Sigma (or $\sum$) Notation

In many branches of Mathematics, especially applied Mathematics and/or Statistics, it routinely occurs that one wants to talk about sums of large numbers of measurements of some quantity. For example, one might one to find the average value of a set of readings, or one might wish to know how a set of readings deviates from its average.

Instead of writing out an expression containing thousands, or millions, or billions, of summands, scientists have developed a shorthand notation:

Instead of $10 + 20 + 30 + 40 + \cdots + 5270 + 5280$ we write $\sum_{i=1}^{528} 10i$,

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In general, instead of

$a_m + a_{m+1} + a_{m+2} + a_{m+3} + \cdots + a_{n-3} + a_{n-2} + a_{n-1} + a_n$,
we write

$\sum_{i=m}^{i=n} a_i$ or, even more briefly, $\sum_{i=m}^{n} a_i$;

instead of

$x_m + x_{m+1} + x_{m+2} + x_{m+3} + \cdots + x_{n-3} + x_{n-2} + x_{n-1} + x_n$,

we write $\sum_{j=m}^{j=n} x_j$ or $\sum_{j=m}^{n} x_j$.

Instead of

$f(m) + f(m+1) + f(m+2) + f(m+3) + \cdots + f(n-3) + f(n-2) + f(n-1) + f(n)$, we write

$\sum_{k=m}^{k=n} f(k)$ or $\sum_{k=m}^{n} f(k)$.

The notation $\sum_{k=m}^{k=n} f(k)$ consists of four components:

first , there is the Σ which tells us that a sum is to be taken over a range of values,

second , there is the $k = m$ under the Σ which tells us that the **summation index** is k and that k starts out at the **lower summation limit** m ,

third , there is the $k = m$, or just m over the Σ which tells us that k stops at the **upper summation limit** n ,

fourth , there is the $f(k)$ which gives us a formula for the summands in terms of the summation index k.

It is required that the summation limits be integers and that the summation index increases by 1 as it runs through all integer values between the lower and upper summation limits.

The **number of summands** is $n - m + 1$, and the **average value** of the summands is

$$\frac{\sum_{i=m}^{n} f(i)}{n - m + 1} =$$

$$\frac{f(m) + f(m+1) + f(m+2) + \cdots + f(n-2) + f(n-1) + f(n)}{n - m + 1}$$

Such averages are often denoted using the Greek letter μ (pronounced “mew”).

Another quantity which is often of interest is the **standard deviation** :

the deviation of a reading $f(i)$ from its mean μ is $|f(i) - \mu|$. The average of these deviations is

$$\frac{\sum_{i=m}^{n} |f(i) - \mu|}{n - m + 1}$$

For very good technical reasons, this is seldom used. Instead, the common practice is to use the square root of the average of the squares of the deviations:

$$\sigma = \sqrt{\frac{\sum_{i=m}^{n} |f(i) - \mu|^2}{n - m + 1}}$$

is called the **standard deviation** , or **root-mean square** of the values $f(i)$.

The lower case Greek letter σ is pronounced "sigma".
The upper case Greek letter Σ is pronounced exactly the same, but it can, and often is, read as "the sum of ... as i runs from m to n".

Examples

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

$$\sum_{i=1}^{10} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 =$$

$$1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 = \frac{10(10 + 1)(21)}{6} = 365$$

$$\sum_{i=1}^{10} 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 10$$

Some Special Sums:

The sums of the powers of the first n integers are of special interest: we write:

$S_n^k = \sum_{i=1}^{n} i^k$,
so that

$S_n^0 = \sum_{i=1}^{n} i^0 = 1^0 + 2^0 + 3^0 + \cdots + (n-1)^0 + n^0 = 1 + 1 + 1 + \cdots + 1 + 1 = n$, just tells us that the sum of n 1's is n, and

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$S_n^1 = \sum_{i=1}^{n} i^1 = 1^1 + 2^1 + 3^1 + \cdots + (n-1)^1 + n^1 = \frac{n(n+1)}{2}$, is just a special case of the formula for an Arithmetic Progression

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$$S_n^2 = \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6},$$

$$S_n^3 = \sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

Also, it is useful to know the basic formula for a

Geometric Progression :

$$\sum_{i=0}^{n} x^i = x^0 + x^1 + x^2 + x^3 + \cdots + x^{n-2} + x^{n-1} + x^n = \frac{x^{n+1} - 1}{x - 1},$$

or

$$\sum_{i=0}^{n} x^i = 1 + x + x^2 + x^3 + \cdots + x^{n-2} + x^{n-1} + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

This is easily derived by computing the product of

$x - 1$ and $1 + x + x^2 + x^3 + \cdots + x^{n-2} + x^{n-1} + x^n$:

$$(x-1)(1 + x + x^2 + x^3 + \cdots + x^{n-2} + x^{n-1} + x^n) =$$

$$x(1 + x + x^2 + x^3 + \cdots + x^{n-2} + x^{n-1} + x^n)$$
$$-(1 + x + x^2 + x^3 + \cdots + x^{n-2} + x^{n-1} + x^n) =$$

$$\begin{aligned} & \quad\;\; x + x^2 + x^3 + \cdots + x^{n-2} + x^{n-1} + x^n + x^{n+1} \\ & -1 - x - x^2 - x^3 - \cdots - x^{n-2} - x^{n-1} - x^n \qquad = x^{n+1} - 1 \end{aligned}$$

Properties

$$\sum_{i=m}^{n}(a_i+b_i)=\sum_{i=m}^{n}a_i+\sum_{i=m}^{n}b_i$$

$$\sum_{i=m}^{n}ca_i=c\sum_{i=m}^{n}a_i$$

$$\sum_{i=m}^{n}a_i=\sum_{i=m}^{p}a_i+\sum_{i=p}^{n}a_i \text{ if } m\le p\le n$$

Note that the ranges of summation in first two equations are identical. If they are different much extra care must be taken.

Examples:

Sums are usually evaluated by reducing them to one or more of the above forms by algebraic manipulation:

$$\sum_{i=0}^{6}2^i=1+2+4+8+16+32+64=\frac{2^{6+1}-1}{2-1}=\frac{128-1}{1}=127$$

$$\sum_{i=0}^{30}5^{i+2}=\sum_{i=1}^{30}5^i5^2=25\sum_{i=1}^{30}5^i=25\frac{5^{30+1}-1}{5-1}=\frac{25}{4}(5^{31}-1)$$

$$\sum_{i=0}^{40}4^{-i+2}=\sum_{i=1}^{40}4^{-i}4^2=16\sum_{i=1}^{40}\left(\frac{1}{4}\right)^i=16\frac{\left(\frac{1}{4}\right)^{40+1}-1}{\left(\frac{1}{4}\right)-1}=16\frac{4^{-41}-1}{-\frac{3}{4}}=\frac{64}{3}(1-4^{-41})$$

$$\sum_{i=10}^{20}2^i=\sum_{i=0}^{10}2^{i+10}=2^{10}\sum_{i=0}^{10}2^i=$$

$$1024(1+2+4+8+16+32+64+128+256+512+1024)=1024\frac{2^{10+1}-1}{2-1}=1024\frac{2048-1}{1}=$$
$$1024\times 2047=2,096,128$$

$$\sum_{i=10}^{20} i = \sum_{i=1}^{20} i - \sum_{i=1}^{9} i = \frac{20(20+1)}{2} - \frac{9(9+1)}{2} = \frac{20(20+1)}{2} - \frac{9(9+1)}{2} = 210 - 45 = 165$$

$$\sum_{i=20}^{40} i^2 = \sum_{i=1}^{40} i^2 - \sum_{i=1}^{19} i^2 = \frac{40(40+1)(2(40)+1)}{6} - \frac{19(19+1)(2(19)+1)}{6} =$$
$$\frac{40(41)(81)}{6} - \frac{19(20)(39)}{6} = 20(41)(27) - 19(10)(13) = 10[2(41)(27) - 19(13)] =$$
$$10[2214 - 247] = 10(1967) = 19,670$$

Telescoping Sums

It often happens that sums can be collapsed nicely, for example:

$$\sum_{k=1}^{n} (k+1)^m - k^m =$$
$$(2^m - 1^m) + (3^m - 2^m) + (4^m - 3^m) + \cdots + (n^m - (n-1)^m) + ((n+1)^m - n^m) =$$
$$(n+1)^m - 1$$

In general, we have a sum of the form

$$\sum_{i=1}^{n} (a_{i+1} - a_1) = (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) + (a_{n+1} - a_n) = a_{n+1} - a_1$$

A Very Useful Technique

It is often necessary to work with two sums whose ranges of summation are slightly different. If the two sums have a differing number of terms, we must split off a few terms from the one with the largest number of terms so as to have a fit:

For example:

$$\sum_{i=1}^{i=n} i^2 + \sum_{i=1}^{n+2} i^2 = \sum_{i=1}^{i=n} i^2 + \sum_{i=1}^{n} i^2 + (n+1)^2 + (n+2)^2 = 2\sum_{i=1}^{i=n} i^2 + (n+1)^2 + (n+2)^2 =$$
$$2\frac{n(n+1)(2n+1)}{6} + (n+1)^2 + (n+2)^2$$

Another possibility is that although the two sums have the same number of terms, the ranges are offset. For example, we might wish to evaluate

$$\sum_{i=1}^{i=n} a_i + \sum_{i=3}^{i=n+2} a_i.$$

Note that both sums have the same number of terms .

We have to adjust one of the sums so that its summation range equals that of the other. We will choose to work on $\sum_{i=3}^{n+2} a_i$ and we will introduce a new summation index $k = i - 2$, so that $k = 1$ when $i = 3$, and $k = n$ when $i = n + 2$.

We of course can solve for i in terms of k: $i = k + 2$, so we replace every occurrence of i in $\sum_{i=3}^{n+2} a_i$ with $k + 2$:

$$\sum_{i=3}^{i=n+2} a_i = \sum_{k+2=3}^{k+2=n+2} a_{k+2} = \sum_{k=1}^{k=n} a_{k+2}.$$

Now it doesn't matter what symbol we use for our summation index, so we write:

$$\sum_{k=1}^{k=n} a_{k+2} = \sum_{i=1}^{i=n} a_{i+2}.$$

Using this in our original expression, we get

$$\sum_{i=1}^{i=n} a_i + \sum_{i=3}^{i=n+2} a_i = \sum_{i=1}^{i=n} a_i + \sum_{i=1}^{i=n} a_{i+2} = \sum_{i=1}^{i=n} (a_i + a_{i+2}).$$

Example:

$$\sum_{i=1}^{i=n} i^2 + \sum_{i=3}^{i=n+2} i^2 = \sum_{i=1}^{i=n} i^2 + (i+2)^2 = \sum_{i=1}^{i=n} i^2 + i^2 + 2i + 1 =$$

$$2\sum_{i=1}^{i=n} i^2 + 2\sum_{i=1}^{i=n} i + \sum_{i=1}^{i=n} 1 = 2\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} + n =$$

$$n\frac{2n^2 + 6n + 7}{3}$$

Area: An Introduction to Integration

Differentiation is the Calculus technique used to analyse how rapidly a function is changing. The basic idea is to look at the ratio

$$\frac{f(x) - f(a)}{x - a}$$

as x approaches a. In so doing, we are focussing on the point $(a, f(a))$, and "differentiating" between it and nearby points $(x, f(x))$. The geometric interpretation of this ratio as the slope of a line is quite useful in understanding the concept, even when the function f models something quite non-geometrical, such as the rate of change of temperature, or the flow of electrons past an electrical junction.

On the other hand, **Integration** is the Calculus technique used to predict how much a function will change over an interval in its domain if we know its rate of change, or derivative. Thus we may have already seen how to predict how money compounded continuously will grow in a bank account, how to predict how the temperature of an object being heated or cooled will change, and how to predict how long it will take to drain tanks of various shapes and sizes. The purely mathematical problem may be stated as that of finding an "antiderivative" — a continuous function F whose "derivative" is some given function f - which is not necessarily continuous.

The key mathematical idea is that the values of the derivative over the whole of an interval, and not just near a point, are important. We take the information at a large number of points(okay, for the mathematical purist, the number is $\mathfrak{c}$, the cardinality of the continuum) and "**integrate**" it to get the amount of change.

As with Differentiation, the geometric interpretation is useful. We think of integration in terms of area underneath a positive curve over an interval. When dealing with negative curves, the concept of signed area will naturally arise. The problem that then arises is that the area under a curve, while quite intuitive, is not easily defined. Let us review what we really know from basic geometry:

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(1) The area of a rectangle is equal to the product of its base times its height.

(2) The area of a triangle is one half its base times its height.

(3) The area of any region bounded by straight lines may be computed by cutting the region up into rectangles and triangles and summing the resulting areas.

The student may recall formulae for areas of circular regions, but would be hard put to prove that a circle can be dissected into pieces that can be arranged to form a square. (Mainly because it's impossible, not because of any failing of the student!) In order to even define the area of regions with curved boundaries we are going to have to develop a lot of mathematical machinery.

Let us agree on some of the properties that the "area" A of a region $\mathcal{R}$ should have:

(1) A should be greater than or equal to the area of any region that $\mathcal{R}$ contains.

(2) A should be less than or equal to the area of any region that contains $\mathcal{R}$.

(3) If $\mathcal{R}$ is divided by a straight line into two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ with respective areas A_1 and A_2 then $A = A_1 + A_2$.

Now we shall look at various ways of estimating the area under a curve which is the graph of a function, using Riemann sums. These are of tremendous theoretical and practical importance. Unfortunately, most Calculus texts don't get around to the practical side. We shall try.

At this point we should look at some Java applets.

Go to http://math.usask.ca/ maclean/Javapage/Riemann.html

Definite Integrals

Definition: A **partition** $\mathcal{P}$ of the closed interval $I = [a, b]$ is a set of closed intervals $I_j = [x_{j-1}, x_j]$ defined by a sequence of numbers

$$x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n$$

where $a = x_0$ and $x_n = b$. We denote the length $x_j - x_{j-1}$ of I_n by Δx_j. The numbers x_j are called **partition points**.

Example: If $a = 1, b = 10$, and we let $x_0 = a = 1$, $x_1 = 2$, $x_3 = 5$, $x_4 = 9$, and $x_5 = 10 = b$, then the partition is $\mathcal{P} = \{[1,2],[2,5],[5,9],[9,10]\} = \{I_1, I_2, I_3, I_4\}$, where $I_1 = [1,2]$, $I_2 = [2,5]$, $I_3 = [5,9]$, $I_4 = [9,10]$, and $\Delta x_1 = 2-1 = 1$, $\Delta x_2 = 5-2 = 3$, $\Delta x_3 = 9-5 = 4$, $\Delta x_4 = 10-9 = 1$.

Definition: A **tagset** T for the partition $\mathcal{P}$ is a set of numbers $\mathsf{T} = \{t_1 \le t_2 \le \cdots \le t_n\}$ which satisfy $t_j \in I_n$.

Example: In the previous example, let $t_1 = 1$, $t_2 = 3$, $t_3 = 7.5$, and $t_4 = 10$, so that $\mathsf{T} = \{1, 3, 7.5, 10\}$.

Definition: $\mathcal{F}([a,b])$ is defined to be the set of all functions with domain $[a, b]$.

Definition: If $f \in \mathcal{F}([a,b])$, and $\mathcal{P}$ is a partition of $[a,b]$ with tagset $\mathcal{T}$, then the **Riemann sum** of f with partition P and tagset T is defined to be

$$S(f, \mathsf{P}, \mathsf{T}) = f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + \cdots + f(t_n)\Delta x_n$$

Example: Using the previous example,

$S(f, \mathsf{P}, \mathsf{T}) = S(f, \{[1,2],[2,5],[5,9],[9,10]\}, \{1, 3, 7.5, 10\}) =$

$f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + f(t_3)\Delta x_3 + f(t_4)\Delta x_4 =$

$f(1)(1) + f(3)(3) + f(7.5)(4) + f(10)(1) =$

$f(1) + 3f(3) + 4f(7.5) + f(10)$

Examples of special Riemann sums:

(1) If we let $t_j = x_{j-1}$, we get the *Left-hand sum* of f associated with the partition P:
$\mathcal{L}(f, \mathsf{P}) = f(x_0)\Delta x_1 + f(x_1)\Delta x_2 + \cdots + f(x_{n-1})\Delta x_n$

(2) If we let $t_j = x_j$, we get the *Right-hand sum* of f associated with the partition P:
$\mathcal{R}(f, \mathsf{P}) = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n$

(3) If we let $t_j = (x_{j-1} + x_j)/2$, we get the *Midpoint sum* of f associated with the partition P:
$\mathcal{M}(f, \mathsf{P}) = f(\frac{x_0+x_1}{2})\Delta x_1 + f(\frac{x_1+x_2}{2})\Delta x_2 + \cdots + f(\frac{x_{n-1}+x_n}{2})\Delta x_n$

Another natural estimate of the area under a curve is the *Trapezoidal Estimate* formed by adding up the areas of the trapezoids with base $[x_{j-1}, x_j]$ and heights $f(x_{j-1})$ and $f(x_j)$:

$$\mathcal{T}(f, \mathsf{P}) = \frac{f(x_0) + f(x_1)}{2}\Delta x_1 + \frac{f(x_1) + f(x_2)}{2}\Delta x_2 + \cdots + \frac{f(x_{n-1}) + f(x_n)}{2}\Delta x_n.$$

It is easily verified that $\mathcal{T}(f, \mathsf{P}) = \dfrac{\mathcal{L}(f, \mathsf{P}) + \mathcal{R}(f, \mathsf{P})}{2}$, that is, that it is the average of the Left-hand and Right-hand sums. It is not itself a Riemann sum.

If we now insist that all the functions f are continuous on the closed interval we can define two more very important Riemann sums:

If s_j is selected so that the minimum m_i of $f(x)$ on the interval I_j occurs at s_j, we get the *Inscribed sum* of f associated with the partition P:
$\mathcal{I}(f, \mathsf{P}) = f(s_1)\Delta x_1 + f(s_2)\Delta x_2 + \cdots + f(s_n)\Delta x_n$
$= m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_n\Delta x_n$

If $f(x) \geq 0$ on $[a, b]$, this is of course the sum of the areas of the rectangles inscribed under the graph of $y = f(x)$ with the intervals I_j as their bases. Note that the numbers s_j need not be unique, since a function can attain its minimum at more than one point in an interval. This does not affect the value of the Inscribed sum. It was necessary to assume that f was continous so as to be certain that it would actually have a minimum value on every interval I_j.

If t_j is selected so that the maximum M_i of $f(x)$ on the interval I_j occurs at t_j, we get the *Exscribed sum* of f associated with the partition P:
$\mathcal{E}(f, \mathsf{P}) = f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + \cdots + f(t_n)\Delta x_n$
$= M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n\Delta x_n$

If $f(x) \geq 0$ on $[a, b]$, this is of course the sum of the areas of the rectangles exscribed over the graph of $y = f(x)$ with the intervals I_j as their bases. Note again that the numbers t_j need not be unique.

It is clear that if f is continuous on $[a, b]$, then for any tagset T we have:

$$\mathcal{I}(f, \mathsf{P}) \leq \mathcal{S}(f, \mathsf{P}, \mathsf{T}) \leq \mathcal{E}(f, \mathsf{P})$$

Definition: The *mesh* or *norm* of the partition P is the number

$$||\mathsf{P}|| = \max\{\Delta x_j : 1 \leq j \leq n\}.$$

Lemma: If f has a continuous derivative on the interval $I_j = [x_{j-1}, x_j]$, then, by the Mean Value Theorem,

$M_j - m_j = f(t_j) - f(s_j) = f'(c_j)(t_j - s_j)$ for some c_j between s_j and t_j.
Thus $M_j - m_j = |f'(c_j)||t_j - s_j| \leq |f'(c_j)|\Delta x_j$.

Theorem: If f is has a continous derivative on $[a, b]$, let $M = \max\{|f'(x)| : x \in [a, b]\}$. Then $\mathcal{E}(f, \mathsf{P}) - \mathcal{I}(f, \mathsf{P}) \leq M(b - a)||\mathsf{P}||$.
Proof: $\mathcal{E}(f, \mathsf{P}) - \mathcal{I}(f, \mathsf{P}) =$
$(M_1 - m_1)\Delta x_1 + (M_2 - m_2)\Delta x_2 + \cdots + (M_n - m_n)\Delta x_n =$
$= |f'(c_1)||t_1 - s_1|\Delta x_1 + |f'(c_2)||t_2 - s_2|\Delta x_2 + \cdots + |f'(c_n)||t_n - s_n|\Delta x_n$
$\leq M||\mathsf{P}||\Delta x_1 + M||\mathsf{P}||\Delta x_2 + \cdots + M||\mathsf{P}||\Delta x_n$
$\leq M||\mathsf{P}||(\Delta x_1 + \cdots + \Delta x_n)$
$= M(b - a)||\mathsf{P}||$ **Q.E.D.**

Corollary: If f has a continuous derivative on $[a, b]$, then for any partition P and tagset T we have:
$\mathcal{E}(f, \mathsf{P}) - \mathcal{R}(f, \mathsf{P}, \mathsf{T}) \leq M(b - a)||\mathsf{P}||$ and $\mathcal{R}(f, \mathsf{P}, \mathsf{T}) - \mathcal{I}(f, \mathsf{P}) \leq M(b - a)||\mathsf{P}||$

Thus at this point we can begin to be convinced that $\lim_{||\mathsf{P}||\to 0} S(f, \mathsf{P}, \mathsf{T})$ should exist if f has a continuous first derivative. As a matter of fact, it can be shown that this limit exists if f is just continuous on $[a, b]$, but this is much more difficult to prove.

Definition: If $\lim_{||\mathsf{P}||\to 0} S(f, \mathsf{P}, \mathsf{T})$ exists, it is called the **definite integral** of f from a to b and is denoted by $\int_a^b f(x)dx$

Properties of the Definite Integral

$$\int_a^b h dx = h(b-a)$$

— if $h > 0$ this is the area of the rectangle with base $[a, b]$ and height h.

$$\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b (f(x) - g(x))\, dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$\int_a^b kf(x)dx = k\int_a^b f(x)dx$, where k is a constant.

If $a > b$ we define $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

Then we have $\int_a^b kf(x)dx = \int_a^c kf(x)dx + \int_c^b kf(x)dx$
for any numbers a, b, c for which all three definite integrals are defined.

Clearly we have $\int_a^a f(x)dx = 0$.

Inequalities of Definite Integrals

If $f(x) \geq 0$ on $[a,b]$ and $a < b$ then $\int_a^b f(x)dx \geq 0$.

Indeed, if f is also continuous at a number c in $[a,b]$ and $f(c) > 0$, we can say that

$\int_a^b f(x)dx > 0$.

We can use these facts to derive some more general inequalities: Suppose $f(x) \geq g(x)$ on $[a,b]$.

Then $f(x) - g(x) \geq 0$ on $[a,b]$, so $\int_a^b (f(x) - g(x))\,dx \geq 0$,

but we also have $\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

so $\int_a^b f(x)dx - \int_a^b g(x)dx \geq 0$, and therefore $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

Thus $f(x) \geq g(x)$ on $[a,b]$ $\quad \Longrightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$

We can use this to get another important chain of inequalities: if $m \leq f(x) \leq M$ on $[a,b]$,

then $\int_a^b m dx \leq \int_a^b f(x)dx$ and $\int_a^b f(x)dx \leq \int_a^b M dx$

But $\int_a^b m dx = m(b-a)$ and $\int_a^b M dx = M(b-a)$, so we have

$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

Dividing every term in this chain of inequalities by the positive number $b - a$ results in

$m \leq \dfrac{\int_a^b f(x)dx}{b-a} \leq M$.

The expression $\dfrac{\int_a^b f(x)dx}{b-a}$ is called the **average value** of $f(x)$ on the interval $[a,b]$, and it should come as no surprise that it lies between the minimum and maximimum values of $f(x)$ on $[a,b]$.

Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, and we let $m = \min\{f(x) | a \leq x \leq b\}$, and $m = \min\{f(x) | a \leq x \leq b\}$, then we know by the Intermediate Value Theorem that

$f(x)$ takes on all values between m and M, so in particular it takes on the *average value* for some number c in $[a, b]$. We therefore have

$f(c) = \dfrac{\int_a^b f(x)dx}{b-a}$, or,

in the usual form in which the Mean Value Theorem for Definite Integrals is stated:

$$\int_a^b f(x)dx = f(c)(b-a)$$

Geometric and Algebraic Areas.

The definite integral $\displaystyle\int_a^b f(x)dx$ is called the **algebraic** area between the graph of $y = f(x)$ and the x-axis. It is the sum of all of the areas of regions where f is positive *minus* the sum of the all the areas of regions where f is negative.

The definite integral $\displaystyle\int_a^b |f(x)|dx$ is called the **geometric** area between the graph of $y = f(x)$ and the x-axis. It is the sum of all of the areas of regions where f is positive *plus* the sum of the all the areas of regions where f is negative.

Clearly $\left|\displaystyle\int_a^b f(x)dx\right| \leq \displaystyle\int_a^b |f(x)|dx$

If f never changes sign on $[a, b]$ then the two quantities will be equal.

If f changes sign on $[a, b]$ and is continuous, then the two quantities will *not* be equal.

Notation

(1) We define $\mathcal{D}(f) = f'$, the derivative of f. $\mathcal{D}$ is usually called the differential operator.

(2) We define $\mathcal{I}(f) = F$, where $F(x) = \int_a^x f(t)dt$, the desired antiderivative F of f for which $F(a) = 0$. $\mathcal{I}$ is usually called an integral operator.

Using this notation, we state, without proof, the:

Fundamental Theorem of Calculus:

If f is continous on $[a,b]$, then $\mathcal{D}(\mathcal{I}(f)) = f$, and if F is differentiable on $[a,b]$ then $\mathcal{I}(\mathcal{D}(F))(b) = \int_a^b \mathcal{D}(F)dx = F(b) - F(a)$.

We *will* prove this with the additional assumption that f has a continuous first derivative.

Notation: instead of writing $F(b) - F(a)$, it will be convenient to use instead the expression

$$F(x)|_a^b,$$

(or even $F(x)|_{x=a}^{x=b}$ when we want to be clear about what is the variable of integration.)

Thus $x^2\Big|_2^5 = 5^2 - 2^2 - 25 - 4 = 21$

Example of an area calculation: suppose we wish to find the area between the x-axis and $y = 3x^2$ from $x = 4$ to $x = 8$.

All we need to do is find an antiderivative $F(x)$ of $3x^2$ — $F(x) = x^3$ will do nicely, and compute

$F(8) - F(4) = 8^3 - 4^3 = 512 - 64 = 448$

The notation generally used looks like this:

$$\int_4^8 3x^2 dx = x^3\Big|_4^8 = 8^3 - 4^3 = 512 - 64 = 448$$

Some Handy Antiderivative, or Indefinite Integral, Formulas:

$$\int (f(x) + g(x))\,dx = \int f(x)dx + \int g(x)dx \qquad \int (f(x) - g(x))\,dx = \int f(x)dx - \int g(x)dx$$

$\int kf(x)dx = k\int f(x)dx$, where k is a constant.

$\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$, if $n \neq -1$

$$\int \sin x dx = -\cos x + C \qquad \int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C \qquad \int \sec x \tan x dx = \sec x + C$$

Definition: The *regular* partition of $[a, b]$ with n intervals is formed by letting

$\Delta x = \dfrac{b-a}{n}$, $x_j = a + j\Delta x$. We have $||\mathsf{P}|| = \Delta x$.

For regular partitions we get nicer formulas for the special Riemann sums we have defined above:

(1) The *Left-hand sum* of f becomes

$$\mathcal{L}(f, \mathsf{P}) = \mathcal{L}_n(f) = \sum_{i=1}^{n} f(x_{i-1})\Delta x = [f(x_0) + f(x_1) + \cdots + f(x_{n-1}]\,\Delta x =$$

$$\left[\sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right)\right]\frac{b-a}{n}$$

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(2) The *Right-hand sum* of f becomes

$$\mathcal{R}(f, \mathsf{P}) = \mathcal{R}_n(f) = \sum_{i=1}^{n} f(x_i)\Delta x = [f(x_1) + f(x_2) + \cdots + f(x_n)]\,\Delta x =$$

$$\left[\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)\right]\frac{b-a}{n}$$

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(3) The *Midpoint sum* of f becomes

$$\mathcal{M}(f, \mathsf{P}) = \mathcal{M}_n(f) = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + f(x_i)}{2}\right)\Delta x =$$

$$\left[f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1} + x_n}{2}\right)\right]\Delta x =$$

$$\left[\sum_{i=1}^{n} f\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{n}\right)\right]\frac{b-a}{n}$$

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(4) The *Inscribed sum* of f becomes $\mathcal{I}(f, \mathsf{P}) = \mathcal{I}_n(f) = [m_1 + m_2 + \cdots + m_n]\Delta x$

(5) The *Exscribed sum* of f becomes $\mathcal{E}(f, \mathsf{P}) = \mathcal{E}_n(f) = [M_1 + M_2 + \cdots + M_n]\Delta x$

The *Trapezoidal Estimate* can now be written as $\mathcal{T}(f, \mathsf{P}) = \mathcal{T}_n(f) = \frac{1}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_n - 1) + f(x_n))\Delta x =$

$$\left[f(a) + 2\sum_{i=1}^{n-1} f\left(a + i\frac{b-a}{n}\right) + f(b)\right]\frac{b-a}{n}$$

Theorem If f has continuous derivative on $[a, b]$,

then $\mathcal{E}_n(f) - \mathcal{I}_n(f) \leq M\frac{(b-a)^2}{n}$, so

$$\lim_{n\to\infty} \mathcal{E}_n(f) - \mathcal{I}_n(f) = 0$$

In plain English, the difference between the sums of exscribed and inscribed rectangles goes to 0 as the number of rectangles approaches infinity. This means that the limit as $n \to \infty$ of any of our sums is equal to the desired value

$$\int_a^b f(x)dx$$

Example:

Let $y = f(x) = x^2$, $a = 0$, $b = 1$. Then $x_i = \frac{i}{n}$

(1) $\mathcal{L}_n(f) = \sum_{i=0}^{n-1}\left(\frac{i}{n}\right)^2\frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n-1} i^2 =$

$$\frac{1}{n^3}\frac{(n-1)(n)(2n-1)}{6} = \frac{1}{n^2}\frac{(n-1)(2n-1)}{6} = \left(1 - \frac{1}{n}\right)\left(\frac{1}{3} - \frac{1}{6n}\right) =$$

$$\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

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(2) $\mathcal{R}_n(f) = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} =$

$$\frac{1}{n^2}\frac{(n+1)(2n+1)}{6} = \left(1 + \frac{1}{n}\right)\left(\frac{1}{3} + \frac{1}{6n}\right) = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

Now $\mathcal{T}_n(f) = \frac{\mathcal{L}_n(f) + \mathcal{R}_n(f)}{2} = \frac{1}{3} + \frac{1}{6n^2}$

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(3) $\mathcal{M}_n(f) = \sum_{i=1}^{n} \left(\frac{2i-1}{2n}\right)^2 \frac{1}{n} = \frac{1}{4n^3} \sum_{i=1}^{n} (2i-1)^2 =$

$\frac{1}{4n^3} \left(\sum_{i=1}^{2n} i^2 - \sum_{i=1}^{n} (2i)^2 \right) = \frac{1}{4n^3} \left(\sum_{i=1}^{2n} i^2 - 4 \sum_{i=1}^{n} i^2 \right) =$

$\frac{1}{4n^3} \left(\frac{2n(2n+1)(2(2n)+1)}{6} - 4\frac{n(n+1)(2n+1)}{6} \right) =$

$\frac{n(2n+1)}{12n^3} \left((2(2n)+1) - 2(n+1) \right) = \frac{2n+1}{12n^2} (4n+1-2n-2) =$

$\frac{2n+1}{12n^2} (2n-1) = \frac{4n^2-1}{12n^2} = \frac{1}{3} - \frac{1}{12n^2}$

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(4) $\mathcal{I}_n(f) = \mathcal{L}_n(f) = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$

(5) $\mathcal{E}_n(f) = \mathcal{R}_n(f) = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$

We note that $\mathcal{E}_n(f) - \mathcal{I}_n(f) = \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) - \left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{n}$

Note that all of these sums go to $\frac{1}{3}$ as $n \to \infty$.

Of course, the easy way to compute $\int_0^1 x^2 dx$ is to use the the Fundamental Theorem of Calculus and the fact that an antiderivative of x^2 is $F(x) = \frac{x^3}{3}$ and simply evaluate

$F(1) - F(0) = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$

The Substitution Rule

Now that we know why we want to be able to find antiderivatives, we realize that we must develop the techniques of finding them.

One of the most important techniques is called the "Method of Substitution".

It is essentially the reverse of the Chain Rule.

The basic idea is that an antidifferentiation problem can be simplified by replacing a complicated expression by substituting a new variable for it.

Example: $\int 5x^2\sqrt[3]{x^3+1}dx$

We see that the expression under the root symbol is a problem, so we let some new symbol reperesent it:

we let $u = x^3 + 1$. Then our problem is to evaluate

$$\int 5x^2\sqrt[3]{x^3+1}dx = \int 5x^2\sqrt[3]{u}dx = \int 5x^2u^{\frac{1}{3}}dx$$

Now this looks simpler, but there is a problem: We have a mixture of variables.

The Umpth Commandment:
Thou shalt not mix thy variables when integrating!

We decided to replace a messy expression involving x, namely x^3+1 with the simpler expression u.

When we do this, we have to replace **every** other x-expression with the appropriate u-expression.

The most critical x-expression that must be looked after is the dx. In this particular example, we have $u = x^3+1$, so if we differentiate and use differentials, we have:

$du = 3x^2dx$, or $dx = \frac{1}{3x^2}du$. Using this in our integral, we have:

$$\int 5x^2\sqrt[3]{x^3+1}dx = \int 5x^2u^{\frac{1}{3}}dx = \int 5x^2u^{\frac{1}{3}}\frac{1}{3x^2}du = \frac{5}{3}\int u^{\frac{1}{3}}du =$$
$$\frac{5}{3}\left(\frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right)+C = \frac{5}{3}\left(\frac{u^{\frac{4}{3}}}{\frac{4}{3}}\right)+C = \frac{5}{3}\left(\frac{3}{4}u^{\frac{4}{3}}\right)+C = \frac{5}{4}u^{\frac{4}{3}} = \frac{5}{4}(x^3+1)^{\frac{4}{3}}+C$$

Having found what we think might be the antiderivative of $5x^2\sqrt[3]{x^3+1}$, we should double check our answer: we let $F(x) = \frac{5}{4}(x^3+1)^{\frac{4}{3}}$, and we compute

$F'(x)$:

$F'(x) = \left(\frac{5}{4}(x^3+1)^{\frac{4}{3}}\right)' = \frac{5}{4}\frac{4}{3}\left((x^3+1)\right)^{\frac{4}{3}-1}(x^3+1)' = \frac{5}{3}\left((x^3+1)\right)^{\frac{1}{3}} 3x^2 = 5\left((x^3+1)\right)^{\frac{1}{3}} x^2 =$
$5x^2\sqrt[3]{x^3+1}$

which is just what we wanted!

The Method of Substitiution is essentially a book-keeping technique. The idea is to substitute one variable for a slightly complicated expression so as to simplify the function being antidifferentiated.

Substitution Rule for Definite Integrals

Now suppose we wish to calculate $\int_0^1 5x^2\sqrt[3]{x^3+1}dx$

When dealing with substitutions in definite integrals, there is a convenient bookkeeping trick: instead of starting with $\int_0^1 5x^2\sqrt[3]{x^3+1}dx$, it is better to use

$\int_{x=0}^{x=1} 5x^2\sqrt[3]{x^3+1}dx$

When we make the substitution $u = x^3+1$, we simply observe that $u = 0^3+1 = 1$ when $x = 0$, and $u = 1^3+1 = 2$ when $x = 0$,

so all we have to do is write

$\int_{x=0}^{x=1} 5x^2\sqrt[3]{x^3+1}dx = \int_{u=1}^{u=2} 5x^2u^{\frac{1}{3}}\frac{1}{3x^2}du = \frac{5}{3}\int_{u=1}^{u=2} u^{\frac{1}{3}}du = \frac{5}{3}\frac{u^{\frac{4}{3}}}{\frac{4}{3}}\Bigg|_{u=1}^{u=2}$

$\frac{5}{4}u^{\frac{4}{3}}\Big|_{u=1}^{u=2} = \frac{5}{4}1^{\frac{4}{3}} - \frac{5}{4}0^{\frac{4}{3}} = \frac{5}{4}$

Example 4.5.6(from Text) "Evaluate $\int_0^4 \sqrt{3x+4}dx$"

We let $u = 3x + 4$, and find $du = 3dx$, so $dx = \frac{1}{3}du$.

Also, $u = 3(0) + 4 = 4$ when $x = 0$, and $u = 3(4) + 4 = 16$ when $x = 4$.

Thus we have

$$\int_0^4 \sqrt{3x+4}dx = \int_{x=0}^{x=4} \sqrt{3x+4}dx = \int_{u=4}^{u=16} \sqrt{u}\left(\frac{1}{3}du\right) = \frac{1}{3}\int_{u=4}^{u=16} u^{\frac{1}{2}}du =$$

$$\frac{1}{3}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_{u=4}^{u=16} = \frac{2}{9}u^{\frac{3}{2}}\Big|_{u=4}^{u=16} = \frac{2}{9}(16)^{\frac{3}{2}} - \frac{2}{9}(4)^{\frac{3}{2}} = \frac{2}{9}\left(2^4\right)^{\frac{3}{2}} - \frac{2}{9}\left(2^2\right)^{\frac{3}{2}} =$$

$$\frac{2}{9}\left(2^{(4\frac{3}{2})}\right) - \frac{2}{9}\left(2^{(2\frac{3}{2})}\right) = \frac{2}{9}\left(2^6\right) - \frac{2}{9}\left(2^3\right) = \frac{2}{9}2^3\left(2^3 - 1\right) = \frac{16}{9}7 = \frac{112}{9}$$