t) = 24\pi\dfrac{10}{9}\dfrac{\text{km}}{\text{min}} = \dfrac{80\pi}{3}\dfrac{\text{km}}{\text{min}} = 1600\pi\dfrac{\text{km}}{\text{hour}}$.