

Indeterminate Forms

L'Hôpital's Rule is used to find limits of quotients where *both* numerator *and* denominator tend to the same *type* of value: either zero or infinite.

We will give a non-rigorous "heuristic" proof of it:

Suppose f and g are differentiable (and therefore continuous) at a and

$\lim_{x \rightarrow a} f(x) = f(a) = 0$ and $\lim_{x \rightarrow a} g(x) = g(a) = 0$ and that we are interested in computing $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$.

By the **Mean Value Theorem**, we may write

$$f(x) = f(a) + f'(c^*(h))(x - a) = f'(c^*(h))(x - a)$$

and

$$g(x) = g(a) + g'(c^{**}(h))(x - a) = g'(c^{**}(h))(x - a)$$

where $c^*(h)$ is the least number for which $f(x) = f(a) + f'(c^*(h))(x - a)$ is true and $c^{**}(h)$ is the least number for which $g(x) = g(a) + g'(c^{**}(h))(x - a)$ is true.

Therefore

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f(a) + f'(c^*(h))(x - a)}{g(a) + g'(c^{**}(h))(x - a)} = \lim_{x \rightarrow a} \frac{f'(c^*(h))}{g'(c^{**}(h))} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

which is one of the standard forms of L'Hôpital's Rule.

Several other forms are given below:

L'Hôpital's Rule (1) If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$, then, if $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ exists, it must equal

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

L'Hôpital's Rule (2) If $\lim_{x \rightarrow a} f(x) = \pm\infty$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$, then, if $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ exists, it must

$$\text{equal } \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

L'Hôpital's Rule (3) If $\lim_{x \rightarrow \infty} f(x) = 0$ and $\lim_{x \rightarrow \infty} g(x) = 0$, then, if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$ exists, it must equal $\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$

L'Hôpital's Rule (4) If $\lim_{x \rightarrow \infty} f(x) = \pm\infty$ and $\lim_{x \rightarrow \infty} g(x) = \pm\infty$, then, if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$ exists, it must equal $\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$

L'Hôpital's Rule (5) If $\lim_{x \rightarrow -\infty} f(x) = 0$ and $\lim_{x \rightarrow -\infty} g(x) = 0$, then, if $\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)}$ exists, it must equal $\lim_{x \rightarrow -\infty} \frac{f'(x)}{g'(x)}$

L'Hôpital's Rule (6) If $\lim_{x \rightarrow -\infty} f(x) = \pm\infty$ and $\lim_{x \rightarrow -\infty} g(x) = \pm\infty$, then, if $\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)}$ exists, it must equal $\lim_{x \rightarrow -\infty} \frac{f'(x)}{g'(x)}$

L'Hôpital's Rule should *not* be used without a little thought beforehand: for example,

$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x^2 - 1)'}{(x - 1)'} = \lim_{x \rightarrow 1} \frac{2x}{1} = 2(1) = 2$ is a correct application of the rule, but

$\lim_{x \rightarrow 0} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 0} \frac{(x^2 - 1)'}{(x - 1)'} = \lim_{x \rightarrow 0} \frac{2x}{1} = 2(0)$ is just plain dumb, because **the rule does not apply:**

neither numerator nor denominator goes to 0! The correct calculation is $\lim_{x \rightarrow 0} \frac{x^2 - 1}{x - 1} = \frac{0^2 - 1}{0 - 1} = \frac{-1}{-1} = 1$, because $\frac{x^2 - 1}{x - 1}$ is continuous at every number except 1.

Example: If $b > 0$, $\lim_{x \rightarrow 0} \frac{b^x - 1}{x} = \lim_{x \rightarrow 0} \frac{(e^{\ln b})^x - 1}{x} = \lim_{x \rightarrow 0} \frac{e^{x \ln b} - 1}{x} =$

$$\lim_{x \rightarrow 0} \frac{(e^{x \ln b} - 1)'}{(x)'} = \lim_{x \rightarrow 0} \frac{\ln b e^{x \ln b}}{1} = \ln b e^{0 \ln b} = \ln b$$

Example: $\lim_{x \rightarrow 0} \frac{\sin x - x}{x} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{1} = \frac{\cos 0 - 1}{1} = \frac{1 - 1}{1} = 0$

Sometimes we must apply L'Hôpital's Rule more than once to get the limit:

Example: $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^2} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{2x} = \lim_{x \rightarrow 0} \frac{-\sin x - 0}{2} = 0$ (2 applications)

Example: $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{-\sin x - 0}{6x} = \lim_{x \rightarrow 0} \frac{-\cos x}{6} = \frac{-\cos 0}{6} = -\frac{1}{6}$ (3 applications)

Example: $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^4} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{4x^3} = \lim_{x \rightarrow 0} \frac{-\sin x - 0}{12x^2} = \lim_{x \rightarrow 0} \frac{-\cos x}{24x} = \text{Does Not Exist}$ (3 applications)

However:

Example: $\lim_{x \rightarrow 0^+} \frac{\sin x - x}{x^4} = \lim_{x \rightarrow 0^+} \frac{\cos x - 1}{4x^3} = \lim_{x \rightarrow 0^+} \frac{-\sin x - 0}{12x^2} = \lim_{x \rightarrow 0^+} \frac{-\cos x}{24x} = -\infty$ (3 applications)

and

Example: $\lim_{x \rightarrow 0^-} \frac{\sin x - x}{x^4} = \lim_{x \rightarrow 0^-} \frac{\cos x - 1}{4x^3} = \lim_{x \rightarrow 0^-} \frac{-\sin x - 0}{12x^2} = \lim_{x \rightarrow 0^-} \frac{-\cos x}{24x} = \infty$ (3 applications)

Sometimes L'Hôpital's Rule has to be applied over and over and over. In the following example it has to be applied n times, where n is an arbitrary positive integer, so there is no limit to the number of times L'Hôpital's Rule may have to be used to compute a limit.

Example: $\lim_{x \rightarrow \infty} \frac{e^x}{x^n} = \lim_{x \rightarrow \infty} \frac{(e^x)'}{(x^n)'} = \lim_{x \rightarrow \infty} \frac{e^x}{nx^{n-1}} = \dots = \lim_{x \rightarrow \infty} \frac{e^x}{n!} = \infty$ (n applications)

Example: $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt[n]{x}} = \lim_{x \rightarrow \infty} \frac{(\ln x)'}{\left(x^{\frac{1}{n}}\right)'} = \lim_{x \rightarrow \infty} \frac{x^{-1}}{\frac{1}{n}x^{\frac{1-n}{n}}} = n \lim_{x \rightarrow \infty} x^{-1-\frac{1-n}{n}} =$

$$\lim_{x \rightarrow \infty} n \lim_{x \rightarrow \infty} x^{-\frac{1}{n}} = n \lim_{x \rightarrow \infty} \frac{1}{x^{\frac{1}{n}}} = 0$$

Example: $\lim_{x \rightarrow \infty} \frac{\sqrt{5x+2}}{\sqrt{x+2}} = \lim_{x \rightarrow \infty} \frac{(5x+2)^{\frac{1}{2}}}{(x+2)^{\frac{1}{2}}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2}(5x+2)^{-\frac{1}{2}}(5)}{\frac{1}{2}(x+2)^{-\frac{1}{2}}(1)} = 5 \lim_{x \rightarrow \infty} \frac{(5x+2)^{-\frac{1}{2}}}{(x+2)^{-\frac{1}{2}}} =$

$$5 \lim_{x \rightarrow \infty} \frac{-\frac{1}{2}(5x+2)^{-\frac{3}{2}}(5)}{-\frac{1}{2}(x+2)^{-\frac{3}{2}}(1)} = 25 \lim_{x \rightarrow \infty} \frac{(5x+2)^{-\frac{3}{2}}}{(x+2)^{-\frac{3}{2}}} = \dots \text{ without end!}$$

Again, it is always best to think a bit before jumping right in with the heavy machinery: the limit laws tell us

$$\lim_{x \rightarrow \infty} \frac{\sqrt{5x+2}}{\sqrt{x+2}} = \lim_{x \rightarrow \infty} \sqrt{\frac{5x+2}{x+2}} = \sqrt{\lim_{x \rightarrow \infty} \frac{5x+2}{x+2}} = \sqrt{5}$$

Example: $\lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\sec x}{\tan x} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\tan x}{\sec x} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\sec^2 x}{\sec x \tan x} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\sec x}{\tan x}$

and we have come full circle! Again, a little thought goes a long way:

$$\lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\sec x}{\tan x} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{1}{\sin x} = \frac{1}{\sin \frac{\pi}{2}} = 1$$

Indeterminate Products

The limit of a product where one factor approaches 0 and the other approaches $\pm\infty$ can often be computed by conversion to a quotient and applying L'Hospital's Rule:

Example: $\lim_{x \rightarrow 0^+} x e^{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{(e^{\frac{1}{x}})'}{(\frac{1}{x})'} = \lim_{x \rightarrow 0^+} \frac{-\frac{1}{x^2} e^{\frac{1}{x}}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} e^{\frac{1}{x}} = \infty$

Example: $\lim_{x \rightarrow 0^-} x e^{\frac{1}{x}} = \lim_{x \rightarrow 0^-} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} = \lim_{x \rightarrow 0^-} \frac{(e^{\frac{1}{x}})'}{(\frac{1}{x})'} = \lim_{x \rightarrow 0^-} \frac{-\frac{1}{x^2} e^{\frac{1}{x}}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^-} e^{\frac{1}{x}} = 0$

Example:

$$\lim_{x \rightarrow 0^-} x \ln x = \lim_{x \rightarrow 0^-} \frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^-} \frac{(\ln x)'}{(\frac{1}{x})'} = \lim_{x \rightarrow 0^-} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^-} -x = 0$$

Indeterminate Differences

The limit of a difference where both factors approach $\pm\infty$ can be computed by conversion to a quotient and applying L'Hospital's Rule:

Example: $\lim_{x \rightarrow 0} (\csc x - \cot x) = \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) = \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{\sin x} \right) =$
 $\lim_{x \rightarrow 0} \left(\frac{(1 - \cos x)'}{(\sin x)'} \right) = \lim_{x \rightarrow 0} \left(\frac{-(-\sin x)}{\cos x} \right) = \lim_{x \rightarrow 0} \tan x = 0$

Indeterminate Powers

The limit of a power such as $\lim_{x \rightarrow a} f(x)^{g(x)}$ can be computed by conversion to a base e power of a product and applying the techniques just introduced:

$$\lim_{x \rightarrow a} f(x)^{g(x)} = \lim_{x \rightarrow a} \left(e^{\ln(f(x))} \right)^{g(x)} = \lim_{x \rightarrow a} e^{g(x) \ln(f(x))} = e^{\lim_{x \rightarrow a} g(x) \ln(f(x))}$$

Example: $\lim_{x \rightarrow 0} x^x = \lim_{x \rightarrow 0} e^{x \ln x} = e^{\lim_{x \rightarrow 0} x \ln x} = e^0 = 1$

Example:

$$\lim_{x \rightarrow \infty} \left(\frac{3x - 4}{3x + 5} \right)^{3x+1}$$

This is most easily handled by looking at the logarithm of the function,

$$\ln \left(\frac{3x - 4}{3x + 5} \right)^{3x+1} = (3x + 1) \ln \left(\frac{3x - 4}{3x + 5} \right) =$$

$$(3x + 1) (\ln(3x - 4) - \ln(3x + 5)) = \frac{\ln(3x - 4) - \ln(3x + 5)}{\frac{1}{3x+1}}$$

and applying L'Hospital's Rule to it:

$$\lim_{x \rightarrow \infty} \frac{\ln(3x - 4) - \ln(3x + 5)}{\frac{1}{3x+1}} = \lim_{x \rightarrow \infty} \frac{(\ln(3x - 4) - \ln(3x + 5))'}{\left(\frac{1}{3x+1} \right)'} =$$

$$\lim_{x \rightarrow \infty} \frac{\frac{3}{3x-4} - \frac{3}{3x+5}}{\frac{-3}{(3x+1)^2}} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{3x-4} - \frac{1}{3x+5} \right)}{\frac{-1}{(3x+1)^2}} =$$

$$-\lim_{x \rightarrow \infty} (3x + 1)^2 \left(\frac{1}{3x - 4} - \frac{1}{3x + 5} \right) =$$

$$-\lim_{x \rightarrow \infty} (3x + 1)^2 \left(\frac{3x + 5}{(3x - 4)(3x + 5)} - \frac{3x - 4}{(3x - 4)(3x + 5)} \right) =$$

$$-\lim_{x \rightarrow \infty} (3x + 1)^2 \left(\frac{9}{(3x - 4)(3x + 5)} \right) = -9 \lim_{x \rightarrow \infty} \left(\frac{(3x + 1)^2}{(3x - 4)(3x + 5)} \right) = -9$$

so the desired limit is e^{-9}

Exercises

(1) Find a value of the constant c that will cause $\lim_{x \rightarrow 0} \frac{\sin x - x - cx^3}{x^4}$ to exist and be equal to a finite number.

(2) Using the value of c from (1), compute $\lim_{x \rightarrow 0} \frac{\sin x - x - cx^3}{x^5}$.

(3) Evaluate $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x}$

(4) Evaluate $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2}$

(5) Evaluate $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x^3}$, $\lim_{x \rightarrow 0^-} \frac{\cos x - 1}{x^3}$, and $\lim_{x \rightarrow 0^+} \frac{\cos x - 1}{x^3}$

(6) Find a value of the constant c that will cause $\lim_{x \rightarrow 0} \frac{\cos x - 1 - cx^3}{x^3}$ to exist and be equal to a finite number.

(7) Using the value of c from (6), compute $\lim_{x \rightarrow 0} \frac{\cos x - 1 - cx^3}{x^4}$