Techniques of Integration

Integration By Parts

There is NO formula for $\int f(x)g(x)dx$.

It almost never happens that $\int f(x)g(x)dx = \left(\int f(x)dx\right)\left(\int g(x)dx\right)$

Notice that $\int df = f(x) + C$. We often shorten this to $\int df = f$ to indicate that the integral and differential operators "cancel" each other.

The Product Rule for derivatives, $$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + \frac{d}{dx}(u)v$$

has no simple counterpart for antiderivatives. It can be restated in terms of differentials as $d(uv) = udv + vdu$, and if we apply indefinite integral signs, we get $\int d(uv) = \int udv + \int vdu$, or $uv = \int udv + \int vdu$.

We usually use the equivalent formula $\boxed{\int udv = uv - \int vdu.}$

Example: Evaluate $\int x \sin x dx$

Solution: Use integration by parts, with $u = x$, and $dv = \sin x dx$.
Then $du = dx$, and $v = -\cos x$, so

$$\int x \sin x dx = \left(\int udv = uv - \int vdu\right) = x(-\cos x) - \int(-\cos x)dx =$$

$$-x\cos x + \int \cos x dx = -x\cos x + \sin x + C$$

We can also use this technique with definite integrals:

Evaluate $\int_0^{\pi} x \sin x dx$

Solution: $\int_0^{\pi} x \sin x dx = \left(\int u dv = uv - \int v du\right) = x(-\cos x)|_0^{\pi} - \int_0^{\pi}(-\cos x)dx =$

$-x \cos x|_0^{\pi} + \int_0^{\pi} \cos x dx = -x \cos x|_0^{\pi} + \sin x|_0^{\pi} = -\pi \cos \pi - (-0 \cos 0) + \sin \pi - \sin 0 =$
$-\pi(-1) = \pi$

Question: How do I know what u and dv should be?

Students first encountering the technique of using the equation $\int u dv = uv - \int v du$ have trouble knowing what to take for u and what to take for dv.

Answer: Get lots of experience. This is an area where we learn a lot from experience. The Integration by Parts technique is characterized by the need to select u from a number of possibilities. Once u has been chosen, dv is determined, and we hope for the best.

The basic idea underlying Integration by Parts is that we hope that in going from $\int u dv$ to $\int v du$ we will end up with a simpler integral to work with. In the example we have just seen, we were lucky.

Let's try it again, the unlucky way:

Example: Evaluate $\int (\sin x) x dx$

Solution: Use integration by parts, with $u = \sin x$, and $dv = x dx$.
Then $du = \cos x dx$, and $v = \dfrac{x^2}{2}$, so

$$\int (\sin x) x dx = \left(\int u dv = uv - \int v du\right) = \sin x \frac{x^2}{2} - \int \frac{x^2}{2} \cos x dx =$$

$$\frac{x^2}{2} \sin x - \frac{1}{2} \int x^2 \cos x dx$$

which involves a tougher looking integral than we started with.

As a rule of thumb, one-third of the possible choices will lead to an easier integral, one-third will lead to a harder one, and one-third will lead to one of equal difficulty.

It is also possible to spin your wheels, and go around in circles, as we shall soon see.

Let's try a strategic approach to our example: u has to be selected so that $udv = x\sin x dx$, so we look at the possible choices for u: x, $\sin x$, or $x\sin x$. Once u is selected, we have $dv = \frac{x\sin x dx}{u}$, and all we have to is find du(easy) and $v = \int dv$ (possibly very hard or impossible). Then we try to decide if we can get anywhere with $\int v du$. Thus, we can let u be any factor of $f(t)$, including 1, and the corresponding dv is determined. (Of course, if we let $u = 1$, the problem of finding v is just our original integration problem, so we will omit it.) If we cannot then find v we know we have a non-viable selection of the pair u and dv.

We shall illustrate the rather inefficient technique of examining all the possibilities and discarding the non-viable ones in the following examples. Organizing our information in a table is helpful:

u	$dv = \frac{x\sin x dx}{u}$	$v = \int dv$	du	$v du$	$\int v du$	Better?
x	$\frac{x\sin x dx}{x} = \sin x dx$	$\int \sin x dx = -\cos x$	dx	$-\cos x dx$	$\int -\cos x dx = -\sin x$	YES!
$\sin x$	$\frac{x\sin x dx}{\sin x} = x dx$	$\int x dx = \frac{x^2}{2}$	$\cos x dx$	$\frac{x^2}{2}\cos x dx$	$\int \frac{x^2}{2}\cos x dx$	NO!
$x\sin x$	$\frac{x\sin x dx}{x\sin x} = dx$	$\int dx = x$	$\sin x + x\cos x$	$x\sin x + x^2\cos x$	$\int x(\sin x + x\cos x)dx$	NO!

There are some simple integrals where little choice is available: knowing which of a large number techniques to use is crucial.

Example: $\int \ln x dx$ obviously requires

$u = \ln x$, $dv = dx$, so that $v = x$ and $du = \frac{dx}{x}$.

$$\int \ln x dx = \left(\int udv = uv - \int vdu\right) = (\ln x)x - \int x\frac{dx}{x} = x\ln x - \int dx =$$

$x\ln x - x + C$

Example: $\int \arctan x\,dx$ also obviously requires

$u = \arctan x$, $dv = dx$, so that $v = x$ and $du = \dfrac{dx}{1+x^2}$

$$\int \arctan x\,dx = \left(\int u\,dv = uv - \int v\,du\right) = (\arctan x)x - \int x\frac{dx}{1+x^2} = x\arctan x - \frac{1}{2}\int \frac{2x\,dx}{1+x^2} = x\arctan x - \frac{1}{2}\ln(1+x^2) + C =$$

$$x\arctan x - \ln\sqrt{1+x^2} + C$$

Indefinite Integration of e^{kt} times another factor

We now look at a family of integrals which show up in LSD, (Linear Systems Design), particularly in the calculation of Laplace Transforms.

We define $\mathcal{G}(G) = \int e^{kt}G(t)\,dt$, where G is any function. The types of *continuous* function G that arise in practical situations are often sums and products of polynomials, exponential functions, and sinusoidal functions. Fortunately, we know how to evaluate these using the technique of integration by parts.

Examples:

(1) $G(t) = c$, a constant. Integration by parts is not needed here. Then $\mathcal{G}(c) = \int e^{kt}G(t)\,dt = \dfrac{c}{k}e^{kt} + C$

(2) $G(t) = t$. Then $\mathcal{G}(t) = \int e^{kt}t\,dt$.
Here $f(t) = e^{kt}t$ has four different possible factorizations:

u	$dv = \frac{f(t)dt}{u}$	v	du	$v\,du$	Viable?
t	$e^{kt}dt$	$\frac{1}{k}e^{kt}$	dt	$\frac{1}{k}e^{kt}dt$	Yes!
e^{kt}	$t\,dt$	t^2	$ke^{kt}dt$	$kt^2e^{kt}dt$	No!
$e^{kt}t$	dt	t	$(t+k)e^{kt}dt$	$t(t+k)e^{kt}dt$	No!

We use the one viable factorization, $u = t$, $dv = e^{kt}dt$:
$\int t(e^{kt}dt) = \int u\,dv = uv - \int v\,du =$
$t\frac{1}{k}e^{kt} - \int \frac{1}{k}e^{kt}dt = \frac{t}{k}e^{kt} - \frac{1}{k^2}e^{kt} + C$
$= \left(\frac{t}{k} - \frac{1}{k^2}\right)e^{kt} + C$

(3) $G(t) = t^2$. Then $\mathcal{G}(t^2) = \int e^{kt} t^2 dt$, so $f(t) = e^{kt} t^2$. This has four possible factorizations:

u	$dv = \frac{f(t)dt}{u}$	v	du	vdu	Viable?
t	$e^{kt} t dt$	$\left(\frac{t}{k} - \frac{1}{k^2}\right) e^{kt}$	dt	$\left(\frac{t}{k} - \frac{1}{k^2}\right) e^{kt} dt$	Yes, but messy
t^2	$e^{kt} dt$	$\frac{1}{k} e^{kt}$	$2t dt$	$\frac{2}{k} t e^{kt} dt$	Yes!
e^{kt}	$t^2 dt$	$\frac{1}{3} t^3$	$k e^{kt} dt$	$\frac{k}{3} t^3 e^{kt} dt$	No!
$e^{kt} t$	$t dt$	$\frac{1}{2} t^2$	$(t+k) e^{kt} dt$	$\frac{1}{2} t^2 (t+k) e^{kt} dt$	No!
$e^{kt} t^2$	dt	t	$(kt^2 + 2t) e^{kt} dt$	$t(kt^2 + 2t) e^{kt} dt$	No!

We see that if we use the factorization $u = t^2$, $dv = e^{kt} dt$, we get $\int t^2 e^{kt} dt = \int u dv = uv - \int v du = \frac{t^2}{k} e^{kt} - \frac{2}{k} \int t e^{kt} dt =$
$\frac{t^2}{k} e^{kt} - \frac{2}{k} \left(\frac{t^2}{k} - \frac{2}{k}\right) e^{kt} =$
$\left(\frac{t^2}{k} - \frac{2t}{k^2} + \frac{2}{k^3}\right) e^{kt} + C$

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(4) We notice how the problem of evaluating $\int t^2 e^{kt}$ was reduced to the evaluation of $\int e^{kt} t dt$, which had just been done. We suspect the existence of a *reduction formula* for $\int e^{kt} t^n dt$. Having been successful in taking $dv = e^{kt} dt$ in the two preceding examples, we decide to do this again, and we have $u = \frac{e^{kt} t^n dt}{e^{kt} dt} = t^n$. We calculate $v = \frac{1}{k} e^{kt}$ and $du = n t^{n-1} dt$, so that:
$\int e^{kt} t^n dt = \int u dv = uv - \int v du = \frac{1}{k} t^n e^{kt} - \frac{n-1}{k} \int e^{kt} t^{n-1} dt$.

Thus we have $\mathcal{G}(t^n) = \frac{1}{k} t^n e^{kt} - \frac{n-1}{k} \mathcal{G}(t^{n-1})$.

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(5) $G(t) = \sin at$. We have $f(t) = e^{kt} \sin at$, so there are just three choices for u:

u	$dv = \frac{f(t)dt}{u}$	v	du	vdu	Viable?
$\sin at$	$e^{kt} dt$	$\frac{1}{k} e^{kt}$	$a \cos at dt$	$\frac{a}{k} e^{kt} \cos at dt$	Yes
e^{kt}	$\sin at dt$	$-\frac{1}{a} \cos at$	$k e^{kt} dt$	$-\frac{k}{a} e^{kt} \cos at dt$	Yes
$e^{kt} \sin at$	dt	t	$(k \sin at + a \cos at) e^{kt} dt$	$t(k \sin at + a \cos at) e^{kt} dt$	No!

The first two choices are both viable, and we see that they both lead to the evaluation of $\int e^{kt} \cos at dt$. We will examine both cases closely:

First Choice: $u_1 = \sin at$, $dv_1 = e^{kt} dt$
We get: $\mathcal{G}(\sin at) = \int e^{kt} \sin at dt = \int u_1 dv_1 = u_1 v_1 - \int v_1 du_1 = \sin at \frac{1}{k} e^{kt} - \int \frac{a}{k} e^{kt} \cos at dt =$
$\sin at \frac{1}{k} e^{kt} - \frac{a}{k} \int e^{kt} \cos at dt =$
$\sin at \frac{1}{k} e^{kt} - \frac{a}{k} \mathcal{G}(\cos at)$
or

$$\mathcal{G}(\sin at) = \sin at \frac{1}{k} e^{kt} - \frac{a}{k} \mathcal{G}(\cos at)$$

In evaluating $\int e^{kt} \cos at dt$ we again have three choices, two of which are viable:

u	$dv = \frac{f(t)dt}{u}$	v	du	vdu	Viable?
$\cos at$	$e^{kt}dt$	$\frac{1}{k}e^{kt}$	$-a\sin at dt$	$-\frac{a}{k}e^{kt}\sin at dt$	Yes
e^{kt}	$\cos at dt$	$\frac{1}{a}\sin at$	$ke^{kt}dt$	$\frac{k}{a}e^{kt}\sin at dt$	Yes
$e^{kt}\cos at$	dt	t	$(k\cos at - a\sin at)e^{kt}dt$	$t(k\sin at + a\cos at)e^{kt}dt$	No!

Again we have two choices. First and Best Choice:
We will let $U_1 = \cos at$, $dV_1 = e^{kt}dt$, and we get:
$\mathcal{G}(\cos at) = \int \cos at e^{kt} dt = \int U_1 dV_1 = U_1V_1 - \int V_1 dU_1 =$
$\cos at \frac{1}{k}e^{kt} - \int \frac{1}{k}e^{kt}(-a\sin at dt) =$
$\frac{1}{k}\cos at e^{kt} + \frac{a}{k}\int e^{kt}\sin at dt =$
$\frac{1}{k}\cos at e^{kt} + \frac{a}{k}\mathcal{G}(\sin at)$,
or

$$\mathcal{G}(\cos at) = \frac{1}{k}\cos at e^{kt} + \frac{a}{k}\mathcal{G}(\sin at)$$

so we are back where we started! However, if we substitute this into the equation

$$\mathcal{G}(\sin at) = \sin at \frac{1}{k}e^{kt} - \frac{a}{k}\mathcal{G}(\cos at)$$

we get $\mathcal{G}(\sin at) = \sin at \frac{1}{k}e^{kt} - \frac{a}{k}\left(\frac{1}{k}\cos at e^{kt} + \frac{a}{k}\mathcal{G}(\sin at)\right) = \sin at \frac{1}{k}e^{kt} - \frac{a}{k^2}\cos at e^{kt} - \frac{a^2}{k^2}\mathcal{G}(\sin at)$
which can be solved for $\mathcal{G}(\sin at)$:

$(1 + \frac{a^2}{k^2})\mathcal{G}(\sin at) = \frac{a^2+k^2}{k^2}\mathcal{G}(\sin at) = \sin at \frac{1}{k}e^{kt} - \frac{a}{k^2}\cos at e^{kt} =$

$e^{kt}(\sin at \frac{k}{k^2} - \frac{a}{k^2}\cos at) = \frac{e^{kt}}{k^2}(k\sin at - a\cos at)$ so

$$\mathcal{G}(\sin at) = \frac{k\sin at - a\cos at}{a^2+k^2}e^{kt}$$

Last and Worst Choice: We will now let $U_2 = e^{kt}$, $dV_2 = \cos at$, and we get:

$\mathcal{G}(\cos at) = e^{kt}(\frac{1}{a}\sin at) - \int \frac{1}{a}\sin at(e^{kt}kdt) = \frac{1}{a}e^{kt}\sin at - \frac{k}{a}\int e^{kt}\sin at dt = \frac{1}{a}e^{kt}\sin at - \frac{k}{a}\mathcal{G}(\sin at)$
This time, however, if we substitute this into the equation

$$\mathcal{G}(\sin at) = \sin at \frac{1}{k}e^{kt} - \frac{a}{k}\mathcal{G}(\cos at)$$

we get

$$\mathcal{G}(\sin at) = \sin at \frac{1}{k}e^{kt} - \frac{a}{k}\left(\frac{1}{a}e^{kt}\sin at - \frac{k}{a}\mathcal{G}(\sin at)\right) = \mathcal{G}(\sin at)$$

so there is no information gained.

Second Choice: $u_2 = e^{kt}$, $dv_2 = \sin at dt$
We get: $\mathcal{G}(\sin at) = \int e^{kt} \sin at dt = \int u_2 dv_2 = u_2 v_2 - \int v_2 du_2 = e^{kt}(-\frac{1}{a}\cos at) - \int \frac{k}{a} e^{kt} \cos at dt =$

$-\frac{1}{a} e^{kt} \cos at - \frac{k}{a} \mathcal{G}(\cos at)$

or

$$\mathcal{G}(\sin at) = -\frac{1}{a} e^{kt} \cos at - \frac{k}{a} \mathcal{G}(\cos at)$$

Using, from above,

$$\mathcal{G}(\cos at) = \frac{1}{k} \cos at e^{kt} + \frac{a}{k} \mathcal{G}(\sin at)$$

we get

$$\mathcal{G}(\sin at) = -\frac{1}{a} e^{kt} \cos at - \frac{k}{a} \left(\frac{1}{k} \cos at e^{kt} + \frac{a}{k} \mathcal{G}(\sin at) \right) = \mathcal{G}(\sin at)$$

so there is no new information. On the other hand, if we use

$$\mathcal{G}(\cos at) = \frac{1}{a} e^{kt} \sin at - \frac{k}{a} \mathcal{G}(\sin at)$$

we get the same value as before.

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(6)$\mathcal{G}(\cos at)$. This is easily derived from the calculations of the previous example.

$$\mathcal{G}(\cos at) = \frac{1}{k} \cos at e^{kt} + \frac{a}{k} \mathcal{G}(\sin at) = \frac{1}{k} \cos at e^{kt} + \frac{a}{k} \left(\frac{k \sin at - a \cos at}{a^2 + k^2} e^{kt} \right) = \frac{a \sin at + k \cos at}{a^2 + k^2} e^{kt}$$

so

$$\mathcal{G}(\cos at) = \frac{a \sin at + k \cos at}{a^2 + k^2} e^{kt}$$

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(7)$\mathcal{G}(t^n \sin at) = \int t^n \sin at e^{kt} dt$
Let $u = t^n \sin at$, $dv = e^{kt} dt$, so that $du = (nt^{n-1} \sin at + at^n \cos at) dt$, and $v = \frac{1}{k} e^{kt}$. Then
$\mathcal{G}(t^n \sin at) = \int t^n \sin at e^{kt} dt = \int u dv = uv - \int v du =$
$t^n \sin at (\frac{1}{k} e^{kt}) - \int \frac{1}{k} e^{kt} ((nt^{n-1} \sin at + at^n \cos at) dt) =$
$\frac{1}{k} t^n \sin at e^{kt} - \frac{n}{k} \mathcal{G}(t^{n-1} \sin at) - \frac{a}{k} \mathcal{G}(t^n \cos at)$
or

$$\mathcal{G}(t^n \sin at) = \frac{1}{k} t^n \sin at e^{kt} - \frac{n}{k} \mathcal{G}(t^{n-1} \sin at) - \frac{a}{k} \mathcal{G}(t^n \cos at)$$

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(8)$\mathcal{G}(t^n \cos at) = \int t^n \cos at e^{kt} dt$. Let $U = t^n \cos at$ and $dV = e^{kt} dt$, so that $dU = (nt^{n-1} \cos at - at^n \sin at)dt$ and $V = \frac{1}{k} e^{kt}$. Then

$\mathcal{G}(t^n \cos at) = \int t^n \cos at e^{kt} dt = \int U dV = UV - \int V du =$
$t^n \cos at \frac{1}{k} e^{kt} - \int e^{kt}((nt^{n-1} \cos at - at^n \sin at)dt) =$
$\frac{1}{k} t^n \cos at e^{kt} - \frac{n}{k} \mathcal{G}(t^{n-1} \cos at) + \frac{a}{k} \mathcal{G}(t^n \sin at)$.

We can now solve for $\mathcal{G}(t^n \sin at)$:

$$\mathcal{G}(t^n \sin at) = \frac{1}{a^2 + k^2} \left[(k \sin at - a \cos at) t^n e^{kt} - n(\mathcal{G}(t^{n-1}(k \sin at - a \cos at))) \right]$$

and hence

$$\mathcal{G}(t^n \cos at) = \frac{1}{a^2 + k^2} \left[(k \cos at - a \sin at) t^n e^{kt} + n(\mathcal{G}(t^{n-1}(k \cos at - a \sin at))) \right]$$

Trigonometric Integrals

It is often necessary to evaluate integrals of the form

$$\int (\sin x)^m (\cos x)^n dx, \text{ where } m \text{ and } n \text{ are integers.}$$

If one of the exponents, either m or n is and odd, there is a straightforward simplification.

Case 1: $m = 2M + 1$ is odd.

Then $(\sin x)^m = (\sin x)^{2M+1} = (\sin^2 x)^M \sin x = (1 - \cos^2 x)^M \sin x$,

so our integral becomes

$$\int (\sin x)^m (\cos x)^n dx = \int (1 - \cos^2 x)^M \sin x (\cos x)^n dx$$

and we may make the substitution $u = \cos x$ with $du = -\sin x dx$ to get

$$\int (\sin x)^m (\cos x)^n dx = \int (1 - \cos^2 x)^M \sin x (\cos x)^n dx = -\int (1 - u^2)^M u^n du$$

Example: $\int (\sin x)^{11} (\cos x)^{10} dx = -\int (1 - u^2)^5 u^{10} du =$

$$-\int \left[1 - 5u^2 + 10(u^2)^2 - 10(u^2)^3 + 5(u^2)^4 - (u^2)^5\right] u^{10} du =$$

$$-\int u^{10} - 5u^{12} + 10u^{14} - 10u^{16} + 5u^{18} - u^{20} du =$$

$$-\left(\frac{u^{11}}{11} - 5\frac{u^{13}}{13} + 10\frac{u^{15}}{15} - 10\frac{u^{17}}{17} + 5\frac{u^{19}}{19} - \frac{u^{21}}{21}\right) + C =$$

$$-\frac{\cos^{11} x}{11} + \frac{5\cos^{13} x}{13} - \frac{2\cos^{15} x}{5} + \frac{10\cos^{17} x}{17} - \frac{5\cos 19x}{19} + \frac{\cos^{21} x}{21} + C$$

Example: $\int (\sin x)^{11}(\cos x)^{-10}dx = -\int (1-u^2)^5 u^{-10}du =$

$-\int \left[1-5u^2+10(u^2)^2-10(u^2)^3+5(u^2)^4-(u^2)^5\right] u^{-10}du =$

$-\int u^{-10}-5u^{-8}+10u^{-6}-10u^{-4}+5u^{-2}-u^0 du =$

$-\left(\frac{u^{-9}}{-9}-5\frac{u^{-7}}{-7}+10\frac{u^{-5}}{-5}-10\frac{u^{-3}}{-3}+5\frac{u^{-1}}{-1}-u\right)+C =$

$-\left(-\frac{\cos^{-9}x}{9}+\frac{5\cos^{7}x}{7}-2\cos^{-5}x+\frac{10\cos^{-3}x}{3}-5\cos^{-1}x-\cos x\right)+C =$

$\frac{1}{9}\sec^9 x-\frac{5}{7}\sec^7 x+2\sec^5 x-\frac{10}{3}\sec^3 x+5\sec x+\cos x+C$

Example: $\int (\sin x)^{-11}(\cos x)^{-10}dx = \int (\sin x)^{-12}(\cos x)^{-10}\sin x dx = -\int (1-u^2)^{-6}u^{-10}du$

can be done, but requires the method of Partial Fractions, which we shall see later.

The situation is similar when the power of $\cos x$ is odd.

Example: $\int (\sin x)^{10}(\cos x)^{11}dx = \int (\sin x)^{10}(\cos x)^{10}\cos x dx =$

$\int (\sin x)^{10}(\cos^2 x)^5\cos x dx = \int (\sin x)^{10}(1-\sin^2 x)^5\cos x dx =$

(letting $u=\sin x$ and $du=\cos x dx$)

$\int u^{10}(1-u^2)^5 du = \int u^{10}\left[1-5u^2+10(u^2)^2-10(u^2)^3+5(u^2)^4-(u^2)^5\right]du =$

$\int u^{10}-5u^{12}+10u^{14}-10u^{16}+5u^{18}-u^{20}du =$

$\frac{u^{11}}{11}-5\frac{u^{13}}{13}+10\frac{u^{15}}{15}-10\frac{u^{17}}{17}+5\frac{u^{19}}{19}-\frac{u^{21}}{21}+C =$

$\frac{1}{11}\sin^{11}x-\frac{5}{13}\sin^{13}x+\frac{2}{5}\sin^{15}x-\frac{10}{17}\sin^{17}x+\frac{5}{19}\sin^{19}-\frac{1}{21}\sin^{21}x+C$

When both powers are odd, it is easiest to select the function with the highest power for substitution:

Good Example: $\int (\sin x)^{11}(\cos x)^3 dx = \int (\sin x)^{11}(\cos x)^2 \cos x dx =$

$\int (\sin x)^{11}(1-\sin^2 x)\cos x dx =$

(letting $u = \sin x$ and $du = \cos x dx$)

$$\int u^{11}(1-u^2)du = \int u^{11} - u^{13} du = \frac{u^{11}}{11} - \frac{u^{14}}{14} + C = \frac{1}{11}\sin^{11} x - \frac{1}{14}\sin^{13} x + C$$

Bad Example: $\int (\sin x)^{11}(\cos x)^3 dx = \int (\sin x)^{10}(\cos x)^3 \sin x dx =$

$\int (1-\cos^2 x)^5(\cos x)^3 \sin x dx =$

(letting $u = \cos x$ and $du = -\sin x dx$)

$$\int (1-u^2)^5 u^3(-du) = \int \left[-1 + 5u^2 - 10(u^2)^2 + 10(u^2)^3 - 5(u^2)^4 + (u^2)^5\right] u^3 du =$$

$$\int \left[-u^3 + 5u^5 - 10u^7 + 10u^9 - 5u^{11} + u^{13}\right] du =$$

$$-\frac{u^4}{4} + 5\frac{u^6}{6} - 10\frac{u^8}{8} + 10\frac{u^{10}}{10} - 5\frac{u^{12}}{12} + \frac{u^{14}}{14} + C =$$

$$-\frac{1}{4}\cos^4 x + \frac{5}{6}\cos^6 x - \frac{5}{4}\cos^8 x + \cos^{10} x - \frac{5}{12}\cos^{12} x + \frac{1}{14}\cos^{14} x + C$$

When neither power is odd, we need to use a double angle formula from trigonometry:

$\cos 2x \equiv 2\cos^2 x - 1 = 1 - 2\sin^2 x$

can be solved for $\cos^2 x$ and $\sin^2 x$:

$\cos^2 x = \dfrac{1 + \cos 2x}{2}$ and $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

Thus, if we want to integrate $\int (\sin x)^m (\cos x)^n dx$, where m and n are even integers, we write $m = 2M$ and $n = 2N$ and we have:

$$\int (\sin x)^m (\cos x)^n dx = \int (\sin x)^{2M} (\cos x)^{2N} dx = \int (\sin^2 x)^M (\cos^2 x)^N dx =$$

$$\int \left(\frac{1 - \cos 2x}{2}\right)^M \left(\frac{1 + \cos 2x}{2}\right)^N dx = 2^{-(M+N)} \int (1 - \cos 2x)^M (1 + \cos 2x)^N \, dx$$

Example: $\int \cos^2 x dx = \int \frac{1+\cos 2x}{2} dx =$

$\frac{1}{2}\int 1+\cos 2x dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C =$

$\frac{x}{2} + \frac{1}{4}\sin 2x + C = \frac{x}{2} + \frac{1}{4}2\sin x\cos x + C = \frac{x+\sin x\cos x}{2} + C$

Example: $\int \sin^2 x dx = \int \frac{1-\cos 2x}{2} dx =$

$\frac{1}{2}\int 1-\cos 2x dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C =$

$\frac{x}{2} - \frac{1}{4}\sin 2x + C = \frac{x}{2} - \frac{1}{4}2\sin x\cos x + C = \frac{x-\sin x\cos x}{2} + C$

It should come as no surprise that $\int \left(\cos^2 x + \sin^2 x\right) dx = x + C$

Example: $\int \sin^2 x\cos^2 x dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2} dx\right) =$

$\frac{1}{4}\int 1-\cos^2 2x dx = \frac{1}{4}\int 1 - \frac{1+\cos 4x}{2} dx = \frac{1}{8}\int 1-\cos 4x dx =$

$\frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + C = \frac{x}{8} - \frac{\sin 4x}{32} + C$

Products of powers of secant and tangent functions

$\sec^m x\tan^n x$ can be expressed as a product of powers of $\sin x$ and $\cos x$, but it is often more convenient to use the identity $\tan^2 x + 1 = \sec^2 x$ and the differentials $d(\tan x) = \sec^2 x dx$ and $d(\sec x) = \sec x\tan x dx$. We assume that m and n are non-negative.

Case 1: The power m of $\sec x$ is positive and even: then $m = 2M$ and we have:

$\int \sec^m x\tan^n x dx = \int (\sec^2 x)^M \tan^n x dx = \int (\sec^2 x)^{M-1}\tan^n x\sec^2 x dx = \int (1+\tan^2 x)^{M-1}\tan^n x\sec^2 x$

and we can make the substitution $u = \tan x$, $du = \sec^2 x dx$, and we get

$\int \left(1+\tan^2 x\right)^{M-1}\tan^n x\sec^2 x dx = \int \left(1+u^2\right)^{M-1} u^n du$

Example: $\int \sec^{10} x \tan^5 x dx = \int \left(1+u^2\right)^4 u^5 du =$

$\int \left(1+4u^2+6u^4+4u^6+u^8\right) u^5 du = \int u^5+4u^7+6u^9+4u^{11}+u^{13} du =$

$\frac{u^6}{6}+4\frac{u^8}{8}+6\frac{u^{10}}{10}+4\frac{u^{12}}{12}+\frac{u^{14}}{14}+C =$

$\frac{1}{6}u^6+\frac{1}{2}u^8+\frac{3}{5}u^{10}+\frac{1}{3}u^{12}+\frac{1}{14}u^{14}+C =$

$\frac{1}{6}\tan^6 x+\frac{1}{2}\tan^8 x+\frac{3}{5}\tan^{10} x+\frac{1}{3}\tan^{12} x+\frac{1}{14}\tan^{14} x+C$

If the power of $\sec x$ is not even, but the power of n of $\tan x$ is odd, so that $n = 2N+1$, and m is positive, we can make the substitution $u = \sec x$, $du = \sec x \tan x dx$, and get

$\int \sec^m x \tan^n x dx = \int \sec^{m-1} x \tan^{2n} x \sec x \tan x dx =$

$\int u^{m-1}\left(\tan^2 x\right)^N du = \int u^{m-1}\left(\sec^2 x - 1\right)^N du =$

$\int u^{m-1}\left(u^2-1\right)^N du$

Example: $\int \sec^5 x \tan^5 x dx = \int u^4(u^2-1)^2 du = \int u^4(u^4-2u^2+1)du =$

$\int u^8-2u^6+u^4 du \frac{u^9}{9}-2\frac{u^7}{7}+\frac{u^5}{5}+C =$

$\frac{1}{9}\sec^9 x-\frac{2}{7}\sec^7 x+\frac{1}{5}\sec^5 x+C$

We are left with the cases where m is odd and n is even, all of which can be reduced to the problem of finding the antiderivative of an odd power of $\sec x$.

Example: $\int \sec x dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx =$

$\int \left(\frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\right) dx = \int \frac{(\sec^2 x + \sec x \tan x)dx}{\sec x + \tan x} =$

$\int \frac{d(\sec x + \tan x)}{\sec x + \tan x} = \ln(\sec x + \tan x) + C$

Example: $\mathcal{I} = \int \sec^3 x\,dx$

Using Integration by Parts, with
$u = \sec x$, $dv = \sec^2 x$, $v = \tan x$, $du = \sec x \tan x\,dx$, we get

$\mathcal{I} = \int \sec^3 x\,dx = \left(\int u\,dv = uv - \int v\,du\right) =$

$\sec x \tan x - \int \tan x \sec x \tan x\,dx = \sec x \tan x - \int \tan^2 x \sec x\,dx =$

$\sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx =$

$\sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx = \sec x \tan x - \mathcal{I} + \int \sec x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C$ so

$2\mathcal{I} = \sec x \tan x + \ln|\sec x + \tan x| + C$ and

$$\int \sec^3 x\,dx = \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C$$

Example: $\int \tan^2 x \sec x\,dx$ appears in the previous calculation, and is one of the simpler cases left:

$\mathcal{I} = \sec x \tan x - \int \tan^2 x \sec x\,dx$ gives us

$\int \tan^2 x \sec x\,dx = \sec x \tan x - \mathcal{I} =$

$\sec x \tan x - \dfrac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C =$

$\dfrac{\sec x \tan x - \ln|\sec x + \tan x|}{2} + C$

Integrals of products of sine and cosine functions with different arguments

The identities:

$$\cos x \cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$$

$$\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

$\sin x \cos y = \frac{1}{2}\left(\sin(x-y) + \sin(x+y)\right)$ may be used:

Example: $\int \cos 5t \cos 7t\,dt = \int \frac{1}{2}\left(\cos(5t+7t) + \cos(5t-7t)\right)dt =$

$\int \frac{1}{2}\left(\cos 12t + \cos(-2t)\right)dt = \frac{1}{2}\left(\frac{1}{12}\sin 12t + \frac{1}{2}\sin 2t\right) + C =$

$\frac{1}{24}\sin 12t + \frac{1}{4}\sin 2t + C$

Trigonometric Substitutions

It is often necessary to evaluate integrals containing expressions of the form

$$a^2 - x^2, \qquad a^2 + x^2, \qquad \text{or} \qquad x^2 - a^2.$$

For example, we may wish to evaluate

$$\int \sqrt{a^2 - x^2}dx \qquad \text{or} \qquad \int \sqrt{a^2 + x^2}dx \qquad \text{or} \qquad \int \sqrt{x^2 - a^2}dx$$

To do this , we need to make the appropriate substitution:

Expression	$x =$	$dx =$	Expression $=$
$a^2 - x^2$	$a\sin\theta$	$a\cos\theta d\theta$	$a^2\cos^2\theta$
$a^2 + x^2$	$a\tan\theta$	$a\sec^2\theta d\theta$	$a^2\sec^2\theta$
$x^2 - a^2$	$a\sec\theta$	$a\sec\theta\tan\theta d\theta$	$a^2\tan^2\theta$

Example: $\int \sqrt{a^2 - x^2}dx$

Letting $x = a\sin\theta$, we have $dx = a\cos\theta d\theta$, so that

$$\int \sqrt{a^2 - x^2}dx = \int \left(\sqrt{a^2 - a^2\sin^2\theta}\right) a\cos\theta d\theta =$$

$$a^2 \int \left(\sqrt{1 - \sin^2\theta}\right) \cos\theta d\theta = a^2 \int \cos^2\theta d\theta =$$

$$a^2 \int \frac{1 + \cos 2\theta}{2} d\theta = \frac{a^2}{2} \int d\theta + \frac{a^2}{2} \int \cos 2\theta d\theta =$$

$$\frac{a^2}{2}\theta + \frac{a^2}{2}\frac{1}{2}\sin 2\theta + C = \frac{a^2\theta}{2} + \frac{a^2}{4} 2\sin\theta\cos\theta + C =$$

$$\frac{a^2}{2}\theta + \frac{a^2}{2}\sin\theta\cos\theta + C = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{2}\frac{x}{a}\sqrt{1 - \left(\frac{x}{a}\right)^2} + C =$$

$$\frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

Let us use this to compute some areas contained in circles of radius a:

First, the area under $y = \sqrt{a^2 - x^2}$ from $-a$ to a is that of a semicircle of radius a:

$$\int_{-a}^{a} \sqrt{a^2 - x^2}dx = \left(\frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2}\right)\Bigg|_{-a}^{a} =$$

$$\left(\frac{a^2}{2}\arcsin\frac{a}{a} + \frac{a}{2}\sqrt{a^2 - a^2}\right) - \left(\frac{a^2}{2}\arcsin\frac{-a}{a} + \frac{-a}{2}\sqrt{a^2 - (-a)^2}\right) =$$

$$\left(\frac{a^2}{2}\arcsin(1) + 0\right) - \left(\frac{a^2}{2}\arcsin(-1) + 0\right) = \frac{a^2}{2}\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = \frac{\pi a^2}{2},$$

as expected.

Second, the area under $y = \sqrt{a^2 - x^2}$ from 0 to a is that of a quarter circle of radius a:

$$\int_{0}^{a} \sqrt{a^2 - x^2}dx = \left(\frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2}\right)\Bigg|_{0}^{a} =$$

$$\left(\frac{a^2}{2}\arcsin\frac{a}{a} + \frac{a}{2}\sqrt{a^2 - a^2}\right) - \left(\frac{a^2}{2}\arcsin\frac{0}{a} + \frac{0}{2}\sqrt{a^2 - 0^2}\right) =$$

$$\left(\frac{a^2}{2}\arcsin(1) + 0\right) - \left(\frac{a^2}{2}\arcsin 0 + 0\right) = \frac{a^2}{2}\frac{\pi}{2} = \frac{\pi a^2}{4} \text{ as expected.}$$

Third, the area under $y = \sqrt{a^2 - x^2}$ from $x = \frac{a}{2}$ to $x = \frac{a\sqrt{3}}{2}$ is:

$$\int_{\frac{a}{2}}^{\frac{a\sqrt{3}}{2}} \sqrt{a^2 - x^2}dx = \left(\frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2}\right)\Bigg|_{\frac{a}{2}}^{\frac{a\sqrt{3}}{2}} =$$

$$\left(\frac{a^2}{2}\arcsin\frac{\frac{a\sqrt{3}}{2}}{a} + \frac{\frac{a\sqrt{3}}{2}}{2}\sqrt{a^2 - \left(\frac{a\sqrt{3}}{2}\right)^2}\right) - \left(\frac{a^2}{2}\arcsin\frac{\frac{a}{2}}{a} + \frac{\frac{a}{2}}{2}\sqrt{a^2 - \left(\frac{a}{2}\right)^2}\right) =$$

$$\left(\frac{a^2}{2}\arcsin\frac{\sqrt{3}}{2} + \frac{a\sqrt{3}}{4}a\sqrt{\frac{1}{4}}\right) - \left(\frac{a^2}{2}\arcsin\frac{1}{2} + \frac{a}{4}a\sqrt{\frac{3}{4}}\right) =$$

$$a^2\left[\left(\frac{1}{2}\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right) - \left(\frac{1}{2}\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right)\right] = a^2\left(\frac{\pi}{6} - \frac{\pi}{12}\right) = \frac{\pi a^2}{12}$$

In general the area under $y = \sqrt{a^2 - x^2}$ from $x = c$ to $x = d$ ($c \le d$) is:

$$\int_c^d \sqrt{a^2 - x^2}dx = \left(\frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2}\right)\Bigg|_c^d =$$

$$\left(\tfrac{a^2}{2}\arcsin\tfrac{d}{a} + \tfrac{d}{2}\sqrt{a^2 - d^2}\right) - \left(\tfrac{a^2}{2}\arcsin\tfrac{c}{a} + \tfrac{c}{2}\sqrt{a^2 - c^2}\right) =$$

$$\tfrac{a^2}{2}\left(\arcsin\tfrac{d}{a} - \arcsin\tfrac{c}{a}\right) + \left(\tfrac{d}{2}\sqrt{a^2 - d^2} - \tfrac{c}{2}\sqrt{a^2 - c^2}\right)$$

Thus the area of the circle $x^2 + y^2 = a^2$ lying between the lines $x = c$ and $x = d$ is

$$a^2\left(\arcsin\frac{d}{a} - \arcsin\frac{c}{a}\right) + \left(d\sqrt{a^2 - d^2} - c\sqrt{a^2 - c^2}\right)$$

Setting $c = 0$, we get a formula for the area A_d between the y-axis and the line $x = d$:

$$A_d = a^2\arcsin\frac{d}{a} + d\sqrt{a^2 - d^2}$$

Example: $\int \sqrt{a^2 + x^2}dx$

We let $x = a\tan\theta$, so that $a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta$,

and $dx = a\sec^2\theta\, d\theta$, so we have

$$\int \sqrt{a^2 + x^2}dx = \int \left(\sqrt{a^2\sec^2\theta}\right) a\sec^2\theta d\theta =$$

$$a^2\int \sec^3\theta d\theta = a^2\frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2} + C$$

Now $\tan\theta = \dfrac{x}{a}$, and $\sec\theta = \dfrac{\sqrt{x^2 + a^2}}{a}$, so our integral in terms of x is:

$$a^2\frac{\frac{\sqrt{x^2+a^2}}{a}\frac{x}{a} + \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right|}{2} + C = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|\frac{\sqrt{x^2 + a^2} + x}{a}\right| + C =$$

$$\frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left(\sqrt{x^2 + a^2} + x\right) + C$$

We have used the previously derived formula

$$\int \sec^3 x\,dx = \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C$$

Example: $\int \sqrt{x^2 - a^2}\,dx$

Letting $x = a\sec\theta$, we have

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$$

(so $\sec\theta = \dfrac{x}{a}$ and $\tan\theta = \dfrac{\sqrt{x^2 - a^2}}{a}$),

and $dx = a\sec\theta\tan\theta\,d\theta$, so

$$\int \sqrt{x^2 - a^2}\,dx = \int \sqrt{a^2\tan^2\theta}\,a\sec\theta\tan\theta\,d\theta = a^2\int \sec\theta\tan^2\theta\,d\theta =$$

$$a^2\frac{\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|}{2} + C =$$

$$a^2\frac{\frac{x}{a}\frac{\sqrt{x^2-a^2}}{a} - \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right|}{2} + C =$$

$$\frac{x\sqrt{x^2 - a^2} - a^2\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|}{2} + C =$$

$$\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + C$$

We have used the previously derived formula

$$\int \tan^2 x\sec x\,dx = \frac{\sec x \tan x - \ln|\sec x + \tan x|}{2} + C$$

It is often necessary to complete squares before using a trig substitution:

Example: $\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{x^2+4x+1+4} = \int \frac{dx}{(x+2)^2+2^2} =$

$\int \frac{du}{u^2+2^2}$

where we have let $u = x+2$.

Now we let $u = 2\tan\theta$ and have $u^2+2^2 = 2^2\sec^2\theta$ and $du = 2\sec^2\theta\, d\theta$,

so we have

$$\int \frac{du}{u^2+2^2} = \int \frac{2\sec^2\theta\, d\theta}{2^2\sec^2\theta} = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C = \frac{1}{2}\arctan\frac{u}{2} + C =$$

$$\frac{1}{2}\arctan\frac{x+2}{2} + C$$

Example:#28, p.454 of Brown *Stewart*: $\int \sqrt{e^{2t}-9}dt$

Note that $e^{2t}-9 = (e^t)^2 - 3^2$, so if we let $x = e^t$, we get

$e^{2t}-9 = (e^t)^2 - 3^2 = x^2 = 3^2$ and we can use the substitution $x = 3\sec\theta$.

We must be *very* careful with the differentials: we have $dx = e^t dt$, so $dt = e^{-t}dx = \frac{1}{e^t}dx = \frac{dx}{x}$

Thus we have $\int \sqrt{e^{2t}-9}dt = \int \sqrt{x^2-3^2}\frac{dx}{x}$

Now we use $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta d\theta$, $x^2-3^2 = 3^2\tan^2\theta$

(so $\sec\theta = \frac{x}{3}$ and $\tan\theta = \frac{\sqrt{x^2-3^2}}{3}$) to get

$$\int \sqrt{x^2-3^2}\frac{dx}{x} = \int \left(\sqrt{3^2\tan^2\theta}\right)\frac{3\sec\theta\tan\theta d\theta}{3\sec\theta} = 3\int \tan^2\theta d\theta = 3\int(\sec^2\theta - 1)d\theta = 3(\tan\theta - \theta) + C = 3\left(\frac{\sqrt{x^2-3^2}}{3} - \operatorname{arcsec}\frac{x}{3}\right) + C =$$

$$\sqrt{x^2-3^2} - 3\operatorname{arcsec}\frac{x}{3} + C = \sqrt{e^{2t}-9} - 3\operatorname{arcsec}\left(\frac{e^t}{3}\right) + C$$

Partial Fractions

This is a technique used to antidifferentiate proper rational functions: quotients of polynomials whose denominator has degree greater than the numerator. Using long division, any rational function can be written as the sum of a polynomial and a proper rational function.

Theorem: Any polynomial with real coefficients can be written as the product of powers of linear and quadratic factors.

We will usually only look at prefactored examples, or those where the factorization is easy.

Example: $\int \frac{1}{u^2-1}du = \int \frac{1}{(u-1)(u+1)}du$

The basic idea is that we wish to write $\frac{1}{(u-1)(u+1)}$ as the sum of the form

$\frac{A}{u-1} + \frac{B}{u+1}$,

where A and B are to be determined so that the equation

$$\frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$

is true for all values of u except -1 and 1.

The ordinary way of finding A and B is to multiply by u^2-1 and to equate the coefficients of the resulting polynomials:

$1 = A(u+1) + B(u-1) = (A+B)u + (A-B)$ gives us two equations in two unknowns:

$A + B = 0$ and $A - B = 1$

which when solved give us $A = \frac{1}{2}$ and $B = -\frac{1}{2}$

A quicker way is to substitute the "illegal" values $u = -1$ and $u = 1$ into the equation $1 = A(u+1) + B(u-1)$

$u = -1$: $1 = A(-1+1) + B(-1-1) = -2B$ so $B = -\dfrac{1}{2}$

$u = 1$: $1 = A(1+1) + B(1-1) = 2A$ so $A = \dfrac{1}{2}$

We then have

$\dfrac{1}{(u-1)(u+1)} = \dfrac{1}{2}\dfrac{1}{u-1} - \dfrac{1}{2}\dfrac{1}{u+1}$, so

$\displaystyle\int \frac{1}{(u-1)(u+1)}du = \frac{1}{2}\int \frac{1}{u-1}du - \frac{1}{2}\int \frac{1}{u+1}du =$

$\dfrac{1}{2}\ln|u-1| - \dfrac{1}{2}\ln|u+1| + C = \ln\sqrt{\left|\dfrac{u-1}{u+1}\right|} + C$

Definition: A **partial fraction** is an expression of the form

$$\frac{A}{(cx+d)^m} \qquad \text{or} \qquad \frac{Bx + +C}{(ax^2+bx+c)^n}.$$

At this point we (theoretically) have learned how to antidifferentiate partial fractions, and the next skill to learn is how to rewrite a proper rational function $\dfrac{P(x)}{Q(x)}$ as the sum of partial fractions.

First Case: $Q(x) = (c_1x+d_1)(c_2x+d_2)\cdots(c_kx+d_k)$ is the product of distinct linear factors.

Then we can write

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(c_1x+d_1)(c_2x+d_2)\cdots(a_kx+d_k)} = \frac{A_1}{c_1x+d_1} + \frac{A_2}{c_2x+d_2} + \cdots + \frac{A_k}{c_kx+d_k}$$

Multiplying both sides of the equation by $Q(x)$, we get:

$P(x) = A_1(c_2x+d_2)\cdots(c_kx+d_k) + A_2(c_1x+d_1)(c_3x+d_3)\cdots(c_kx+d_k) + \cdots + A_k(c_1x+d_1)\cdots(c_{k-1}x+d_{k-1})$

which is easily solved by substituting the values $x_i = -\dfrac{d_i}{c_i}$, $i = 1, 2, 3, \ldots, k$.

Example2, p456 of Brown *Stewart*: $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{A_1}{x} + \frac{A_2}{x+2} + \frac{A_3}{2x-1}$ leads to

$x^2+2x-1 = A_1(x+2)(2x-1) + A_2x(2x-1) + A_3x(x+2)$

and we then substitute $x = 0, -2$, and $\frac{1}{2}$:

$x = 0$: $0^2 + 2(0) - 1 = -1 =$

$A_1(0+2)(2(0)-1) + A_2(0)(2(0)-1) + A_3(0)((0)+2) = -2A_1$, so $A_1 = \frac{1}{2}$

$x = -2$: $(-2)^2 + 2(-2) - 1 = -1 =$

$A_1(-2+2)(2(-2)-1) + A_2(-2)(2(-2)-1) + A_3(-2)((-2)+2) = 10A_2$, so $A_2 = -\frac{1}{10}$

$x = \frac{1}{2}$: $(\frac{1}{2})^2 + 2(\frac{1}{2}) - 1 = \frac{1}{4} =$

$A_1(\frac{1}{2}+2)(2(\frac{1}{2})-1) + A_2(\frac{1}{2})(2(\frac{1}{2})-1) + A_3(\frac{1}{2})((\frac{1}{2})+2) = \frac{5}{4}A_3$, so $A_3 = \frac{1}{5}$

Thus $\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{\frac{1}{2}}{x} + \frac{-\frac{1}{10}}{x+2} + \frac{\frac{1}{5}}{2x-1}$

and therefore

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx = \int \left(\frac{\frac{1}{2}}{x} + \frac{-\frac{1}{10}}{x+2} + \frac{\frac{1}{5}}{2x-1} \right) dx =$$

$$\frac{1}{2}\int \frac{1}{x}dx - \frac{1}{10}\int \frac{1}{x+2}dx + \frac{1}{5}\int \frac{1}{2x-1}dx =$$

$$\frac{1}{2}\ln|x| - \frac{1}{10}\ln|x+2| + \frac{1}{10}\ln|2x-1| + C$$

Example: $\int \sec x\, dx = \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{\cos x\, dx}{1-\sin^2 x} =$

$\int \frac{1}{1-u^2} du = -\int \frac{1}{u^2-1} = -\ln\sqrt{\frac{u-1}{u+1}} + C = \ln\sqrt{\frac{u+1}{u-1}} + C =$

$\frac{1}{2}\ln\frac{\sin x + 1}{|\sin x - 1|} + C =$

$\ln\sqrt{\frac{\sin x+1}{\sin x-1}} + C = \ln\sqrt{\left(\frac{\sin x+1}{\sin x-1}\right)\left(\frac{\sin x+1}{\sin x+1}\right)} + C =$

$\ln\sqrt{\frac{(\sin x+1)^2}{\sin^2 x-1}} + C = \ln\sqrt{\frac{(\sin x+1)^2}{\cos^2 x}} + C = \ln\left|\frac{\sin x+1}{\cos x}\right| + C =$

$\ln\left|\frac{\sin x}{\cos x} + \frac{1}{\cos x}\right| + C = \ln|\tan x + \sec x| + C$

Second Case: $Q(x) = (c_1x+d_1)^{m_1}(c_2x+d_2)^{m_2}\cdots(c_kx+d_x)^{m_k}$

We then have to write
$\frac{P(x)}{Q(x)} =$

$$\frac{A_{1,m_1}}{(c_1x+d_1)^{m_1}} + \frac{A_{1,m_1-1}}{(c_1x+d_1)^{m_1-1}} + \cdots + \frac{A_{1,1}}{c_1x+d_1} +$$

$$\frac{A_{2,m_2}}{(c_2x+d_2)^{m_2}} + \frac{A_{2,m_2-1}}{(c_2x+d_2)^{m_2-1}} + \cdots + \frac{A_{2,1}}{c_2x+d_2} +$$

$$\vdots$$

$$\frac{A_{k,m_k}}{(c_kx+d_k)^{m_k}} + \frac{A_{k,m_k-1}}{(c_kx+d_k)^{m_k-1}} + \cdots + \frac{A_{k,1}}{c_kx+d_k}$$

Example:

$$\int \frac{1}{(1-x^2)^2}dx = \int \frac{1}{(x-1)^2(x+1)^2}dx$$

We write $\dfrac{1}{(x-1)^2(x+1)^2} = \dfrac{A_{1,2}}{(x-1)^2} + \dfrac{A_{1,1}}{x-1} + \dfrac{A_{22}}{(x+1)^2} + \dfrac{A_{2,1}}{x+1}$,

and multiply by $(x-1)^2(x+1)^2$ to get

$$1 = A_{1,2}(x+1)^2 + A_{1,1}(x-1)(x+1)^2 + A_{2,2}(x-1)^2 + A_{2,1}(x-1)^2(x+1)$$

Substituting $x = -1$ and $x = 1$ gets us two constants quickly:

$x = -1$: $1 = A_{1,2}(-1+1)^2 + A_{1,1}(-1-1)(-1+1)^2 + A_{2,2}(-1-1)^2 + A_{2,1}(-1-1)^2(-1+1) = 4A_{2,2}$

so $A_{2,2} = \dfrac{1}{4}$

$x = 1$: $1 = A_{1,2}(1+1)^2 + A_{1,1}(1-1)(1+1)^2 + A_{2,2}(1-1)^2 + A_{2,1}(1-1)^2(1+1) = 4A_{1,2}$

so $A_{1,2} = \dfrac{1}{4}$

The quickest way to get the two remaining coefficients is to insert the coefficients just obtained into the original equation:

$$1 = A_{1,2}(x+1)^2 + A_{1,1}(x-1)(x+1)^2 + A_{2,2}(x-1)^2 + A_{2,1}(x-1)^2(x+1)$$

becomes

$$1 = \frac{1}{4}(x+1)^2 + A_{1,1}(x-1)(x+1)^2 + \frac{1}{4}(x-1)^2 + A_{2,1}(x-1)^2(x+1)$$

or

$$1 = (x+1)^2\left[\frac{1}{4} + A_{1,1}(x-1)\right] + (x-1)^2\left[\frac{1}{4} + A_{2,1}(x+1)\right]$$

Differentiating, we get

$$0 = 2(x+1)\left[\frac{1}{4} + A_{1,1}(x-1)\right] + (x+1)^2A_{1,1} + 2(x-1)\left[\frac{1}{4} + A_{2,1}(x+1)\right] + (x-1)^2A_{2,1}$$

Again we substitute the "illegal" values of x:

$x = -1$:

$$0 = 2(-1+1)\left[\frac{1}{4} + A_{1,1}(-1-1)\right] + (-1+1)^2 A_{1,1} + 2(-1-1)\left[\frac{1}{4} + A_{2,1}(-1+1)\right] + (-1-1)^2 A_{2,1}$$

simplifies to $0 = -4\left[\frac{1}{4}\right] + 4A_{2,1} = -1 + 4A_{2,1}$, so $A_{2,1} = \frac{1}{4}$

$x = 1$:

$$0 = 2(1+1)\left[\frac{1}{4} + A_{1,1}(1-1)\right] + (1+1)^2 A_{1,1} + 2(1-1)\left[\frac{1}{4} + A_{2,1}(1+1)\right] + (1-1)^2 A_{2,1}$$

simplifies to $0 = 4\left[\frac{1}{4}\right] + 4A_{1,1} = 1 + 4A_{1,1}$, so $A_{1,1} = -\frac{1}{4}$

Thus we have

$$\frac{1}{(x-1)^2(x+1)^2} = \frac{\frac{1}{4}}{(x-1)^2} + \frac{-\frac{1}{4}}{x-1} + \frac{\frac{1}{4}}{(x+1)^2} + \frac{\frac{1}{4}}{x+1} =$$

$$\frac{1}{4}\left[(x-1)^{-2} - (x-1)^{-1} + (x+1)^{-2} + (x+1)^{-1}\right]$$

Therefore:

$$\int \frac{1}{(1-x^2)^2}dx = \int \frac{1}{4}\left[(x-1)^{-2} - (x-1)^{-1} + (x+1)^{-2} + (x+1)^{-1}\right] dx =$$

$$\frac{1}{4}\left[\frac{(x-1)^{-2+1}}{-2+1} - \ln|x-1| + \frac{(x+1)^{-2+1}}{-2+1} + \ln|x+1|\right] + C =$$

$$\frac{1}{4}\left[-\frac{1}{x-1} - \frac{1}{x+1} + \ln\left|\frac{x+1}{x-1}\right|\right] + C = \frac{x}{2(1-x^2)} + \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| + C$$

Example: $\mathcal{I} = \int \sec^3 x dx = \int \frac{1}{\cos^3 x} dx = \int \frac{\cos x}{\cos^4 x} dx = \int \frac{\cos x}{(\cos^2 x)^2} dx =$

$\int \frac{\cos x}{(1-\sin^2 x)^2} dx = \int \frac{1}{(1-u^2)^2} du = \frac{u}{2(1-u^2)} + \frac{1}{4} \ln \left| \frac{u+1}{u-1} \right| + C =$

$\frac{\sin x}{2(1-\sin^2 x)} + \frac{1}{4} \ln \left| \frac{\sin x + 1}{\sin x - 1} \right| + C = \frac{\sin x}{2\cos^2 x} + \frac{1}{4} \ln \left| \frac{\sin x + 1}{\sin x - 1} \right| + C$

where we have made the substitution $u = \sin x$.

By trigonometric manipulation, this can be worked into the somewhat useless formula we have already derived using a complicated Integration by Parts calculation:

$$\int \sec^3 x dx = \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C$$

Example: $\int (\sin x)^{-11} (\cos x)^{-10} dx = \int (\sin x)^{-12} (\cos x)^{-10} \sin x dx =$

$-\int (1-u^2)^{-6} u^{-10} du = -\int \frac{1}{(u-1)^6 (u+1)^6 u^{10}} du$, (where $u = \sin x$)

We write

$\frac{1}{(u-1)^6 (u+1)^6 u^{10}} =$

$\frac{A_{1,6}}{(u-1)^6} + \frac{A_{1,5}}{(u-1)^5} + \frac{A_{1,4}}{(u-1)^4} + \frac{A_{1,3}}{(u-1)^3} + \frac{A_{1,2}}{(u-1)^2} + \frac{A_{1,1}}{u-1} +$

$\frac{A_{2,6}}{(u+1)^6} + \frac{A_{2,5}}{(u+1)^5} + \frac{A_{2,4}}{(u+1)^4} + \frac{A_{2,3}}{(u+1)^3} + \frac{A_{2,2}}{(u+1)^2} + \frac{A_{2,1}}{u+1} +$

$\frac{A_{3,10}}{u^{10}} + \frac{A_{3,9}}{u^9} + \frac{A_{3,8}}{u^8} + \frac{A_{3,7}}{u^7} + \frac{A_{3,6}}{u^6} + \frac{A_{3,5}}{u^5} + \frac{A_{3,4}}{u^4} + \frac{A_{3,3}}{u^3} + \frac{A_{3,2}}{u^2} + \frac{A_{3,1}}{u}$

Third Case: $Q(x) = (a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2) \cdots (a_kx^2 + b_kx + c_k)$ is the product of distinct irreducible quadratic factors.

Then we can write

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2) \cdots (a_kx^2 + b_kx + c_k)} =$$

$$\frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{a_2x^2 + b_2x + c_2} + \cdots + \frac{A_kx + B_k}{a_kx^2 + b_kx + c_k}$$

Example: $\displaystyle\int \frac{1}{(x^2+1)(x^2+4)}dx = \int \frac{A_1x + B_1}{x^2+1}dx + \int \frac{A_2x + B_2}{x^2+4}dx$

has to be solved for $A_1, B_1, A_2,$ and B_2. Multiplying by the common denominator $(x^2+1)(x^2+4)$, we get

$$1 = (A_1x + B_1)(x^2 + 4) + (A_2x + B_2)(x^2 + 1)$$

There are two ways of solving this for the desired constants. One is to use the fact that the coefficients of the two polynomials must be equal so as to get 4 equations in 4 unknowns:

$$1 = 0x^3 + 0x^2 + 0x + 1 = (A_1x + B_1)(x^2 + 4) + (A_2x + B_2)(x^2 + 1) =$$

$$A_1x^3 + B_1x^2 + 4A_1x + 4B_1 + A_2x^3 + B_2x^2 + A_2x + B_2 =$$

$(A_1 + A_2)x^3 + (B_1 + B_2)x^2 + (4A_1 + A_2)x + (4B_1 + B_2)$ gives us

$$A_1 + A_2 = 0,\ B_1 + B_2 = 0,\ 4A_1 + A_2 = 0,\ 4B_1 + B_2 = 1$$

Using $B_2 = -B_1$, we get $3B_1 = 1$, so $B_1 = \frac{1}{3}$ and $B_2 = -\frac{1}{3}$.

Usin $A_2 = -A_1$, we get $3A_1 = 0$, so $A_1 = A_2 = 0$. Thus we have

$$\int \frac{1}{(x^2+1)(x^2+4)}dx = \frac{1}{3}\int \frac{1}{x^2+1}dx - \frac{1}{3}\int \frac{1}{x^2+4}dx =$$

$$\frac{1}{3}\arctan x - \frac{1}{3}\frac{1}{2}\arctan\frac{x}{2} + C = \frac{1}{3}\arctan x - \frac{1}{6}\arctan\frac{x}{2} + C$$

An alternative methof of solving
$1 = (A_1x + B_1)(x^2 + 4) + (A_2x + B_2)(x^2 + 1)$ for the desired constants is to substitute artfully selected values of x. In this case we shall use complex numbers:

x=i: Then we have

$$1 = (A_1 i + B_1)(i^2 + 4) + (A_2 i + B_2)(i^2 + 1) =$$

$$(A_1 i + B_1)(-1 + 4) + (A_2 i + B_2)(-1 + 1) = 3(A_1 i + B_1)$$

or $1 + 0i = 3B_1 + 3A_1 i$, which immediately gives us $A_1 = 0$ and $B_1 = \dfrac{1}{3}$

---

x=2i: Then we have

$$1 = (A_1 2i + B_1)((2i)^2 + 4) + (A_2 2i + B_2)((2i)^2 + 1) =$$

$$(2A_1 i + B_1)(-4 + 4) + (2A_2 i + B_2)(-4 + 1) = -3(2A_2 + B_2)$$

or $1 + 0i = -3B_2 - 6A_2 i$, which immediately gives us $A_2 = 0$ and $B_2 = -\dfrac{1}{3}$

Fourth Case:

$Q(x) = (a_1x^2 + b_1x + c_1)^{n_1}(a_2x^2 + b_2x + c_2)^{n_2} \cdots (a_kx^2 + b_kx + c_k)^{n_k}$ is the product of not necessarily distinct irreducible quadratic factors.

Then we can write

$$\frac{P(x)}{(a_1x^2 + b_1x + c_1)^{n_1}(a_2x^2 + b_2x + c_2)^{n_2} \cdots (a_kx^2 + b_kx + c_k)^{n_k}} =$$

$$\frac{A_{1,n_1}x + B_{1,n_1}}{(a_1x^2 + b_1x + c_1)^{n_1}} + \frac{A_{1,n_1-1}x + B_{1,n_1-1}}{(a_1x^2 + b_1x + c_1)^{n_1-1}} + \cdots + \frac{A_{1,2}x + B_{1,2}}{(a_1x^2 + b_1x + c_1)^2} + \frac{A_{1,1}x + B_{1,1}}{a_1x^2 + b_1x + c_1} +$$
$$\frac{A_{2,n_1}x + B_{2,n_2}}{(a_2x^2 + b_2x + c_2)^{n_1}} + \frac{A_{2,n_2-1}x + B_{2,n_2-1}}{(a_2x^2 + b_2x + c_2)^{n_2-1}} + \cdots + \frac{A_{2,2}x + B_{2,2}}{(a_2x^2 + b_2x + c_2)^2} + \frac{A_{2,1}x + B_{2,1}}{a_2x^2 + b_2x + c_2} +$$
$$+ \vdots +$$
$$\frac{A_{k,n_k}x + B_{k,n_k}}{(a_kx^2 + b_kx + c_k)^{n_k}} + \frac{A_{k,n_k-1}x + B_{k,n_k-1}}{(a_kx^2 + b_kx + c_k)^{n_k-1}} + \cdots + \frac{A_{k,2}x + B_{k,2}}{(a_kx^2 + b_kx + c_k)^2} + \frac{A_{k,1}x + B_{k,1}}{a_kx^2 + b_kx + c_k}$$

Example: $\int \frac{1}{(x^2+1)^2(x^2+4)^2}dx$

We write $\frac{1}{(x^2+1)^2(x^2+4)^2} =$

$$\frac{A_{1,2}x + B_{1,2}}{(x^2+1)^2} + \frac{A_{1,1}x + B_{1,1}}{x^2+1} + \frac{A_{2,2}x + B_{2,2}}{(x^2+4)^2} + \frac{A_{2,1}x + B_{2,1}}{x^2+4}$$

and multiply by the common denominator to get

$$\begin{aligned} 1 = \\ &(A_{1,2}x + B_{1,2})(x^2+4)^2 + (A_{1,1}x + B_{1,1})(x^2+1)(x^2+4)^2 + \\ &(A_{2,2}x + B_{2,2})(x^2+1)^2 + (A_{2,1}x + B_{2,1})(x^2+1)^2(x^2+4) \end{aligned}$$

which will result in 8 equations in 8 unknown constants. Most reasonable people would have this computation done by a CAS: a Computer Algebra System.

Improper Integrals

So far, definite integrals have been used to compute areas of *finite regions*. It is possible to extend the notion of area to regions lying under the graph of a function over infinite intervals, and to functions which have vertical asymptotes.

Definition: Improper Integrals of Type I are defined to be those of the form:

$$\int_a^\infty f(x)dx = \lim_{T\to\infty}\int_a^T f(x)dx \qquad \int_{-\infty}^a f(x)dx = \lim_{T\to-\infty}\int_T^a f(x)dx$$

$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$, *if both limits exist.*

Example: $\int_1^\infty x^{-2}dx = \lim_{T\to\infty}\int_1^T x^{-2}dx = \lim_{T\to\infty}\left.\frac{x^{-1}}{-1}\right|_1^T = \lim_{T\to\infty}\left.\frac{-1}{x}\right|_1^T = \lim_{T\to\infty}\left(\frac{-1}{T}-\frac{-1}{1}\right) =$
$1-\lim_{T\to\infty}\frac{1}{T} =$

$1-0 = 1$

Example: $\int_1^\infty x^{-1}dx = \lim_{T\to\infty}\int_1^T x^{-1}dx = \lim_{T\to\infty}\ln|x||_1^T = \lim_{T\to\infty}\ln T-\ln 1 = \infty$

Example: $\int_{-\infty}^{-1} x^{-1}dx = \lim_{T\to-\infty}\int_{-\infty}^{-1} x^{-1}dx = \lim_{T\to-\infty}\ln|x||_T^{-1} = \lim_{T\to-\infty}\ln|-1|-\ln|T| = -\infty$

If the limit defining an improper integral exists, we say that the integral **converges** or is **convergent**, otherwise we say it **diverges** or is **divergent**.

Example: If s is considered to be a constant,

$$\int_0^\infty e^{-sx}dx = \lim_{T\to\infty}\int_0^T e^{-sx}dx = \lim_{T\to\infty}\left.\frac{1}{-s}e^{-sx}\right|_0^T = \lim_{T\to\infty}\frac{1}{-s}e^{-s(T)}-\frac{1}{-s}e^{-s(0)} = \frac{1}{s}-\lim_{T\to\infty}\frac{1}{-s}e^{-sT} = \frac{1}{s}$$

Definition: If f is defined on the interval $0, \infty)$, and treating s as a constant in integrating, we can define a function of $F(s)$ of s by letting

$$F(s) = \int_0^{\infty} e^{-sx} f(x)dx = \lim_{T\to\infty} \int_0^T f(x)e^{-sx}dx$$

This function is called the **Laplace Transform** of f. In the previous example we showed that the Laplace transform of the constant function 1 is $\frac{1}{s}$. Most of the integration techniques learned in this course will be used far more often in practical applications of Calculus via the Laplace transform than in the computation of areas.

Example:(closely related to Example 2, p.489 of brown Stewart)

$$\int_0^{\infty} xe^{-sx}dx = \lim_{T\to\infty} \int_0^T xe^{-sx}dx$$

We integrate $\int xe^{-sx}dx$ by parts with $u = x$, $dv = e^{-sx}dx$, so that $du = dx$ and $v = -\frac{1}{s}e^{-sx}$:

$$\int xe^{-sx}dx = \int udv = uv - \int vdu = x\left(-\frac{1}{s}e^{-sx}\right) - \int -\frac{1}{s}e^{-sx}dx =$$

$$-\frac{x}{s}e^{-sx} + \frac{1}{s}\int e^{-sx}dx = -\frac{x}{s}e^{-sx} - \frac{1}{s^2}e^{-sx}$$

so

$$\lim_{T\to\infty} \int_0^T xe^{-sx}dx = \lim_{T\to\infty} \left(-\frac{x}{s}e^{-sx} - \frac{1}{s^2}e^{-sx}\right)\Big|_0^T =$$

$$\lim_{T\to\infty} \left(-\frac{T}{s}e^{-sT} - \frac{1}{s^2}e^{-sT}\right) - \left(-\frac{0}{s}e^{-sx} - \frac{1}{s^2}e^{-s(0)}\right) = \frac{1}{s^2}$$

Note: Examples 3 & 4 of *brown Stewart* 7.9 are very important:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{T\to-\infty} \int_T^0 \frac{dx}{1+x^2} + \lim_{T\to\infty} \int_0^T \frac{dx}{1+x^2} =$$

$$\lim_{T\to-\infty} \arctan x\,\Big|_T^0 + \lim_{T\to\infty} \arctan x\,\Big|_0^T =$$

$$\lim_{T\to-\infty} (\arctan 0 - \arctan T) + \lim_{T\to\infty} (\arctan T - \arctan 0) =$$

$$0 - \lim_{T\to-\infty} \arctan T + \lim_{T\to\infty} \arctan T - 0 =$$

$$\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{T\to\infty} \int_1^T x^{-p}\,dx = \lim_{T\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg|_1^T =$$

$$\lim_{T\to\infty} \frac{T^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} =$$

$$\frac{1}{p-1} + \lim_{T\to\infty} \frac{T^{1-p}}{1-p} =$$

$$\frac{1}{p-1} \text{ if and only if } p > 1$$

The other type of improper integral occurs when the integrand $f(x)$ has a vertical asymptote at say $x = c$.

Then, assuming $a < c < b$, we define

$$\int_a^c f(x)dx = \lim_{T\to c^-} \int_a^T f(x)dx \qquad \int_c^b f(x)dx = \lim_{T\to c^+} \int_T^b f(x)dx$$

and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \textit{ (if both integrals exist).}$$

Comparison Theorems: If we have $f(x) \leq g(x)$ on the (possibly infinite) interval (a, b), then we know that $\displaystyle\int_a^b f(x)dx \leq \int_a^b g(x)dx$. Thus we can conclude for proper integrals that if $\int_a^b f(x)dx$ diverges, then so must $\int_a^b g(x)dx$, and conversely, if $\int_a^b g(x)dx$ converges, then so must $\int_a^b f(x)dx$.

Example: The region $\mathcal{R} = \left\{(x, y) | x \geq 1, 0 \leq y \leq \dfrac{1}{x}\right\}$ is revolved about the x-axis to form a solid object, known as **Gabriel's Horn.** It volume is

$$V = \int_1^\infty \frac{\pi}{x^2}dx = \pi \lim_{T\to\infty} \int_1^T x^{-2}dx = \pi \lim_{T\to\infty} \left.\frac{x^{-1}}{-1}\right|_1^T = \pi \lim_{T\to\infty} \left.\frac{-1}{x}\right|_1^T = \pi \lim_{T\to\infty} \frac{-1}{T} - \frac{-1}{1} = \pi$$

However, as we shall soon see, the formula for its surface area is

$$S = \int_1^\infty 2\pi \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}dx \geq \int_1^\infty 2\pi \frac{1}{x} = \infty \text{ !!!}$$

Thus we now have a pattern for a paint bucket whose volume is finite, but which cannot be painted because its surface area is infinite!

Approximate Integration

We have already seen how Riemann sums are approximations to definite integrals. Recall that the *regular* partition P of $[a, b]$ with n intervals is formed by letting

$\Delta x = \dfrac{b-a}{n}$, $x_j = a + j\Delta x$, so that $x_0 = a$,,$x_1 = a + \Delta x$, ..., $x_{n-1} = b - \Delta x$, $x_n = b$. We have $||\mathsf{P}|| = \Delta x$.

The formulas for the special Riemann sums are:

(1) The *Left-hand sum* of f:

$$\mathcal{L}_n(f) = \sum_{i=1}^{n} f(x_{i-1})\Delta x = [f(x_0) + f(x_1) + \cdots + f(x_{n-1}]\,\Delta x =$$

$$\left[\sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right)\right]\frac{b-a}{n}$$

---

(2) The *Right-hand sum* of f:

$$\mathcal{R}_n(f) = \sum_{i=1}^{n} f(x_i)\Delta x = [f(x_1) + f(x_2) + \cdots + f(x_n)]\,\Delta x =$$

$$\left[\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)\right]\frac{b-a}{n}$$

---

(3) The *Midpoint sum* :

$$\mathcal{M}_n(f) = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + f(x_i)}{2}\right)\Delta x =$$

$$\left[f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1} + x_n}{2}\right)\right]\Delta x =$$

$$\left[\sum_{i=1}^{n} f\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{n}\right)\right]\frac{b-a}{n}$$

---

(4) The *Inscribed sum* of f: $\mathcal{I}_n(f) = [m_1 + m_2 + \cdots + m_n]\Delta x$

(5) The *Exscribed sum* of f: $\mathcal{E}_n(f) = [M_1 + M_2 + \cdots + M_n]\Delta x$

The *Trapezoidal Estimate* or *Trapezoidal Rule* $\mathcal{T}_n(f) = \frac{1}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_n - 1) + f(x_n))\Delta x =$

$$\left[f(a) + 2\sum_{i=1}^{n-1} f\left(a + i\frac{b-a}{n}\right) + f(b)\right]\frac{b-a}{n}$$

which is the sum of the areas of the trapezoids passing $(x_{i-1}, 0)(x_{i-1}, f(x_{i-1}))(x_i, f(x_i))(x_i, 0)$ turns out to be the average of the left and right hand sums:

$$\mathcal{L}_n(f) + \mathcal{R}_n(f) = \sum_{i=1}^{n} f(x_{i-1})\Delta x + \sum_{i=1}^{n} f(x_i)\Delta x =$$

$$\left[f(a) + 2\sum_{i=1}^{n-1} f\left(a + i\frac{b-a}{n}\right) + f(b)\right]\frac{b-a}{n} = \mathcal{T}_n(f)$$

We would like to know how close these sums are to the actual value of $\int_a^b f(x)dx$. If A is an approximation to $\int_a^b f(x)dx$, we define $E(A) = \left|\int_a^b f(x)dx - A)\right|$

So far, we only know the following theorem:

Theorem If f has continuous derivative f' on $[a, b]$, and $|f'(x)| < M$ for all x in $[a, b]$, then $\mathcal{E}_n(f) - \mathcal{I}_n(f) \leq M\frac{(b-a)^2}{n}$, so

$$\lim_{n\to\infty} \mathcal{E}_n(f) - \mathcal{I}_n(f) = 0$$

This was used to prove part of the Fundamental Theorem of Calculus, but is not particulary useful because of the difficulty of finding the Incribed and Exscribed Sums. We can make some use of it by noting that for any Riemann sum $\mathcal{R}_n(f) = \sum_{i=1}^{n} f(t_i)\Delta x = \frac{b-a}{n}\left[\sum_{i=1}^{n} f(t_i)\right]$ over a regular partition we have $\mathcal{I}_n(f) \leq\leq \mathcal{E}_n(f)$, so the theorem tells us that $E(\mathcal{R}_n(f)) \leq M\frac{(b-a)^2}{n}$.

In terms of the examples just preceding the Substitution Method, we have, letting $y = f(x) = x^2$, $a = 0$, $b = 1$, (so that $\int_a^b f(x)dx = \frac{1}{3}$),

$\mathcal{L}_n(f) = \dfrac{1}{3} - \dfrac{1}{2n} + \dfrac{1}{6n^2}$, so $E(\mathcal{L}_n(f)) = \dfrac{1}{2n} - \dfrac{1}{6n^2}$

$\mathcal{R}_n(f) = \dfrac{1}{3} + \dfrac{1}{2n} + \dfrac{1}{6n^2}$, so $E(\mathcal{R}_n(f)) = \dfrac{1}{2n} + \dfrac{1}{6n^2}$

$\mathcal{T}_n(f) = \dfrac{\mathcal{L}_n(f) + \mathcal{R}_n(f)}{2} = \dfrac{1}{3} + \dfrac{1}{6n^2}$, so $E(\mathcal{T}_n(f)) = \dfrac{1}{6n^2}$

$\mathcal{M}_n(f) = \dfrac{1}{3} - \dfrac{1}{12n^2}$, so $E(\mathcal{M}_n(f)) = \dfrac{1}{12n^2}$

We state without proof some useful facts: If f'' exists and is continuous on $[a, b]$, and if $M_2 = \max\{|f''(x)| : a \le x \le b\}$, then

$E(\mathcal{T}_n(f)) \le M_2 \dfrac{(b-a)^3}{12n^2}$ and $E(\mathcal{M}_n(f)) \le M_2 \dfrac{(b-a)^3}{24n^2}$

In our example, we have $f''(x) = 2$, so we take $M_2 = 2$ and get

$E(\mathcal{T}_n(f)) \le 2\dfrac{(1-0)^3}{12n^2} = \dfrac{1}{6n^2}$ and $E(\mathcal{M}_n(f)) \le M_2 \dfrac{(b-a)^3}{12n^2} = 2\dfrac{(1-0)^3}{24n^2} = \dfrac{1}{12n^2}$

Example: Use the Trapezoidal and Midpoint Rules to estimate $\int_0^3 \sqrt{4^2 - x^2}dx$ to within 0.001 of the true value.

Solution: We have $f(x) = \left(4^2 - x^2\right)^{\frac{1}{2}}$,

$f'(x) = \dfrac{1}{2}\left(16 - x^2\right)^{-\frac{1}{2}}(-2x) = -\dfrac{x}{(16-x^2)^{\frac{1}{2}}}$,

$f''(x) = -\dfrac{(16-x^2)^{\frac{1}{2}}(1) - (x)\frac{1}{2}(16-x^2)^{-\frac{1}{2}}(-2x)}{16 - x^2} = -\dfrac{16 - x^2 + x^2}{(16-x^2)^{\frac{3}{2}}} = -\dfrac{16}{(16-x^2)^{\frac{3}{2}}}$

so $M_2 = \max\left\{\left|-\dfrac{16}{(16-x^2)^{\frac{3}{2}}}\right| : 0 \le x \le 3\right\} = 16\max\left\{\dfrac{1}{(16-x^2)^{\frac{3}{2}}} : 0 \le x \le 3\right\} = \dfrac{16}{7\sqrt{7}}$

Thus $E(\mathcal{T}_n(f)) \le \dfrac{16}{7\sqrt{7}}\dfrac{(3-0)^3}{12n^2} = \dfrac{36}{7\sqrt{7}n^2} < 0.001$ if $n^2 > \dfrac{36000}{7\sqrt{7}}$ or $n > \sqrt{\dfrac{36000}{7\sqrt{7}}} \doteq 44.1$. We take $n = 45$ and get $\mathcal{L}_{45}(f) = 10.797845$, $\mathcal{R}_{45}(f) = 10.707562$, so $\mathcal{T}_{45}(f) = \dfrac{10.797845 + 10.707562}{2} = 10.752704$.

For the midpoint rule, we take $n = 23$ and get $\mathcal{M}_{23}(f) = 10.753927$. The actual value (to 15 decimal places) is 10.753123598448735.

Simpson's Rule

A more sophisticated approach is make n be even and to approximate the graph of $y = f(x)$ on the intervals $[x_i, x_{i+2}]$ with the parabolas passing throught the points $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1})$, and $(x_{i+2}, f(x_{i+2})$. The result is the approximation

$$S_n(f) = [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]\,\frac{b-a}{3n}$$

If f'''' exists and is continuous on $[a, b]$, and if $M_4 = \max\{|f''''(x)| : a \le x \le b\}$,

$$E(S_n(f)) \le M_4 \frac{(b-a)^5}{180n^4}$$

Simpson's Rule can be expressed in terms of the Trapezoidal and Midpoint Rules:

$$S_n(f) = \frac{\mathcal{T}_n(f) + 2\mathcal{M}_n(f)}{3}$$

In our first example, we have $f''''(x) = 0$, so we take $M_4 = 0$ and get

$$E(S_n(f)) \le M_4 \frac{(b-a)^5}{180n^4} = 0$$

Using $S_n(f) = \dfrac{\mathcal{T}_n(f) + 2\mathcal{M}_n(f)}{3}$, we get $S_n(f) = \dfrac{\frac{1}{3} + \frac{1}{6n^2} + 2(\frac{1}{3} - \frac{1}{12n^2})}{3} = \dfrac{1}{3}$.

In our second example, we have, using $f''(x) = -16(16 - x^2)^{-\frac{3}{2}}$,

$$f'''(x) = -16\left(-\frac{3}{2}\right)(16 - x^2)^{-\frac{5}{2}}(-2x) = -48\frac{x}{(16 - x^2)^{\frac{5}{2}}}$$

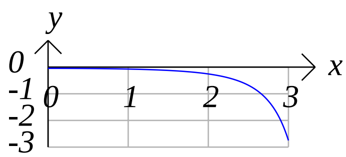
$$f''''(x) = -48\frac{(16 - x^2)^{\frac{5}{2}}(1) - x\frac{5}{2}(16 - x^2)^{\frac{3}{2}}(-2x)}{(16 - x^2)^5} = -48\frac{16 - x^2 + 5x^2}{(16 - x^2)^{\frac{7}{2}}} = -192\frac{4 + x^2}{(16 - x^2)^{\frac{7}{2}}}$$

Thus $M_4 = \max\left\{\left|-192\dfrac{4 + x^2}{(16 - x^2)^{\frac{7}{2}}}\right| : 0 \le x \le 3\right\} = 192 \max\left\{\left|\dfrac{4 + x^2}{(16 - x^2)^{\frac{7}{2}}}\right| : 0 \le x \le 3\right\} \le 192\dfrac{4 + 3^2}{7^{\frac{7}{2}}} = \dfrac{2496}{343\sqrt{7}} \doteq 2.75$

Thus $E(S_n(f)) \le M_4\dfrac{(3-0)^5}{180n^4} = M_4\dfrac{9}{20n^4} < 2.75\dfrac{9}{20n^4} = \dfrac{1.237}{n^4} < 0.001$ if $n^4 > 1237$ or $n > 5.9$.

We take $n = 6$ and get: $\mathcal{L}_6(f) = 11.06815$, $R_6(f) = 10.39103$, so $\mathcal{T}_6(f) = \dfrac{11.06815 + 10.39103}{2} = 10.72959$. Also, $\mathcal{M}_6(f) = 10.764584$, so $S_6(f) = \dfrac{\mathcal{T}_6(f) + 2S_6(f)}{3} = \dfrac{10.72959 + 2(10.764584)}{3} = \dfrac{32.258758}{3} = 10.752919$

Is it worth any extra effort to get a better estimate of M_4? The answer is that it depends on difficult the expression for $f''''(x)$ is. We obtained a good guess for the maximum value of $|f''''(x)$ by substituting $x = 3$ into it. It is now easy to sketch the graphs of functions with computers:



The graph confirms the suspicion that the maximum value of $|f''''(x)|$ occurs at $x = 3$.

Example: Evaluate $\int_0^1 e^{-x^2} dx$ to 5 decimal place accuracy.

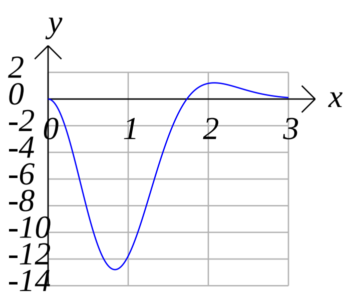
Solution: The error must be less than $\frac{1}{2}0.00001 = \dfrac{1}{20,000}$. With $f(x) = e^{-x^2}$, we have

$f'(x) = -2xe^{-x^2}$, and

$f''(x) = -2e^{-x^2} - 2x(-2x)e^{-x^2} = -2e^{-x^2}\left(1 - 2x^2\right)$,

$f'''(x) = -2\left[(-2x)e^{-x^2}\left(1 - 2x^2\right) + e^{-x^2}(-4x)\right] = 4e^{-x^2}(3x - 2x^3)$,

$f''''(x) = 16x^2e^{-x^2}(x^2 - 3)$



We have $M_4 \le 13$, so $E(S_n(f)) \le M_4 \dfrac{(1-0)^5}{180n^4} = \le \dfrac{13}{180n^4} < \dfrac{1}{20,000}$ if

$n^4 > \dfrac{260,000}{180} = \dfrac{26,000}{18} = \dfrac{13,000}{9} \doteq 1444.4$ or $n > 6.16$, so we take $n = 8$.

Then $\mathcal{L}_8(f) \doteq 0.78537315$, $\mathcal{R}_8(f) \doteq 0.7063581$,

so $\mathcal{T}_8(f) = \dfrac{\mathcal{L}_8(f) + \mathcal{R}_8(f)}{2} = 0.74586564$.

Also $\mathcal{M}_8(f) = 0.7473036$,and $S_8(f) = 0.74682426$.

As a matter of fact,$E(S_8(f)) \le \dfrac{13}{180(10)^4} \doteq 0.000007$, so we expect six decimal place accuracy of $S_8(f)$.

Using $n = 100$, we have

$\mathcal{L}_{100}(f) = 0.7499786$,

$R_{100}(f) = 0.7436574$,

$\mathcal{T}_{100}(f) = 0.746818$,

$\mathcal{M}_{100}(f) = 0.7468272$,

$S_{100}(f) = 0.7468242574357303$.

We have $E(S_{100}(f)) \le \dfrac{13}{180(100)^4} \doteq 0.0000000007$, so our first 9 decimal places are accurate, i.e.,

$\displaystyle\int_0^1 e^{-x^2}dx = 0.746824257$ is a true statement.