Inverse Functions

Recall the **Vertical Line Test** which tells us whether or not a curve can be the graph of a function:

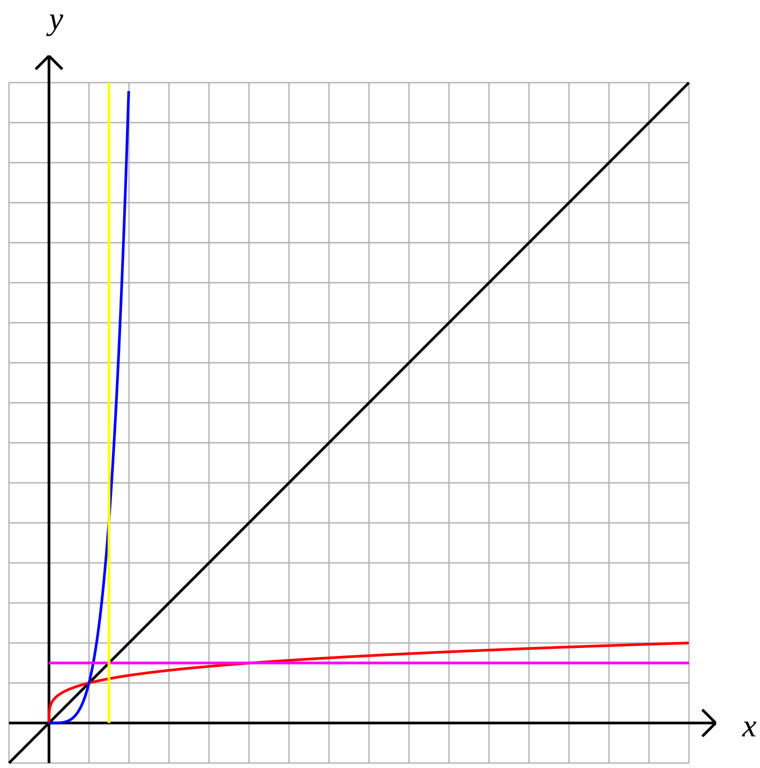
If no vertical line intersects the curve more than once, then the curve is the graph of a function.

There is also the **Horizontal Line Test** :

If no horizontal line intersects the curve more than once, then the curve is the graph of a one-to-one function.

One-to-one functions are functions which do not achieve any value more than once on a specified interval. Any function which is strictly increasing or strictly decreasing on an interval is one-to-one on that interval.

Since the reflection of the graph of a one-to-one function f in the line $y = x$ clearly passes the Vertical Line Test, it is the graph of a new function, which we call the **inverse** of f and which we denote by f^{-1} or by f^{inv}. The domain of f^{-1} is the range of f and vice-versa.



It is best to look at a Java Applet .

We always havethe so-called **Cancellation Equations:**

$f^{-1}(f(x)) = x$ for all x in the domain of f,

and $f(f^{-1}(y)) = y$ for all y in the range of f.

Theorem: The inverse function of a one-to-one continuous function is also continuous.

Differentiation of the Cancellation Equations (using the Chain Rule) yields useful information about the derivatives of inverse functions, when they exist:

$$\frac{d}{dy}\left(f(f^{-1}(y))\right) = \frac{d}{dy}(y) = 1 \iff$$

$$f'(f^{-1}(y))\frac{d}{dy}\left(f^{-1}(y)\right) = 1 \iff$$

$$\frac{d}{dy}\left(f^{-1}(y)\right) = \frac{1}{f'(f^{-1}(y))}$$

Note that there is a problem if the denominator is 0.

or

$$\frac{d}{dy}\left(f(f^{inv}(y))\right) = \frac{d}{dy}(y) = 1 \iff$$

$$f'(f^{inv}(y))\frac{d}{dy}\left(f^{inv}(y)\right) = 1 \iff$$

$$\frac{d}{dy}\left(f^{inv}(y)\right) = \frac{1}{f'(f^{inv}(y))}$$

Examples: (1) $f(x) = 2x + 1$, so $f'(x) = 2$, therefore $\frac{d}{dy}\left(f^{inv}(y)\right) = \frac{1}{f'(f^{inv}(y))} = \frac{1}{2}$

(2) $f(x) = x^3$, so $f'(x) = 3x^2$, and thus

$$\frac{d}{dy}\left(f^{inv}(y)\right) = \frac{1}{f'(f^{inv}(y))} = \frac{1}{3\left(f^{inv}(y)\right)^2}$$

which is rather unsatisfactory as an answer. We have to find a formula for $f^{inv}(y)$ to complete the calculation:

Using the cancellation equation $f(f^{inv}(y)) = y$, we have $\left(f^{inv}(y)\right)^3 = y$, so $f^{inv}(y) = y^{\frac{1}{3}}$.

Therefore

$$\frac{d}{dy}\left(f^{inv}(y)\right) = \frac{1}{3\left(f^{inv}(y)\right)^2} = \frac{1}{3\left(y^{\frac{1}{3}}\right)^2} = \frac{1}{3}y^{-\frac{2}{3}}$$

In most cases, it is impossible to explicitly calculate a formula for the inverse function. We will look at a number of extremely important cases where this can be done.